ON THE CONTINUITY OF BROWNIAN MOTION WITH A MULTIDIMENSIONAL PARAMETER

TUNEKITI SIRAO

§ 1. Introduction

A stochastic process $X(A, \omega)$ is called Brownian motion with an N-dimensional parameter when it satisfies the following conditions:

1) For any positive integer n and any set of points $A_1, A_2, \ldots, A_n$ in an N-dimensional Euclidian space E_N, the joint variable $\langle X_i = X(A_i)\,;\ i=1, 2, \ldots, n\rangle$ is subject to an n-dimensional Gaussian distribution having the vector $\mathbf{0}$ as its mean vector.

2) $$E(X_i X_j) = \{\mathrm{dis}\,(O, A_i) + \mathrm{dis}\,(O, A_j) - \mathrm{dis}\,(A_i, A_j)\}/2,$$

where $E(X)$, O, and $\mathrm{dis}\,(A, B)$ denote the expectation of X, the origin of E_N, and the Euclidian distance between A and B respectively.

3) For almost every sample point ω, $X(A, \omega)$ is continuous in A and $X(O, \omega)=0$. The random variables $X(A)-X(B)$ evidently form Wiener process if A moves on some demi-straight line with the terminal point B. In this paper, we study the continuity of Brownian motion process with an N-dimensional parameter.

Let us begin with the definitions of the concepts of upper class and lower class with respect to $\{X(A)\,;\ A\in E_N\}$. Let $\psi(t)$ be a non-negative and non-decreasing function defined for large t's.

i) If the set of A satisfying

$$X(A, \omega) > (\mathrm{dis}\,(O, A))^{1/2}\,\psi(\mathrm{dis}\,(O, A))$$

is bounded (unbounded) for almost all ω, we say that $\psi(t)$ belongs to the upper (lower) class with respect to $\{X(A)\,;\ A\in E_N\}$ at ∞ and denote it by $\psi(t)\in \mathfrak{U}_N^\infty$ ($\psi(t)\in \mathfrak{L}_N^\infty$).

ii) If the set of A satisfying

$$X(A, \omega) > (\mathrm{dis}\,(O, A))^{1/2}\,\psi(1/\mathrm{dis}\,(O, A))$$

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is separated (not separated) from O for almost all ω, we say that $\psi(t)$ belongs to the upper (lower) class with respect to $\{X(A);\ A\in E_N\}$ at O and denote it by $\psi(t)\in\mathfrak{U}_N^{\circ}$ $(\psi(t)\in\mathfrak{L}_N^{\circ})$.

According to the theorem of projective invarince, $\psi(t)$ belongs to $\mathfrak{U}_N^{\infty}$ $(\mathfrak{L}_N^{\infty})$ if and only if $\psi(t)$ belongs to $\mathfrak{U}_N^{\circ}$ $(\mathfrak{L}_N^{\circ})$. Therefore, we have only to discuss the behavior of $X(A)$ in the neighborhood of O.

For Wiener process, namely for Brownian motion with 1-dimensional parameter, we have the following criterion of Kolmogorov [1]: a monotone non-decreasing function $\varphi(t)$ belongs to $\mathfrak{U}_1^{\circ}$ $(\mathfrak{L}_1^{\circ})$ if and only if

$$\int^{\infty}\frac{1}{t}\,\psi(t)\,e^{-\frac{1}{2}\psi^2(t)}\,dt<+\infty\,(=+\infty).$$

This criterion shows that the function

$$\psi(t)=\{2\ \log_{(2)}t+3\ \log_{(3)}t+\cdots+2\ \log_{(n-1)}t+(2+\delta)\ \log_{(n)}t\}^{1/2}$$

belongs to $\mathfrak{U}_1^{\circ}$ for $\delta>0$ and belongs to $\mathfrak{L}_1^{\circ}$ for $\delta\leqq 0$, where $\log_{(n)}t$ denotes the n-time iterated logarithm. We shall extend this result to Brownian motion with an N-dimensional parameter using Chung-Erdös' method in §3.

Secondly, we define similar concepts with regard to the uniform continuity of $X(A)$. Let $\varphi(t)$ be a non-negative, continuous, and non-decreasing function defined in some finite interval $(O,\ T)$, and $f(A)$ be a function defined on some region in E_N.

If there exists a positive number ε such that dis $(A,\ B)\leqq\varepsilon$ implies

$$|f(A)-f(B)|\leqq\varphi(\mathrm{dis}\,(A,\ B)),$$

we say that $f(A)$ satisfies Lipschitz's condition relative to $\varphi(t)$. We put now $\psi(t)=\varphi(1/t)\ t^{1/2}$ and consider the cube $U_N=\{A=(a_1,\ a_2,\ \ldots,\ a_N);\ \max_{1\leqq i\leqq N}|a_i|\leqq 1\}$. If the process $X(A,\ \omega)$ with the parameter domain U_N satisfies (does not satisfy) Lipschitz's condition relative to $\psi(t)$ for almost all ω, we say that $\varphi(t)$ belongs to the upper (lower) class with regard to the uniform continuity of $\{X(A);\ A\in U_N\}$, and denote it by $\varphi(t)\in\mathfrak{U}_N^{u}$ $(\mathfrak{L}_N^{u})$.

P. Lévy remarked in his book [2] that the concepts of upper class and lower class with regard to the uniform continuity of $X(A)$ are meaningful only for the process with a bounded parameter domain. Accordingly, it is sufficient to define the concepts for $\{X(A);\ A\in U_N\}$.

For Wiener process, P. Lévy [3] proved that the function

$$\xi(t) = \{2\, c \log t\}^{1/2}$$

belongs to $\mathfrak{U}_1^u$ for $c>1$ and belongs to $\mathfrak{L}_1^u$ for $c<1$. Recently K. L. Chung, P. Erdös, and T. Sirao [4] proved a final form of the criterion which reads: $\varphi(t)$ belongs to $\mathfrak{U}_1^u$ ($\mathfrak{L}_1^u$) if and only if the integral

$$\int^{\infty} \varphi^3(t)\, e^{-\frac{1}{2}\varphi^2(t)}\, dt$$

is convergent (divergent). In virtue of this criterion, we can easily see that the function

$$\varphi(t) = \{2 \log t + 5 \log_{(2)} t + 2 \log_{(3)} t + \cdots + 2 \log_{(n-1)} t + (2+\delta) \log_{(n)} t\}^{1/2}$$

belongs to $\mathfrak{U}_1^u$ for $\delta>0$ and belongs to $\mathfrak{L}_1^u$ for $\delta \leqq 0$.

Also, for Brownian motion with an N-dimensional parameter, P. Lévy [5] proved that the function

$$\eta(t) = \{2\, Nc \log t\}^{1/2}$$

belongs to $\mathfrak{U}_N^u$ for $c>1$ and belongs to $\mathfrak{L}_N^u$ for $c<1$. This result was improved by T. Hida [6] as follows:

$$\zeta(t) = \{2\, N \log t + c \log_{(2)} t\}^{1/2}$$

belongs to $\mathfrak{U}_N^u$ for $c>8\,N+1$ and belongs to $\mathfrak{L}_N^u$ for $c<1$. In § 2, the author proves a final form of the criterion, a generalization of Chung-Erdös-Sirao's result, for Brownian motion with an N-dimensional parameter. We shall here use the same method as in the 1-dimensional case [4] with some device of computation which will be necessary to overcome the difficulty due to high dimensionality.

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§ 2. Uniform continuity of Brownian motion with an N-dimensional parameter

Concerning the uniform continuity of $X(A)$, we have

THEOREM 1. *Let $\varphi(t)$ be a non-negative, continuous, and non-decreasing function defined for large t's. Then $\varphi(t)$ belongs to $\mathfrak{U}_N^u$ or $\mathfrak{L}_N^u$ according as the*

integral

$$\int^{\infty} t^{N-1} \varphi^{4N-1}(t)\, e^{-\frac{1}{2}\varphi^2(t)}\, dt \tag{1}$$

is convergent or divergent.

In virtue of this theorem, we obtain easily

COR. 1. *The function*

$$\varphi(t) = \{2N \log t + (4N+1) \log_{(2)} t + 2 \log_{(3)} t + \cdots + 2 \log_{(n-1)} t + (2+\delta) \log_{(n)} t\}^{1/2}$$

belongs to $\mathfrak{U}_N^u$ *for* $\delta > 0$ *and belongs to* $\mathfrak{L}_N^u$ *for* $\delta \leqq 0$.

By $\log^+_{(n)} t$, let us denote $\log_{(n)} t$ so long as it is defined and positive, and 0 elsewhere. Namely,

$$\log^+_{(n)} t = \begin{cases} \log_{(n)} t & \text{for } a_n < t < +\infty \\ 0 & \text{for } 0 < t \leqq a_n, \end{cases} \tag{2}$$

where a_n is defined by $\log_{(n)} a_{n+1} = 1$ and $a_1 = 1$. Then we obtain

COR. 2. *The function*

$$\varphi_\infty(t) = \{2N \log^+ t + (4N+1) \log^+_{(2)} t + 2 \sum_{n=3}^{\infty} \log^+_{(n)} t\}^{1/2}$$

belongs to $\mathfrak{L}_N^u$.

