GALOIS GROUP OF THE MAXIMAL ABELIAN EXTENSION OVER AN ALGEBRAIC NUMBER FIELD

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The aim of the present work is to determine the Galois group of the maximal abelian extension Ω_A over an algebraic number field Ω of finite degree, which we fix once for all.

Let χ be a continuous character of the Galois group of Ω_A/Ω. Then, by class field theory, the character χ is also regarded as a character of the idèle group of Ω. We call such a χ a *character of* Ω. For our purpose, it suffices to determine the group X_l of the characters of Ω whose orders are powers of a prime number l.

Let L be the group of the characters χ of Ω with $\chi^l = 1$; set $L_\nu = L \cap X_l^{l^\nu}$, where $\nu = 1, 2, \ldots$. We denote by υ_ν the largest number of independent elements of the factor group $L_{\nu-1}/L_\nu$. A character $\chi \in X_l$ is said to be divisible if, for any power l^r of l, there is a character $\psi \in X_l$ such that we have $\chi = \psi^{l^r}$. We denote by $X'_{l,\infty}$ the group of all divisible characters in X_l. Let now $Z(l, \infty)$ be the group of the roots of unity whose orders are powers of l. Then $X'_{l,\infty}$ has the unique subgroup $X_{l,\infty}$ such that $X_{l,\infty}$ is the direct product of finite number of groups all isomorphic to $Z(l, \infty)$ and that $X'_{l,\infty}/X_{l,\infty}$ is a finite group. Call the number $\dim X_l$ of direct factors of $X_{l,\infty}$ the *dimension* of X_l and let there be $\upsilon_{\infty,\nu}$ cyclic factors of order l^ν in the direct decomposition of $X'_{l,\infty}/X_{l,\infty}$ into cyclic groups. Then, the structure of X_l is completely determined by υ_ν, $\upsilon_{\infty,\nu}$ and by $\dim X_l$. This conclusion, together with the above one concerning the structure of $X'_{l,\infty}$, is brought by the results of Kaplansky [3], in which υ_ν, $\upsilon_{\infty,\nu}$ are called the *Ulm invariants* of X_l. Thus the problem is reduced to the determination of υ_ν, $\upsilon_{\infty,\nu}$ and $\dim X_l$.

Let ζ_l be a primitive l-th root of unity and let ν_l be the natural number such that the field $\Omega(\zeta_l)$ contains a primitive l^{ν_l}-th root of unity but no primitive l^{ν_l+1}-th root of unity. On the other hand, let $\mathfrak{l}_1, \mathfrak{l}_2, \ldots$ be all the prime factors of l in Ω and let $\mathfrak{e}_{l,\nu}$ be the group of the units of Ω which are l^ν-th powers in every $\mathfrak{l}_i$-completion $\Omega_{\mathfrak{l}_i}$ of Ω. Then, we can prove that there is a natural number

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μ_l such that we have $l^{\mu_l}=(\mathfrak{e}_{l,\nu}:\mathfrak{e}_{l,\nu+1})$ for every sufficiently large ν. Using these constants ν_l, μ_l, the determination of υ_ν and $\dim X_l$ is done. Namely, we have $\upsilon_\nu=0$ for $\nu<\nu_l$, $\upsilon_\nu=\infty$ for $\nu\geqq\nu_l$ and $\dim X_l=N-\mu_l$, where N is the absolute degree of Ω.

We determine also the number l^{c_ν} of the elements of $X'_{l,\infty}$ whose orders divide l^ν. It is shown that we have $\upsilon_{\infty,\nu}=2c_\nu-c_{\nu-1}-c_{\nu+1}$. The number $\upsilon_{\infty,\nu}$ has, however, no simple expression as υ_ν or as $\dim X_l$. Assume, for example, that $l\neq 2$. Let h_ν be the number of the ideal classes of Ω whose orders divide l^ν and let $\mathfrak{w}_i$ be the group of roots of unity in $\Omega_{\mathfrak{l}_i}$. Furthermore, let $B^{(\nu)}$ be the group of $\beta\in\Omega^{\times}$ [1] such that the principal ideal (β) is the l^ν-th power of an ideal of Ω, and let $B_*^{(\nu)}$ be the group of $\beta\in B^{(\nu)}$ such that β is in $\mathfrak{w}_i\Omega_{\mathfrak{l}_i}^{\times l^\nu}$ for every i. Then we have $l^{c_\nu}=h_\nu\cdot l^{N\nu}\cdot(B^{(\nu)}:B_*^{(\nu)})$ and therefore

$$l^{\upsilon_{\infty,\nu}}=\frac{h_\nu^2}{h_{\nu-1}h_{\nu+1}}\cdot\frac{(B^{(\nu-1)}:B_*^{(\nu-1)})(B^{(\nu+1)}:B_*^{(\nu+1)})}{(B^{(\nu)}:B_*^{(\nu)})^2}.$$

§ 1. Preliminaries

1. In order that a homomorphism f_B, into a finite abelian group $\mathfrak{A}$, of a subgroup B of a finite abelian group A is the restriction to B of a homomorphism f of A into $\mathfrak{A}$, it is necessary and sufficient that we have $f_B(B\cap A^m)\subset\mathfrak{A}^m$ for every natural number m. In particular, if $\mathfrak{A}$ is a cyclic group $\mathfrak{Z}$ whose order is a power l^ν of a prime number l, then the above condition becomes $f_B(B\cap A^{l^\nu})=1$.