Proof. By the definition of $\log^+_{(n)} t$, we have

$$\begin{aligned} \int_{a_2}^{\infty} t^{N-1} \varphi_\infty^{4N-1}(t)\, e^{-\frac{1}{2}\varphi^2_\infty(t)} dt &= \sum_{n=2}^{\infty} \int_{a_n}^{a_{n+1}} t^{N-1} \varphi_\infty^{4N-1}(t)\, e^{-\frac{1}{2}\varphi^2_\infty(t)} dt \\ &> (2N)^N \sum_{n=2}^{\infty} \int_{a_n}^{a_{n+1}} (t \prod_{k=1}^{n-1} \log_{(k)} t)^{-1} dt \\ &= (2N)^N \sum_{n=2}^{\infty} [\log_{(n)} t]_{a_n}^{a_{n+1}} = +\infty. \end{aligned} \tag{3}$$

So our assertion follows from Theorem 1.

Before going into the proof of Theorem 1, we state

LEMMA 1. *Theorem* 1 *holds, if it holds under the following condition:*

$$(2N \log t - 10N \log_{(2)} t)^{1/2} \leqq \varphi(t) \leqq (2N \log t + 10N \log_{(2)} t)^{1/2}. \tag{4}$$

Proof. If we put

$$\hat{\varphi}(t) = \min\{\max(\varphi(t), \varphi_1(t)), \varphi_2(t)\}, \tag{5}$$

where

$$\varphi_1(t) = \{2\,N \log t - 10\,N \log_{(2)} t\}^{1/2},$$
$$\varphi_2(t) = \{2\,N \log t + 10\,N \log_{(2)} t\}^{1/2},$$

then $\hat{\varphi}(t)$ satisfies the condition (4).

First, let us consider the case in which the integral (1) for $\varphi(t)$ is convergent. If there exists a monotone increasing sequence $\{t_n\}$ such that $\varphi(t_n)$ is less than $\varphi_1(t_n)$, and t_n tends to infinity with n, we have

$$\begin{aligned}\int^{\infty} t^{N-1}\varphi^{4N-1}(t)\ e^{-\frac{1}{2}\varphi^2(t)}\,dt &> \int_{t_1}^{t_n} t^{N-1}\varphi^{4N-1}(t)\ e^{-\frac{1}{2}\varphi^2(t)}\,dt \\ &\geqq \int_{t_1}^{t_n} t^{N-1}\varphi^{4N-1}(t_n)\ e^{-\frac{1}{2}\varphi^2(t_n)}\,dt \\ &\geqq c\,t_n^N\,\varphi^{4N-1}(t_n)\ e^{-\frac{1}{2}\varphi^2(t_n)} \qquad (6)\\ &\geqq c\,t_n^N\,\varphi_1^{4N-1}(t_n)\ e^{-\frac{1}{2}\varphi_1^2(t_n)} \\ &= c(\log t_n)^{7N-\frac{1}{2}}\end{aligned}$$

because $\varphi(t)$ is monotone non-decreasing, where c is a suitably chosen positive constant. Since $\log t_n$ tends to infinity with n, no such $\{t_n\}$ can exist in the present case. Therefore, $\varphi(t) > \varphi_1(t)$ and also $\varphi(t) \geqq \hat{\varphi}(t)$ for sufficiently large t's. Moreover, the integral (1) for $\varphi_2(t)$ is convergent, so the integral (1) for $\hat{\varphi}(t)$ is convergent and $\hat{\varphi}(t)$ belongs to $\mathfrak{U}_N^u$ if Theorem 1 holds under the condition (4). As $\varphi(t) \geqq \hat{\varphi}(t)$ for sufficiently large t's, $\varphi(t)$ belongs to $\mathfrak{U}_N^u$.

Secondly, let us consider the case in which the integral (1) for $\varphi(t)$ is divergent. If there is an increasing sequence $\{t_n\}$ such that $\varphi(t_n) < \varphi_1(t_n)$ and t_n tends to infinity with n, we have

$$\begin{aligned}\int^{\infty} t^{N-1}\hat{\varphi}^{4N-1}(t)\ e^{-\frac{1}{2}\hat{\varphi}^2(t)}\,dt &\geqq c t_n^N\,\hat{\varphi}(t_n)\ e^{-\frac{1}{2}\hat{\varphi}^2(t_n)} \\ &= c t_n^N\,\varphi_1(t_n)\ e^{-\frac{1}{2}\varphi_1^2(t_n)} \qquad (7)\\ &= c(\log t_n)^{7N-\frac{1}{2}}\end{aligned}$$

because $\hat{\varphi}(t)$ is monotone non-decreasing and $\hat{\varphi}(t_n) = \varphi_1(t_n)$, where c is a suitably chosen positive constant. On the contrary, if $\varphi_1(t)$ is less than $\varphi(t)$ for large t's, then $\varphi(t) \geqq \hat{\varphi}(t)$ for large t's and hence there exists a positive constant c such that

$$\int^{\infty} t^{N-1}\hat{\varphi}^{4N-1}(t)\ e^{-\frac{1}{2}\hat{\varphi}^2(t)}\,dt \geqq c\int^{\infty} t^{N-1}\varphi^{4N-1}(t)\ e^{-\frac{1}{2}\varphi^2(t)}\,dt = +\infty. \qquad (8)$$

Now (7) and (8) show that the integral for $\hat{\varphi}(t)$ is divergent in the present case. Namely, $\hat{\varphi}(t)$ belongs to $\mathfrak{L}_N^u$, if Theorem 1 holds under the condition (4), i.e. for almost all ω, there exists a sequence $\{(A_n, B_n);\ A_n, B_n \in U_N\}$ in which $\mathrm{dis}(A_n, B_n)$ tends to 0 as n increases to infinity and satisfying the condition

$$(9) \qquad |X(A_n, \omega) - X(B_n, \omega)| > (\mathrm{dis}(A_n, B_n))^{1/2}\ \hat{\varphi}(1/\mathrm{dis}(A_n, B_n)).$$

Moreover, $\varphi_2(t)$ belongs to $\mathfrak{U}_N^u$ if Theorem 1 holds under the condition (4). Hence, for almost all ω, there exists a positive number ε such that $\mathrm{dis}(A, B) < \varepsilon$ implies

$$(10) \qquad |X(A, \omega) - X(B, \omega)| < (\mathrm{dis}(A, B))^{1/2}\ \varphi_2(1/\mathrm{dis}(A, B)).$$

From (9) and (10), follows the inequality

$$\hat{\varphi}(1/\mathrm{dis}(A_n, B_n)) < \varphi_2(1/\mathrm{dis}(A_n, B_n))$$

for large n's. By the definition of $\hat{\varphi}(t)$, we obtain

$$(11) \qquad \varphi(1/\mathrm{dis}(A_n, B_n)) \leqq \hat{\varphi}(1/\mathrm{dis}(A_n, B_n)).$$

Here (9) and (11) show that $\varphi(t)$ belongs to $\mathfrak{L}_N^u$.
Thus Lemma 1 has been proved.

Proof of Theorem 1

a) The convergent case

First, we remark that it suffices to prove, for almost all ω, the existence of ε' such that $\mathrm{dis}(A, B) \leqq \varepsilon'$ implies the inequality

$$(12) \qquad X(A, \omega) - X(B, \omega) < (\mathrm{dis}(A, B))^{1/2}\ \varphi(1/\mathrm{dis}(A, B)).$$

In fact, if this assertion holds then, for almost all ω, there exists a positive ε'' such that $\mathrm{dis}(A, B) \leqq \varepsilon''$ implies

$$(13) \qquad -\{X(A, \omega) - X(B, \omega)\} < (\mathrm{dis}(A, B))^{1/2}\ \varphi(1/\mathrm{dis}(A, B))$$

because the process $\{X(A);\ A \in U_N\}$ is symmetric. Taking $\min(\varepsilon', \varepsilon'')$ for ε *in the definition of* $\mathfrak{U}_N^u$, *we have Theorem 1 in the present case. Therefore, we* may consider only the difference of $X(A)$ and $X(B)$ instead of its absolute value.

By $E^p_{\langle k_1, \dots, k_N; l_1, \dots, l_N\rangle}$ (shortly $E^p_{\langle k_i, l_i\rangle}$), we denote the following event:

$$(14) \qquad X(A) - X(B) > (\mathrm{dis}(A, B))^{1/2}\ \varphi(1/\mathrm{dis}(A, B)),$$

where $A = \langle (k_1 + l_1)/2^p, \ldots, (k_N + l_N)/2^p \rangle$ and $B = \langle k_1/2^p, \ldots, k_N/2^p \rangle$ are points in U_N. Then we have for large p's that

$$P(E^p_{\langle k_i, l_i \rangle}) \sim e^{-\frac{1}{2}\varphi^2(2^p/(\sum_{i=1}^N l_i^2)^{1/2})}/\varphi(2^p/(\sum_{i=1}^N l_i^2)^{1/2}). \tag{15}$$

Summing up the above probability for $p = 1, 2, \ldots$; $k_i = \pm 1, \pm 2, \ldots, \pm 2^p$ $(i = 1, 2, \ldots, N)$ and for all lattice points $\langle (k_1 + l_1)/2^p, \ldots, (k_N + l_N)/2^p \rangle$ satisfying $p/3 < (\sum_{i=1}^N l_i^2)^{1/2} \leqq p$, we obtain

$$\sum_{p=1}^{\infty} \sum_{\langle k_i \rangle} \sum_{\langle l_i \rangle} P(E^p_{\langle k_i, l_i \rangle}) = 0(1) \sum_{p=1}^{\infty} \sum_{\langle k_i \rangle} \sum_{\langle l_i \rangle} \frac{1}{\varphi(2^p/(\sum_{i=1}^N l_i^2)^{1/2})} e^{-\frac{1}{2}\varphi^2(2^p/(\sum_{i=1}^N l_i^2)^{1/2})}.$$

By the monotony of $\varphi(t)$ and Lemma 1, we have

$$\begin{aligned}
\sum_{p=1}^{\infty} \sum_{\langle k_i \rangle} \sum_{\langle l_i \rangle} P(E^p_{\langle k_i, l_i \rangle}) &= 0(1) \sum_{p=1}^{\infty} \sum_{\langle k_i \rangle} \sum_{\langle l_i \rangle} \frac{1}{\varphi(2^p/(\sum_{i=1}^N l_i^2)^{1/2})} e^{-\frac{1}{2}\varphi^2(2^p/(\sum_{i=1}^N l_i^2)^{1/2})} \\
&= 0(1) \sum_{p-1}^{\infty} \sum_{\langle k_i \rangle} \frac{p^N}{\varphi(2^p/p)} e^{-\frac{1}{2}\varphi^2(2^p/p)} \\
&= 0(1) \sum_{p=1}^{\infty} \frac{2^{pN} p^N}{\varphi(2^p/p)} e^{-\frac{1}{2}\varphi^2(2^p/p)} \\
&= 0(1) \sum_{p=1}^{\infty} \left(\frac{2^p}{p}\right)^{N-1} \left(\frac{2^{p+1}}{p+1} - \frac{2^p}{p}\right) \varphi^{4N-1}(2^p/p)\, e^{-\frac{1}{2}\varphi^2(2^p/p)} \\
&= 0(1) \int^{\infty} t^{N-1} \varphi^{4N-1}(t)\, e^{-\frac{1}{2}\varphi^2(t)}\, dt < +\infty.
\end{aligned} \tag{16}$$

Now let us take an event $E^p_{\langle k_i, l_i \rangle}$ appearing in the summand of (16) and fix it. By $F^{(1)}_{\langle m_1^{(1)}, \ldots, m_N^{(1)};\, n_1^{(1)}, \ldots, n_n^{(1)} \rangle}$ (shortly $F^{(1)}_{\langle m_i^{(1)}, n_i^{(1)} \rangle}$), we denote the following event:

$$\begin{aligned}
&X(A_{\langle m_i^{(1)} \rangle}) - X(B_{\langle n_i^{(1)} \rangle}) > (\mathrm{dis}\,(A_{\langle m_i^{(1)} \rangle}, B_{\langle n_i^{(1)} \rangle}))^{1/2} \\
&\times \left\{ \varphi(2^p/(\sum_{i=1}^N l_i^2)^{1/2}) + \frac{2\,NC}{\varphi(2^p/(\sum_{i=1}^N l_i^2)^{1/2})} \right\}, \quad m_i^{(1)}, n_i^{(1)} = 0, \pm 1, \pm 2, \ldots, \pm e^c,
\end{aligned} \tag{17}$$

where $A_{\langle m_i^{(1)} \rangle} = \langle (k_1 + l_1 + m_1^{(1)} e^{-c})/2^p, \ldots, (k_N + l_N + m_N^{(1)} e^{-c})/2^p \rangle$ and $B_{\langle n_i^{(1)} \rangle} = \langle (k_1 + n_1^{(1)} e^{-c})/2^p, \ldots, (k_N + n_N^{(1)} e^{-c})/2^p \rangle$ are points in U_N and c is a suitably chosen constant which makes e^c an integer. For sufficiently large c and p, it follows that

$$\begin{aligned}
\sum_{\langle m_i^{(1)}, n_i^{(1)} \rangle} P(F^{(1)}_{\langle m_i^{(1)}, n_i^{(1)} \rangle}) &= 0(1)\, e^{-\frac{1}{2}\varphi^2(2^p/(\sum_{i=1}^N l_i^2)^{1/2})}/\varphi(2^p/(\sum_{i=1}^N l_i^2)^{1/2}) \\
&= 0(1)\, P(E^p_{\langle k_i, l_i \rangle}).
\end{aligned} \tag{18}$$

Also we define $F^{(k)}_{\langle m_i^{(k)},\, n_i^{(k)}\rangle}$ as follows:

$$X(A_{\langle m_i^{(k)}\rangle}) - X(B_{\langle n_i^{(k)}\rangle}) >$$

$$(19)\qquad (\mathrm{dis}\,(A_{\langle m_i^{(k)}\rangle},\ B_{\langle n_i^{(k)}\rangle}))^{1/2}\{\varphi(2^p/(\sum_{i=1}^{N} l_i^2)^{1/2}) + \frac{2NC}{\varphi(2^p/(\sum_{i=1}^{N} l_i^2)^{1/2})}\sum_{r=0}^{k-1} 1/2^r\},$$

$$m_i^{(k)},\ n_i^{(k)},\ = 0,\ \pm 1,\ \pm 2,\ \ldots,\ \pm e^{kc},$$

where $A_{\langle m_i^{(k)}\rangle} = \langle (k_1 + l_1 + m_1^{(k)} e^{-kc})/2^p,\ \ldots,\ (k_N + l_N + m_N^{(k)} e^{-kc})/2^p\rangle$ and $B_{\langle n_i^{(k)}\rangle} = \langle (k_1 + n_1^{(k)} e^{-kc})/2^p,\ \ldots,\ (k_N + n_N^{(k)} e^{-kc})/2^p\rangle$. Then we have

$$(20)\qquad P(\bigcup_{\langle m_i^{(k)},\, n_i^{(k)}\rangle} F^{(k)}_{\langle m_i^{(k)},\, n_i^{(k)}\rangle}) \leqq P(\bigcup_{\langle m_i^{(k-1)},\, n_i^{(k-1)}\rangle} F^{(k-1)}_{\langle m_i^{(k-1)},\, n_i^{(k-1)}\rangle})$$
$$+ \sum_{\langle m_i^{(k)},\, n_i^{(k)}\rangle} P\{(\bigcap_{\langle m_i^{(k-1)},\, n_i^{(k-1)}\rangle} F^{(k-1)'}_{\langle m_i^{(k-1)},\, n_i^{(k-1)}\rangle}) \cap F^{(k)}_{\langle m_i^{(k)},\, n_i^{(k)}\rangle}\},$$

where F' denotes the complement of F for any event F, $F \cap G$ denotes the event that both F and G hold, and $F \cup G$ denotes the event that F or G holds, for any pair of events F and G.

To estimate the second term in the right side of (20), we use the following:

LEMMA 2. *Let U and V be two random variables whose joint distribution is a 2-dimensional Gaussian distribution and each of them is subject to the 1-dimensional standard Gaussian distribution, and let ρ denote the correlation coefficient between U and V. The function*

$$F(a,\ b;\ \rho) \equiv P(U < a,\ V > b)$$

is monotone decreasing as a function of ρ for fixed a and b $(0 < a < b)$.

Proof. Let W be a random variable independent of V and subject to the 1-dimensional standard Gaussian distribution. Since $(U,\ V)$ and $((1-\rho^2)^{1/2} W + \rho V,\ V)$ are subject to the same distribution, we have

$$F(a,\ b;\ \rho) = P((1-\rho^2)^{1/2} W + \rho V < a,\ V > b)$$
$$= \frac{1}{(2\pi)^{1/2}} \int_b^\infty P(W < (a - \rho v)/(1-\rho^2)^{1/2})\, e^{-\frac{1}{2} v^2} dv.$$

This equality shows Lemma 2, because $(a - \rho v)/(1-\rho^2)^{1/2}$ is monotone decreasing in ρ in the present case.

Let us take a pair of points $(A_{\langle m_{i_0}^{(k-1)}\rangle},\ B_{\langle n_{i_0}^{(k-1)}\rangle})$ satisfying the following

conditions:

$$(A.1)\qquad \begin{aligned} &\mathrm{dis}\,(A_{\langle m_{i_0}^{(k-1)}\rangle},\ A_{\langle m_i^{(k)}\rangle}) \leqq N^{1/2} e^{-(k-1)c}/2^{p+1}, \\ &\mathrm{dis}\,(B_{\langle n_{i_0}^{(k-1)}\rangle},\ B_{\langle n_{i_0}^{(k)}\rangle}) \leqq N^{1/2} e^{-(k-1)c}/2^{p+1}. \end{aligned}$$

From the definition of Brownian motion with an N-dimensional parameter, for the correlation coefficient ρ between $(X(A_{\langle m_{i_0}^{(k-1)}\rangle} - X(B_{\langle n_{i_0}^{(k-1)}\rangle}))$ and $(X(A_{\langle m_i^{(k)}\rangle}) - X(B_{\langle n_i^{(k)}\rangle}))$ holds

$$\begin{aligned} \rho = \{&\mathrm{dis}\,(A,\ B') + \mathrm{dis}\,(A',\ B) - \mathrm{dis}\,(A,\ A') \\ &- \mathrm{dis}\,(B,\ B')\}/2\{\mathrm{dis}\,(A,\ B)\mathrm{dis}\,(A',B')\}^{1/2}, \end{aligned}$$

where $A = A_{\langle m_i^{(k)}\rangle}$, $B = B_{\langle n_i^{(k)}\rangle}$, $A' = A_{\langle m_{i_0}^{(k-1)}\rangle}$, and $B' = B_{\langle n_{i_0}^{(k-1)}\rangle}$. Using (A.1) and the condition $\mathrm{dis}\,(A,\ B) > 2^{-p}p/3$, we have

$$\begin{aligned} \rho > [&\mathrm{dis}\,(A,\ B) - \mathrm{dis}\,(A,\ A') - \mathrm{dis}\,(B,\ B')]\ [\mathrm{dis}\,(A,\ B)\ \{\mathrm{dis}\,(A,\ B) \\ &+ \mathrm{dis}\,(A,\ A') + \mathrm{dis}\,(B,\ B')\}]^{-1/2} > \rho_0, \end{aligned}$$

where $\rho_0 = 1 - (9\,N^{1/2})/2\,pe^{(k-1)c}$.

Now we return to the estimation of the right side of (20). In virtue of Lemma 2, we obtain, using φ for $\varphi(2^p/(\sum_{i=1}^{N} l_i^2)^{1/2})$,

$$\begin{aligned} &P\{(\bigcap_{\langle m_i^{(k-1)},\ n_i^{(k-1)}\rangle} F^{(k-1)'}_{\langle m_i^{(k-1)},\ n_i^{(k-1)}\rangle}) \cap F_{\langle m_i^{(k)},\ n_i^{(k)}\rangle}\} < P\{F^{(k-1)'}_{\langle m_{i_0}^{(k-1)},\ n_{i_0}^{(k-1)}\rangle} \cap F^{(k)}_{\langle m_i^{(k)},\ n_i^{(k)}\rangle}\} \\ &= P\Big[\ X(A_{\langle m_{i_0}^{(k-1)}\rangle}) - X(B_{\langle n_{i_0}^{(k-1)}\rangle}) \leqq (\mathrm{dis}\,(A_{\langle m_{i_0}^{(k-1)}\rangle},\ B_{\langle n_{i_0}^{(k-1)}\rangle}))^{1/2} \\ &\qquad\qquad \times \Big\{\varphi + \frac{2\,NC}{\varphi}\sum_{r=0}^{k-2} 1/2^r\Big\}, \\ &\qquad X(A_{\langle m_i^{(k)}\rangle}) - X(B_{\langle n_i^{(k)}\rangle}) > (\mathrm{dis}\,(A_{\langle m_i^{(k)}\rangle},\ B_{\langle n_i^{(k)}\rangle}))^{1/2}\Big\{\varphi + \frac{2\,NC}{\varphi}\sum_{r=0}^{k-1} 1/2^r\Big\}\Big] \\ &< P\Big\{(1-\rho_0^2)^{1/2} X + \rho_0 Y < \varphi + \frac{2\,NC}{\varphi}\sum_{r=0}^{k-2} 1/2^r,\ Y > \varphi + \frac{2\,NC}{\varphi}\sum_{r=0}^{k-1} 1/2^r\Big\} \\ &< P\Big\{(1-\rho_0^2)^{1/2} X < -\frac{NC}{2^{k-1}\varphi},\ \ Y > \varphi + \frac{2\,NC}{\varphi}\sum_{r=0}^{k-1} 1/2^r\Big\} \\ &< e^{-2kc(N+1)} P(\bigcup_{\langle m_i^{(1)},\ n_i^{(1)}\rangle} F^{(1)}_{\langle m_i^{(1)},\ n_i^{(1)}\rangle}), \end{aligned} \tag{21}$$

where X and Y are mutually independent random variables subject to the 1-dimensional standard Gaussian distribution. Combining (20) and (21), we have

$$P(\bigcup_{\langle m_i^{(k)},\ n_i^{(k)}\rangle} F^{(k)}_{\langle m_i^{(k)},\ n_i^{(k)}\rangle}) < \{1 + e^{-c} + \cdots + e^{-kc}\} \sum_{\langle m_i^{(1)},\ n_i^{(1)}\rangle} P(F^{(1)}_{\langle m_i^{(1)},\ n_i^{(1)}\rangle}). \tag{22}$$

Let us denote by $\tilde{E}^{p}_{\langle k_i, l_i\rangle}$ the following event:

$$\max_{A,B}\{(X(A)-X(B))/(\mathrm{dis}\,(A,B))^{1/2}\}>\varphi(2^p/(\sum_{i=1}^{N} l_i^2)^{1/2}) + 4NC/\varphi(2^p/(\sum_{i=1}^{N} l_i^2)^{1/2}), \tag{23}$$

where A and B run over the cubes $[(k_1+l_1-1)/2^p, (k_1+l_1+1)/2^p; \ldots; (k_N+l_N-1)/2^p, (k_N+l_N+1)/2^p]$ and $[(k_1-1)/2^p, (k_1+1)/2^p; \ldots; (k_N-1)/2^p, (k_N+1)/2^p]$ respectively. Since $X(A)$ is continuous, we have by (18), (22)

$$\begin{aligned} P(\tilde{E}^{p}_{\langle k_i, l_i\rangle}) &\leqq \liminf P(\bigcup_{\langle m_i^{(k)}, n_i^{(k)}\rangle} F^{(k)}_{\langle m_i^{(k)}, n_i^{(k)}\rangle}) \\ &= 0(1)\, P(E^{p}_{\langle k_i, l_i\rangle}). \end{aligned} \tag{24}$$

From (16) and (24) it follows that

$$\sum_{p=1}^{\infty}\sum_{\langle k_i\rangle}\sum_{\langle l_i\rangle} P(\tilde{E}^{p}_{\langle k_i, l_i\rangle}) < +\infty. \tag{25}$$

According to Borel-Cantelli's lemma in the convergent case, (25) shows that only finitely many events $\tilde{E}^{p}_{\langle k_i, l_i\rangle}$ appearing in (25) can occur for almost all ω. In other words, for almost all ω, there exists p_0 such that no $\tilde{E}^{p}_{\langle k_i, l_i\rangle}$ can occur for p's larger than p_0.

Now, for any pair of points (A, B) of $\mathrm{dis}\,(A, B) < (p_0 - N^{1/2})/2^{p_0}$, we choose p such that

$$(p+1-N^{1/2})/2^{p+1} < \mathrm{dis}\,(A, B) \leqq (p-N^{1/2})/2^p. \tag{26}$$

Evidently $p_0 \leqq p$ and $(p-N^{1/2})/2 < 2^p\ (\mathrm{dis}\,(A, B)) \leqq p - N^{1/2}$. For A and B, let us choose from all pairs of lattice points C_p and D_p of the form $(k_1/2^p, \ldots, k_N/2^p)$, satisfying $\mathrm{dis}\,(C_p, D_p) \geqq \mathrm{dis}\,(A, B)$, a pair (A_p, B_p) which minimizes $\mathrm{dis}\,(A, C_p) + \mathrm{dis}\,(B, D_p)$. The event

$$X(A_p) - X(B_p) > (\mathrm{dis}\,(A_p, B_p))^{1/2}\varphi(1/\mathrm{dis}\,(A_p, B_p)) \tag{27}$$

is identical with some $E^{p}_{\langle k_i, l_i\rangle}$ appearing in the summation of (16). Considering the corresponding $\tilde{E}^{p}_{\langle k_i, l_i\rangle}$, for almost all ω, we obtain

$$\begin{aligned} X(A) - X(B) &\leqq (\mathrm{dis}\,(A, B))^{1/2}\left\{\varphi(1/\mathrm{dis}\,(A_p, B_p)) + \frac{4NC}{\varphi(1/\mathrm{dis}\,(A_p, B_p)}\right\} \\ &\leqq (\mathrm{dis}\,(A, B))^{1/2}\left\{\varphi(1/\mathrm{dis}\,(A, B)) + \frac{4NC}{\varphi(1/\mathrm{dis}\,(A, B))}\right\} \end{aligned} \tag{28}$$

because $\varphi(t) + 4Nc/\varphi(t)$ is monotone non-decreasing for large t's.

Hence the function $\varphi(t)+4\,Nc/\varphi(t)$ belongs to $\mathfrak{U}_N^u$ by its definition. Since this result is obtained by assumption of the convergence of the integral (1) only, the same result should also be obtained for $\tilde{\varphi}(t)=\varphi(t)-5\,Nc/\varphi(t)$ because $\tilde{\varphi}(t)$ is non-decreasing for sufficiently large t's and the integral for $\tilde{\varphi}(t)$ is convergent. Moreover, it is easily seen that the inequality

$$\tilde{\varphi}(t)+4\,Nc/\tilde{\varphi}(t)<\varphi(t) \tag{29}$$

holds for large t's. Hence by (29), we see that $\varphi(t)$ belongs to $\mathfrak{U}_N^u$.

Thus Theorem 1 has been proved for the convergent case.

b) The divergent case.

Let $E^p_{\langle k_i,\,l_i\rangle}$ be the event defined by (14). Because $\varphi(t)$ is monotone non-decreasing, by Lemma 1, we have

$$\begin{aligned}
&\sum_{p=1}^{\infty}\sum_{\langle k_i\rangle}\sum_{\langle l_i\rangle}P(E^p_{\langle k_i,\,l_i\rangle})\\
&=0(1)\sum_{p=1}^{\infty}\sum_{\langle k_i\rangle}\sum_{\langle l_i\rangle}\frac{1}{\varphi(2^p/(\sum_{i=1}^{N}l_i^2)^{1/2})}e^{-\frac{1}{2}\varphi^2(2^p/(\sum_{i=1}^{N}l_i^2)^{1/2})}\\
&=0(1)\sum_{p=1}^{\infty}\sum_{\langle k_i\rangle}\sum_{\langle l_i\rangle}\frac{1}{\varphi(2^{p+1}/p)}e^{-\frac{1}{2}\varphi^2(2^{p+1}/p)}\\
&=0(1)\sum_{p=1}^{\infty}\frac{2^{pN}p^N}{\varphi(2^{p+1}/p)}e^{-\frac{1}{2}\varphi^2(2^{p+1}/p)}\\
&=0(1)\int^{\infty}t^{N-1}\varphi^{4N-1}(t)\,e^{-\frac{1}{2}\varphi^2(t)}dt=+\infty,
\end{aligned} \tag{30}$$

where $\sum_{\langle l_i\rangle}$ and $\sum_{\langle k_i\rangle}$ denote the summation for all lattice points $\langle(k_1+l_1)/2^p,\ldots,(k_N+l_N)/2^p\rangle$ satisfying $p/2<(\sum_{i=1}^{N}l_i^2)^{1/2}\leqq p$ and for all lattice points $\langle k_1/2^p,\ldots,k_N/2^p\rangle$ satisfying $\max_{1\leqq i\leqq N}|k_i|\leqq 2^p$, respectively. By the definition of $\mathfrak{L}_N^u$, $\varphi(t)$ belongs to $\mathfrak{L}_N^u$ if $E^p_{\langle k_i,\,l_i\rangle}$ occurs "*infinitely often*" for almost all ω. To prove that this is the case, we use the following due to K. L. Chung and P. Erdös [7].

LEMMA 3. *Let $\{E_k\}$ be an infinite sequence of events satisfying the following conditions:*

(i) $$\sum_{k=1}^{\infty}P(E_k)=+\infty.$$

(ii) *For every pair of positive integers h and n satisfying $n\geqq h$, there exists $C(h)>0$ and $H(n,h)>n$ such that for every $m\geqq H(n,h)$ holds*

$$P(E_m/E'_h\cap\cdots\cap E'_n)>C(h)\,P(E_m),$$

where $P(F/E)$ denote the conditional probability of F on the hypothesis E.

(iii) *There exist two absolute constants c_1 and c_2 with the following property: to each E_j there corresponds a set of events $E_{j_1}, \ldots, E_{j_s}$ belonging to $\{E_k\}$ such that*

$$\text{(a)} \qquad \sum_{i=1}^{s} P(E_j \cap E_{j_i}) < c_1 P(E_j)$$

and that for any other E_k than $E_{j_i}(1 \leqq i \leqq s)$ which stands after E_j in the sequence (viz. $k > j$)

$$\text{(b)} \qquad P(E_j \cap E_k) < c_2\, P(E_j)\, P(E_k).$$

The probability that infinitely many events E_k occur is equal to 1.

Because (30) shows that the sequence $\{E^p_{\langle k_i, l_i\rangle}\}$ satisfies the condition (i) in Lemma 3, it suffices to prove that the sequence satisfies also (ii) and (iii). For this purpose, we enumerate the events $E^p_{\langle k_i, l_i\rangle}$ in the order that $E^p_{\langle k_i, l_i\rangle}$ stands before $E^{p'}_{\langle k_{i'}, l_{i'}\rangle}$ if and only if one of the following four conditions holds:

$$(\alpha) \qquad p < p',$$

$$(\beta) \qquad p = p' \text{ and } \sum_{i=1}^{N} l_i'^2 < \sum_{i=1}^{N} l_i^2,$$

$$(\gamma) \qquad p = p',\ \sum_{i=1}^{N} l_i'^2 = \sum_{i=1}^{N} l_i^2,\ k_j = k_j'(j = 1, 2, \ldots, i-1) \text{ and } k_i < k_i' \text{ for some } i(\leqq N),$$

$$(\delta) \qquad p = p',\ \sum_{i=1}^{N} l_i'^2 = \sum_{i=1}^{N} l_i^2,\ k_i = k_i'(i = 1, 2, \ldots, N),\ l_j = l_j'(j = 1, 2, \ldots, i-1), \text{ and } l_i < l_i' \text{ for some } i(\leqq N).$$

Let $\{E_n\}$ be the newly obtained sequence of events. This special ordering is employed for the convenience of later computations.

Put

$$U_n = X((k_1 + l_1)/2^p, \ldots, (k_N + l_N)/2^p) - X(k_1/2^p, \ldots, k_N/2^p)$$

for $E_n = E^p_{\langle k_i, l_i\rangle}$. Then a simple computation shows that, for any positive integer n, we have

$$(31) \qquad \lim_{m \to \infty} \rho(U_n, U_m) = 0.$$

If we denote by $E_n(a)$ the event that $U_n + a$ is positive, $P(E_n(a))$ tends to 1 as a increases to infinity. Therefore, for each pair of positive integer h and n satisfying $n \geqq h$, we can choose $a_{h,n}$ such that

$$P\{\bigcap_{l=h}^{n}(E'_l \cap E_l(a_{h,n}))\} \geqq P(\bigcap_{l=h}^{n} E'_l)/2. \tag{32}$$

Then holds

$$\begin{aligned} P(E_m/E'_h \cap \cdots \cap E'_n) &= P(E_m \cap E'_h \cap \cdots \cap E'_n)/P(E'_h \cap \cdots \cap E'_n) \\ &\geqq P\{E_m \cap (\bigcap_{l=h}^{n}(E'_l \cap E_l(a_{h,n})))\}/2\, P\{\bigcap_{l=h}^{n}(E'_l \cap E_l(a_{h,n}))\} \\ &= P\{E_m/\bigcap_{l=h}^{n}(E'_l \cap E_l(a_{h,n}))\}/2. \end{aligned} \tag{33}$$

Let $\{X_1, \ldots, X_n, Y_m;\ m \in \mathfrak{M}\}$ be a Gaussian system satisfying the conditions

$$E(X_i) = E(Y_m) = 0,\ E(X_i^2) = E(Y_m^2) = 1,\ i = 1, 2, \ldots;\ n,\ m \in \mathfrak{M}.$$

For any bounded Borel sets $B_1, \ldots, B_n$, we define $\varepsilon(m, B_m) = \varepsilon(\rho_{1,m}, \ldots, \rho_{n,m};\ B)$ by

$$P(Y_m \in B_m/X_i \in B_i,\ i = 1, 2, \ldots, n) = (1 + \varepsilon(m, B_m))\ P(Y_m \in B_m),$$

where $\rho_{i,m} = \rho(X_i, Y_m)$, and B_m denotes a Borel set contained in the interval $[-\rho_m^{-s}, \rho_m^{-s}]$ with $s < 1$, ρ_m being max $(|\rho_{i,m}|;\ 1 \leqq i \leqq n)$. B_m may vary with m. Then we have

LEMMA 4. *$\varepsilon(m, B_m) \to 0$ as $\rho_m \to 0$.*

Proof. Let $p_m(X_1, \ldots, X_n)$ denote the conditional expectation of Y_m for given values of $X_1, \ldots, X_{n-1}$, and X_n. Then the expectation of $p_m(X_1, \ldots, X_n)$ is 0 and its variance tends to 0 with ρ_m. Since the Gaussian distribution with mean vector $\mathbf{0}$ is determined by its covariance matrix, we have

$$\begin{aligned} &P(X_i \in B_i,\ i = 1, 2, \ldots, n \text{ and } Y_m \in B_m) \\ &\quad = P(X_i \in B_i,\ i = 1, 2, \ldots, n \text{ and } (1-\alpha^2)^{1/2} Z + p_m(X_1, \ldots, X_n) \in B), \end{aligned}$$

where $\alpha^2 = E(p_m^2(X_1, \ldots, X_n))$ and Z denotes the random variable independent of $\langle X_1, \ldots, X_n\rangle$ and subject to the 1-dimensional standard Gaussian distribution. Denoting by $P_{\langle X_i\rangle}$ the probability law of $\langle X_1, \ldots, X_n\rangle$, we have

$$\begin{aligned} &P(X_i \in B_i,\ i = 1, 2, \ldots, n \text{ and } Y_m \in B_m) \\ &= \int_{\substack{x_i \in B_i \\ 1 \leqq i \leqq n}} \left\{ \int_{z \in B_m} \frac{1}{(2\pi(1-\alpha^2))^{1/2}} e^{-\frac{1}{2(1-\alpha^2)}(z - p_m(x_1, \ldots, x_n))^2} dz \right\} P_{\langle X_i\rangle}(dx_1, \ldots, dx_n) \\ &= \int_{\substack{x_i \in B_i \\ 1 \leqq i \leqq n}} \left\{ \int_{z \in B_m} \frac{1}{(2\pi(1-\alpha^2))^{1/2}} e^{-\frac{1}{2} z^2 + 0} dz \right\} P_{\langle X_i\rangle}(dx_1, \ldots, dx_n), \end{aligned} \tag{A.2}$$

where

$$\theta = -\{\alpha^2 z^2 - 2\, z p_m(x_1, \ldots, x_n) + p_m^2(x_1, \ldots, x_n)\}/2(1-\alpha^2). \tag{A.3}$$

α and $p_m(x_1, \ldots, x_n)$ are at most of the same order as ρ_m. So, by (A.2), (A.3), and the restriction imposed on B_m, we obtain Lemma 4.

Now we apply Lemma 4 to the estimation of the right side of (33). If $E_m = E^{p'}_{\langle k'_i, l'_i\rangle}$ and $E_n = E^{p}_{\langle k_i, l_i\rangle}$, then $\max_{h \leqq l \leqq n} |\rho(U_l, U_m)|$ is at most $(p'/2^{p'-p-1})$. Hence $\varphi(2^{p'}/(\sum_{i=1}^{N} l_i'^2)^{1/2}) < (\max_{h \leqq l \leqq n} |\rho(U_l, U_m)|)^{-2/3}$ for large m's. On the other hand, for large m's, we have

$$P(E_m) < 2\, P(G_m), \tag{A.4}$$

where G_m denotes the event

$$\varphi(2^{p'}/(\sum_{i=1}^{N} l_i'^2)^{1/2}) < U_m/(E(U_m^2))^{1/2} < 2\,\varphi(2^{p'}/(\sum_{i=1}^{N} l_i'^2)^{1/2}).$$

From Lemma 4, it follows that

$$\begin{aligned} P(E_m/\bigcap_{l=h}^{n}(E'_l \cap E_l(a_{h,n}))) &> P(G_m/\bigcap_{l=h}^{n}(E'_l \cap E_l(a_{h,n}))) \\ &> P(G_m)/2\,; \end{aligned} \tag{A.5}$$

we get the last inequality, taking $U_l/(E(U_l^2))^{1/2}$ and $U_m/(E(U_m^2))^{1/2}$ for X_l and Y_m in Lemma 4, respectively. By (33), (A.4), and (A.5), we can see that

$$\lim_{m\to\infty} \frac{P(E_m/E'_h \cap \cdots \cap E'_n)}{P(E_m)} \geqq 1/8,$$

which proves (ii).

To verify (iii), we use the following lemma given in [4].

LEMMA 5. *Let U and V be two random variables whose joint distribution is a 2-dimensional Gaussian distribution and each of them is subject to the standard 1-dimensional Gaussian distribution.*

(i) *If $\rho(U, V) < 1/ab$, there exists a positive constant c such that*

$$P(U > a,\ V > b) \leqq c\, P(U > a)\, P(V > b).$$

(ii) *There exist two positive constant d and δ such that for $a > 0$ holds*

$$P(U > a,\ V > a) \leqq d e^{-\delta(1-\rho^2)a^2} P(U > a),$$

where ρ denotes $\rho(U, V)$.

For each $E_j = E^{p}_{\langle k_i, l_i\rangle}$, we choose a sequence $\{E_{j_i} = E^{p'}_{\langle k_{i'}, l_{i'}\rangle}\,;\ i = 1, 2, \ldots, s\}$

of all the events satisfying $j_i \geqq j$ and $\rho(U_j, U_{j_i}) \geqq \{\varphi(2^p/(\sum_{i=1}^{N} l_i^2)^{1/2}) \times \varphi(2^{p'}/\sum_{i=1}^{N} l_i'^2)^{1/2})\}^{-1}$. For any event E_k other than $E_{j_i}(1 \leqq i \leqq s)$ and standing after E_j, by (i) of Lemma 5 and definition of E_j and E_k, we have

$$P(E_j \cap E_k) < c\, P(E_j)\, P(E_k), \tag{34}$$

where c is an absolute constant. Thus the sequence $\{E_n\}$ satisfies the condition (b) of (iii) in Lemma 3.

In order to verify the condition (a) of (iii), we divide the sum of $P(E_j \cap E_{j_i})$ according to the magnitude of the correlation coefficient $\rho(U_j, U_{j_i})$ into two summations as follows:

$$\sum_{i=1}^{s} P(E_j \cap E_{j_i}) = \sum{}' P(E_j \cap E_{j_i}) + \sum{}'' P(E_j \cap E_{j_i}), \tag{35}$$

where $\sum'$ expresses the summation over i's such that $\rho(U_j, U_{j_i})$ is larger than $(1-p^{-1/2})^{1/2}$ and $\sum''$ expresses the summation of the other probabilities. Let A, B, A', and B' be the parameter points of random variables employed in the definition of E_j and E_{j_i}, i.e. $U_j = X(A) - X(B)$ and $U_{j_i} = X(A') - X(B')$. Then, for E_{j_i} summed up in $\sum'$, we can show that there exists a positive integer k less than $p^{1/2}$ and satisfying the following inequality:

$$(1-k/p)^{1/2} \leqq \rho(U_j, U_{j_i}) < (1-(k-1)/p)^{1/2}, \tag{36}$$

where $\rho(U_j, U_{j_i})$ can be computed as

$$\rho = \{\mathrm{dis}\,(A, B') + \mathrm{dis}\,(A', B) - \mathrm{dis}\,(A, A') - \mathrm{dis}\,(B, B')\}/2\{\mathrm{dis}\,(A, B)\,\mathrm{dis}\,(A', B')\}^{1/2}.$$

Now, for given A and B we estimate the number of pairs of points A' and B' satisfying the inequality (36). Since the correlation coefficient $\rho(U_j, U_{j_i})$ is less than $[\min\{\mathrm{dis}\,(A, B), \mathrm{dis}\,(A', B')\}]\,[\mathrm{dis}\,(A, B)\,\mathrm{dis}\,(A', B')]^{-1/2}$, it follows from the definition of the ordering of the sequence $\{E_n\}$ that

$$(1-k/p)\ \mathrm{dis}\,(A, B) \leqq \mathrm{dis}\,(A', B') \leqq \mathrm{dis}\,(A, B). \tag{37}$$

We can also see that $(\mathrm{dis}\,(A, B') - \mathrm{dis}\,(B, B'))$ and $(\mathrm{dis}\,(A', B) - \mathrm{dis}\,(A, A'))$ are less than $\mathrm{dis}\,(A, B)$. Hence, by (36) and (37), the inequalities

$$\begin{aligned} &(1-2k/p)\ \mathrm{dis}\,(A, B) \leqq \mathrm{dis}\,(A', B) - \mathrm{dis}\,(A, A'), \\ &(1-2k/p)\ \mathrm{dis}\,(A, B) \leqq \mathrm{dis}\,(A, B') - \mathrm{dis}\,(B, B') \end{aligned} \tag{38}$$

hold for large p's. (37) and (38) show that the corresponding superscript p'

of E_{j_i} is at most $(p+1)$ and also that for given A and B, the numbers of such points A' and B' are at most of order k^N. Moreover, it follows from Lemma 1 and (ii) of Lemma 5 that for $E_{j_i}=E^{p'}_{\langle k_{i'}, l_{i'}\rangle}$ summed up in $\sum'$ holds

$$
\begin{aligned}
P(E_j \cap E_{j_i}) &= P\{U_j > (\mathrm{dis}\,(A,\ B))^{1/2}\varphi(1/\mathrm{dis}\,(A,\ B)), \\
&\qquad\qquad U_{j_i} > (\mathrm{dis}\,(A',\ B'))^{1/2}\varphi(1/\mathrm{dis}\,(A',\ B'))\} \\
&\leqq P\{U_j > (\mathrm{dis}\,(A,\ B))^{1/2}\varphi(1/\mathrm{dis}\,(A,\ B)), \\
&\qquad\qquad U_{j_i} > (\mathrm{dis}\,(A',\ B'))^{1/2}\varphi(1/\mathrm{dis}\,(A,\ B))\} \\
&\leqq de^{-\delta(1-\rho^2(U_j,\,U_{j_i}))p}P(E_j) \\
&\leqq d'e^{-\delta k}P(E_j),
\end{aligned}
\tag{39}
$$

where d, δ, and d' are absolute constants. Considering the number of E_{j_i}, we see that there exist two positive constants c_1 and c_2 satisfying

$$
\begin{aligned}
\sum{}' P(E_j \cap E_{j_i}) &< c_1 \sum_{k=1}^{\infty} k^{2N} e^{-\delta k} P(E_j) \\
&= c_2 P(E_j).
\end{aligned}
\tag{40}
$$

To estimate $\sum''$, we consider first the magnitude of superscript p' of $E_{j_i}=E^{p'}_{\langle k_{i'}, l_{i'}\rangle}$ summed up in $\sum''$. The restriction imposed on $\rho(U_j,\ U_{j_i})$ implies that

$$
p' < p + 5 \log p. \tag{41}
$$

Moreover, simple computation shows that if one of the two distances, dis $(A,\ A')$ and dis $(B,\ B')$, between the corresponding parameter points employed in the definitions of U_j and U_k is larger than $p^2/2^p$, then E_k is not among $E_{j_i}(1 \leqq i \leqq s)$. Hence, for given E_j, the number of E_{j_i} with fixed superscript p' is at most of order p^{4N}. By Lemma 1 and Lemma 5, we have

$$
\begin{aligned}
\sum{}'' P(E_j \cap E_{j_i}) &< \sum{}'' P\{U_j > (\mathrm{dis}\,(A,\ B))^{1/2}\varphi(1/\mathrm{dis}\,(A,\ B)), \\
&\qquad\qquad U_{j_i} > (\mathrm{dis}\,(A',\ B'))^{1/2}\varphi(1/\mathrm{dis}\,(A,\ B))\} \\
&\leqq dP(E_j) \sum{}'' e^{-\delta(1-\rho^2(U_j,\,U_{j_i}))p},
\end{aligned}
\tag{42}
$$

where d and δ are positive constants. Since the correlation coefficient $\rho(U_j,\ U_{j_i})$ is less than $(1-p^{-1/2})^{1/2}$ in the present case, the estimation for the number of E_{j_i}'s shows that

$$
\begin{aligned}
\sum{}'' P(E_j \cap E_{j_i}) &\leqq dP(E_j) \sum{}'' e^{-\delta p^{1/2}} \\
&< dP(E_j)\ (p + 5 \log p)^{4N+1} e^{-\delta p^{1/2}} \\
&< c_3 P(E_j),
\end{aligned}
\tag{43}
$$

where c_3 is a suitably chosen positive constant.

Now (a) of (iii) in Lemma 3 follows from (35), (40), and (43).

Thus we have proved completely the divergent case.

§ 3. Local continuity of Brownian motion with an N-dimensional parameter

In this section, we study the continuity of $X(A)$ at the origin O of E_N.

THEOREM 2. *Let $\psi(t)$ be a non-negative and monotone non-decreasing function defined for large t's. Then $\psi(t)$ belongs to $\mathfrak{U}_N^0$ or $\mathfrak{L}_N^0$ according as the integral*

$$\int^{\infty} \frac{1}{t} \psi^{2N-1}(t) e^{-\frac{1}{2}\psi^2(t)} dt \tag{44}$$

is convergent or divergent.

COR. 4. *The function*

$$\psi(t) = \{2 \log_{(2)} t + (2N+1) \log_{(3)} t + 2 \log_{(4)} t + \cdots + 2 \log_{(n-1)} t + (2+\delta) \log_{(n)} t\}^{1/2}$$

belongs to $\mathfrak{U}_N^0$ for $\delta > 0$ and belongs to $\mathfrak{L}_N^0$ for $\delta \leqq 0$.

COR. 5. *The function*

$$\psi_\infty(t) = \{2 \log_{(2)}^+ t + (2N+1) \log_{(3)}^+ t + 2 \sum_{n=4}^{\infty} \log_{(n)}^+ t\}^{1/2},$$

belongs to $\mathfrak{L}_N^0$, where $\log_{(n)}^+ t$ denotes the function defined in § 2.

Cor. 4 and Cor. 5 follow from Theorem 2 immediately.

As we remarked in the introduction, Theorem 2 assures the following theorem:

THEOREM 3. *Let $\psi(t)$ be a function given in Theorem 2. Then $\psi(t)$ belongs to $\mathfrak{U}_N^\infty$ or $\mathfrak{L}_N^\infty$ according as the integral (44) is convergent or divergent.*

COR. 6. *The function $\psi(t)$ defined in Cor. 4 belongs to $\mathfrak{U}_N^\infty$ for $\delta > 0$ and belongs to $\mathfrak{L}_N^\infty$ for $\delta \leqq 0$.*

COR. 7. *The function $\psi_\infty(t)$ defined in Cor. 5 belongs to $\mathfrak{L}_N^\infty$.*

The proof of Theorem 2 can be given in a parallel way to the proof of Theorem 1.

LEMMA 6. *Theorem 2 holds, if it holds under the following condition:*

$$(2\log_{(2)} t)^{1/2} \leqq \psi(t) \leqq (3\log_{(2)} t)^{1/2}. \tag{45}$$

Proof. We assume that Theorem 2 holds for $\psi(t)$ satisfying (45) and put

$$\hat{\psi}(t) = \min\{\max(\psi(t), \psi_1(t)), \psi_2(t)\}, \tag{46}$$

where

$$\psi_1(t) = (2\log_{(2)} t)^{1/2},$$
$$\psi_2(t) = (3\log_{(2)} t)^{1/2}.$$

Evidently, $\hat{\psi}(t)$ satisfies the condition (45).

If there exists a monotone increasing sequence $\{t_n\}$ such that $\psi(t_n) < \psi_1(t_n)$ and t_n tends to infinity with n, we have

$$\begin{aligned}\int_{t_1}^{\infty} \frac{1}{t}\psi^{2N-1}(t)\, e^{-\frac{1}{2}\psi^2(t)}dt &> \int_{t_1}^{t_n} \frac{1}{t}\psi^{2N-1}(t)\, e^{-\frac{1}{2}\psi^2(t)}dt \\ &\geqq c\log t_n\, \psi^{2N-1}(t_n)\, e^{-\frac{1}{2}\psi^2(t_n)} \\ &\geqq c\log t_n\, \psi_1^{2N-1}(t_n)\, e^{-\frac{1}{2}\psi_1^2(t_n)} \\ &= c\,(2\log_{(2)} t_n)^{N-\frac{1}{2}}\end{aligned} \tag{47}$$

because $\psi(t)$ is monotone non-decreasing, where c is a suitably chosen positive constant. Also (47) holds for $\hat{\psi}(t)$, because $\hat{\psi}(t)$ is monotone non-decreasing and $\hat{\psi}(t_n) = \psi_1(t_n)$. Hence the integrals (44) for $\psi(t)$ and $\hat{\psi}(t)$ diverge simultaneously in the present case. On the contrary, if $\psi_1(t)$ is less than $\psi(t)$ for large t's, then $\psi(t) \geqq \hat{\psi}(t)$ for large t's, hence there is a positive constant c such that

$$\begin{aligned}\int^{\infty} \frac{1}{t}\psi^{2N-1}(t)\, e^{-\frac{1}{2}\psi^2(t)}dt &\leqq c\int^{\infty} \frac{1}{t}\hat{\psi}^{2N-1}(t)\, e^{-\frac{1}{2}\psi^2(t)}dt \\ &\leqq c\Big\{\int^{\infty} \frac{1}{t}\psi^{2N-1}(t)\, e^{-\frac{1}{2}\psi(t)}dt + \\ &\qquad \int^{\infty} \frac{1}{t}\psi_2^{2N-2}(t)\, e^{-\frac{1}{2}\psi_1(t)}dt\Big\}.\end{aligned} \tag{48}$$

So the integrals (44) for $\psi(t)$ and $\hat{\psi}(t)$ diverge or converge simultaneously.

First, let us consider the case in which the integral for $\psi(t)$ is convergent. Considering (47) we see that the set of t's where $\psi(t)$ is less than $\psi(t)$ is bounded. Therefore, $\psi(t) > \psi_1(t)$ and accordingly $\psi(t) \geqq \hat{\psi}(t)$ for sufficiently large t's. So $\psi(t)$ belongs to $\mathfrak{U}_N^{\circ}$ because $\hat{\psi}(t)$ belongs to $\mathfrak{U}_N^{\circ}$ by our assumption. Secondly, we consider the case in which the integral for $\psi(t)$ is divergent. By what has been above stated, the integral for $\hat{\psi}(t)$ is divergent and so $\hat{\psi}(t)$ belongs to $\mathfrak{L}_N^{\circ}$ byour as sumption. Hence there exists a sequence

$\{A_n\}$ such that

$$(49)\qquad |X(A_n)| > (\mathrm{dis}\,(O,\ A_n))^{1/2}\hat{\psi}(1/\mathrm{dis}\,(O,\ A_n)),$$
$$\mathrm{dis}\,(O,\ A_n) \to O \text{ as } n \to +\infty.$$

Moreover, $\psi_2(t)$ belongs to $\mathfrak{U}_N^\circ$ because $\psi_2(t)$ satisfies the condition (45). So, for large n's holds

$$\hat{\psi}(1/\mathrm{dis}\,(O,\ A_n)) < \psi_2(1/\mathrm{dis}\,(O,\ A_n)),$$

hence

$$(50)\qquad \psi(1/\mathrm{dis}\,(O,\ A_n)) \leqq \hat{\psi}(1/\mathrm{dis}\,(O,\ A_n)).$$

Here (49) and (50) show that $\psi(t)$ belongs to $\mathfrak{L}_N^\circ$.

Thus Lemma 6 has been proved.

Proof of Theorem 2.

a) The convergent case.

Let us denote by $E^p_{\langle k_1,\ldots,k_N\rangle}$ (shortly $E^p_{\langle k_i\rangle}$), the following event:

$$(51)\qquad X(k/2^p,\ \ldots,\ k_N/2^p) > ((\sum_{i=1}^N k_i^2)^{1/2}/2^p)\,\psi(2^p/(\sum_{i=1}^N k_i^2)^{1/2}).$$
$$k_i = \pm 1,\ \pm 2,\ \ldots,\ \pm 2^p,\ i = 1,\ 2,\ \ldots,\ N.$$

Summing up $P(E^p_{\langle k_i\rangle})$ for $p = 1, 2, \ldots$, and for all lattice points $\langle k_1/2^p, \ldots, k_N/2^p\rangle$ satisfying $(\log p)/3 < (\sum_{i=1}^N k_i^2)^{1/2} \leqq \log p$, we have by Lemma 6 that

$$\begin{aligned}\sum_{p=1}^\infty \sum_{\langle k_i\rangle} P(E^p_{\langle k_i\rangle}) &= 0(1)\sum_{p=1}^\infty \sum_{\langle k_i\rangle} \frac{1}{\psi(2^p/(\sum_{i=1}^N k_i^2)^{1/2})}\, e^{-\frac{1}{2}\psi^2(2^p/(\sum_{i=1}^N k_i^2)^{1/2})}\\ &= 0(1)\sum_{p=1}^\infty \frac{(\log p)^N}{\psi(2^p/\log p)}\, e^{-\frac{1}{2}\psi^2(2^p/\log p)}\\ &= 0(1)\sum_{p=1}^\infty \psi^{2N-1}(2^p/\log p)\, e^{-\frac{1}{2}\psi^2(2^p/\log p)}\\ &= 0(1)\int^\infty \frac{1}{t}\,\psi^{2N-1}(t)\, e^{-\frac{1}{2}\psi^2(t)}\,dt < +\infty.\end{aligned}\qquad (52)$$

By $\tilde{E}^p_{\langle k_1,\ldots,k_N\rangle}$ (shortly $\tilde{E}^p_{\langle k_i\rangle}$), we denote the following event:

$$\max_A X(A)/(\mathrm{dis}\,(O,\ A))^{1/2} > \psi(2^p/(\sum_{i=1}^N k_i^2)^{1/2}) + \frac{c}{\psi(2^p/(\sum_{i=1}^N k_i^2)^{1/2})},$$

where A runs over the cube $[(k_1-1)2^p,\ (k_1+1)/2^p;\ \ldots;\ (k_N-1)/2^p,\ (k_N+1)/2^p]$. For sufficiently large c and p's, we have by a similar way as in §2 that

$$P(\tilde{E}^p_{\langle k_i\rangle}) = 0(1)\ P(E^p_{\langle k_i\rangle}).$$

From (52) it follows that

$$\sum_{p=1}^{\infty} \sum_{\langle k_i \rangle} P(\tilde{E}^{p}_{\langle k_i \rangle}) < +\infty. \tag{53}$$

According to Borel-Cantelli's lemma in the convergent case, (53) shows that only finitely many events $\tilde{E}^{p}_{\langle k_i \rangle}$ appearing in (53) can occur for almost all ω. Namely, for almost all ω, there exists p_0 such that no $\tilde{E}^{p}_{\langle k_i \rangle}$ can occur for p's larger than p_0.

Now, for any point A of $\mathrm{dis}\,(O, A) < (\log p_0 - N^{1/2})/2^{p_0}$, we choose p such that

$$(\log (p+1) - N^{1/2})/2^{p+1} < \mathrm{dis}\,(O, A) < (\log p - N^{1/2})/2^{p}.$$

By the same way as in §2, we have

$$X(A) \leqq (\mathrm{dis}\,(O, A))^{1/2} \{\psi(1/\mathrm{dis}\,(O, A)) + 2c/\psi(1/\mathrm{dis}\,(O, A))\}.$$

Thus $\psi(t) + 2c/\psi(t)$ belongs to $\mathfrak{U}^{\circ}_{N}$ and we can prove by the same procedure as in §2 that $\psi(t)$ belongs to $\mathfrak{U}^{\circ}_{N}$.

b) The divergent case.

Let $E^{p}_{\langle k_i \rangle}$ be the same event as in the convergent case. By Lemma 6, we have

$$\sum_{p=1}^{\infty} \sum_{\langle k_i \rangle} P(E^{p}_{\langle k_i \rangle}) = 0(1) \int^{\infty} \frac{1}{t} \psi^{2N-1}(t)\, e^{-\frac{1}{2}\psi^2(t)}\, dt = +\infty, \tag{54}$$

where $\sum_{\langle k_i \rangle}$ denotes the summation for all lattice points $\langle k_1/2^{p}, \ldots, k_N/2^{p} \rangle$ satisfying $(\log p)/2 < (\sum_{i=1}^{N} k_i^2)^{1/2} \leqq \log p$. Hence it suffices to prove that the sequence $\{E^{p}_{\langle k_i \rangle}\}$ satisfies the condition (ii) and (iii) in Lemma 3. To prove that this is the case, we enumerate the events $E^{p}_{\langle k_i \rangle}$ by the same method as in §2 and denote the new sequence by $\{E_n\}$. Then it is clear that by a similar consideration as in §2, (ii) is satisfied in the present case. Next, for each $E_j = E^{p}_{\langle k_i \rangle}$, we choose a sequence $\{E_{j_i} = E^{p'}_{\langle k_{i'} \rangle};\ i = 1, 2, \ldots, s\}$ of the events satisfying $j_i > j$ and

$$\rho(U_j, U_{j_i}) > 1/\{\psi(2^{p}/(\sum_{i=1}^{N} k_i^2)^{1/2})\ \psi(2^{p'}/(\sum_{i=1}^{N} k_i'^2)^{1/2})\}, \tag{55}$$

where U_j and U_{j_i} denote the random variables $X(k_1/2^{p}, \ldots, k_N/2^{p})$ and $X(k_1'/2^{p'}, \ldots, k_N'/2^{p'})$ respectively. For any event E_k other than $E_{j_i} (1 \leqq i \leqq s)$ and standing after E_j, we can apply Lemma 5 and accordingly (b) of (iii) holds.

To verify (a) of (iii), we employ the same method as in §2. We divide

the sum of $P(E_j \cap E_{j_i})$ by the magnitude of the corresponding correlation coefficient $\rho(U_j, U_{j_i})$ into two summations as follows:

$$\sum_{i=1}^{s} P(E_j \cap E_{j_i}) = \sum{}' P(E_j \cap E_{j_i}) + \sum{}'' P(E_j \cap E_{j_i}). \tag{56}$$

where $\sum'$ expresses the summation over i's such that $\rho(U_j, U_{j_i})$ is larger than $(1-(\log p)^{-1/2})^{1/2}$, and $\sum''$ expresses the summation of the other probabilities. Let A and B be the points $(k_1/2^p, \ldots, k_N/2^p)$ and $(k_1'/2p', \ldots, k_N'/2^{p'})$ respectively. Then, for E_{j_i} summed up $\sum'$, we can show that there exists a positive integer k less than $(\log p)^{1/2}$ and satisfying the following inequality:

$$(1-k/\log p)^{1/2} \leqq \rho(U_j, U_{j_i}) < (1-(k-1)/\log p)^{1/2}, \tag{57}$$

where

$$\rho(U_j, U_{j_i}) = \{\mathrm{dis}\,(O, A) + \mathrm{dis}\,(O, B) - \mathrm{dis}\,(A, B)\}/2\{\mathrm{dis}\,(O, A)\ \mathrm{dis}\,(O, B)\}^{1/2}.$$

Since $\rho(U_j, U_{j_i})$ is less than $\{\min\,(\mathrm{dis}\,(O, A), \mathrm{dis}\,(O, B))\}\,\{\mathrm{dis}\,(O, A)\ \mathrm{dis}\,(O, B)\}^{-1/2}$, it follows from (57) and the definition of ordering of the sequence $\{E_n\}$ that

$$(1-k/\log p)\ \mathrm{dis}\,(O, A) \leqq \mathrm{dis}\,(O, B) \leqq \mathrm{dis}\,(O, A). \tag{58}$$

From (57) and (58) it follows that for large p's

$$\mathrm{dis}\,(A, B) < 2\,k\ \mathrm{dis}\,(O, A)/\log p. \tag{59}$$

Now (58) shows that the superscript p' of $E_{j_i} = E^{p'}_{\langle k_{i'}\rangle}$ summed up in $\sum'$ is at most $p+1$. Also (59) shows that for given E_j, the number of such E_{j_i}'s is at most of order k^N. Therefore, by Lemma 5, Lemma 6, (57), and (58) holds

$$\begin{aligned}
\sum{}' P(E_j \cap E_{j_i}) &\leqq \sum{}' P\{U_j > (\mathrm{dis}\,(O, A))^{1/2}\ \psi(1/\mathrm{dis}\,(O, A)), \\
&\qquad\qquad U_{j_i} > (\mathrm{dis}\,(O, B))^{1/2}\ \psi(1/\mathrm{dis}\,(O, A))\} \\
&\leqq c_1 \sum_{k=1}^{\infty} k^N e^{-\delta(1-\rho^2(U_i, U_{j_i}))\,\psi^2(1/\mathrm{dis}(O, A))} P(E_j) \\
&\leqq c_2 \sum_{k=1}^{\infty} k^N e^{-\delta' k} P(E_j) \\
&= c_3\, P(E_j),
\end{aligned} \tag{60}$$

where c_1, c_2, c_3, δ, and δ' are positive constants. On the other hand, if the superscript p' of $E_n = E^{p'}_{\langle k_{i'}\rangle}$ is larger than $\log p + 5 \log_{(2)} p$, then $\rho(U_j, U_n)$ is less than $\{\psi(2^p/(\sum_{i=1}^{N} k_i^2)^{1/2})\ \psi(2^p/(\sum_{i=1}^{N} k_i'^2)^{1/2})\}^{-1}$. Hence, by Lemma 5 and Lemma

6, we have for large p's

$$
\begin{aligned}
\sum{}'' P(E_j \cap E_{j_i}) &\leqq \sum{}'' P\{U_j > (\mathrm{dis}\,(O,\ A))^{1/2}\ \psi(1/\mathrm{dis}\,(O,\ A)), \\
&\qquad\qquad U_{j_i} > (\mathrm{dis}\,(O,\ B))^{1/2}\ \psi(1/\mathrm{dis}\,(O,\ A))\} \\
&\leqq d \sum{}'' e^{-\delta(1-\rho^2(U_j, U_{j_i}))\,\psi^2(1/\mathrm{dis}(O,A))} P(E_j) \qquad (61) \\
&\leqq d(\log p + 5 \log_{(2)} p)^{2N+1} e^{-\delta'(\log p)^{1/2}} P(E_j) \\
&< P(E_j),
\end{aligned}
$$

where d, δ, and δ' are positive constants. (60) and (61) show that the sequence $\{E_n\}$ satisfies the consition (a) of (iii).

Thus we have proved Theorem 2.

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Shizuoka University