Let now $\mathbf{I}$, $\mathbf{U}$ be the idèle group and the unit idèle group[2] of Ω, respectively, and denote by $\Omega^\times$ the principal idèle group of Ω. Then we see at once that a character[3] $\chi_{\mathbf{U}}$ of $\mathbf{U}$ is the restriction to $\mathbf{U}$ of a character χ' with $\chi'(\Omega^\times\mathbf{I}^{l^\nu})=1$ of $\Omega^\times\mathbf{I}^{l^\nu}\mathbf{U}$ if and only if we have $\chi_{\mathbf{U}}(\Omega\mathbf{I}^{l^\nu}\cap\mathbf{U})=1$. Moreover, if the latter condition is satisfied, then $\chi_{\mathbf{U}}$ determines χ uniquely and, from what is described above, χ' is the restriction to $\Omega^\times\mathbf{I}^{l^\nu}\mathbf{U}$ of a character χ with $\chi^{l^\nu}=1$ of Ω.

Let $\mathfrak{S}$ be a finite set of places of Ω and $\chi_{\mathbf{U}}$ be a character of $\mathbf{U}$ such that $\chi^{l^\nu}=1$ and that the $\mathfrak{q}$-component[4] of $\chi_{\mathbf{U}}$ is trivial for every place $\mathfrak{q}\notin S$. Then

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1) Throughout the paper, we use the mark $\times$ to stand for the multiplicative group of non-zero elements of a field.

2) In this paper, we settle no sign condition for the real infinite components of unit idèles, somewhat differently from the definition of Weil [5].

3) This means an ordinary character of the topological abelian group.

4) This is naturally defined by means of local components of idèles.

χ_{U} is, in a natural way, regarded as a character of the group $U_{\mathfrak{S},\nu} = \prod_{\mathfrak{p}\in\mathfrak{S}} U_{\mathfrak{p}}/U_{\mathfrak{p}}^{l^\nu}$, where $U_{\mathfrak{p}}$ is the unit group of the $\mathfrak{p}$-completion $\Omega_{\mathfrak{p}}$ of Ω. On the other hand, set $B^{(\nu)} = \Omega^\times \cap \mathbf{I}^{l^\nu}\mathbf{U}$; then $B^{(\nu)}$ consists of the numbers β of $\Omega^\times$ such that the principal ideal (β) is the l^ν-th power of an ideal of Ω, and, setting $\beta = \mathbf{a}^{l^\nu}\mathbf{u}$ $(\mathbf{a} \in \mathbf{I}, \mathbf{u} \in \mathbf{U})$, the mapping $\beta \to \mathbf{u}$ followed by the natural mapping of $\mathbf{u}$ into $U_{\mathfrak{S},\nu}$ gives rise to a homomorphism $\iota_{\mathfrak{S},\nu}$ of $B^{(\nu)}$ into $U_{\mathfrak{S},\nu}$. Since the natural image of $\Omega^\times \mathbf{I}^{l^\nu} \cap \mathbf{U}$ into $U_{\mathfrak{S},\nu}$ coincides with $\iota_{\mathfrak{S},\nu}(B^{(\nu)})$, we have

LEMMA 1. *Let l^ν be a power of a prime number l and let $\mathfrak{S}$ be a finite set of places of Ω. Then the restriction to* U *of a character χ with $\chi^{l^\nu} = 1$ of Ω unramified*[5] *at every place of Ω outside $\mathfrak{S}$ is characterized as a character χ_{U} with $\chi_{\mathrm{U}}^{l^\nu}$ of* U *which has trivial $\mathfrak{q}$-component for every place $\mathfrak{q} \notin \mathfrak{S}$ and which satisfies $\chi_{\mathrm{U}}(\iota_{\mathfrak{S},\nu}(B^{(\nu)})) = 1$.*

Let $U_{\mathfrak{S},\nu}$ be as above. Lemma 1 implies

LEMMA 2. *Let V be any subgroup of $U_{\mathfrak{S},\nu}$ and let h_ν be the l^ν-class number of Ω, i.e., the index $(\mathbf{I} : \Omega^\times \mathbf{I}^{l^\nu}\mathbf{U})$. Then the number of all characters, with $\chi^{l^\nu} = 1$ and with $\chi_{\mathrm{U}}(V) = 1$, of Ω unramified at every $\mathfrak{q} \notin \mathfrak{S}$ is equal to $h_\nu \cdot (U_{\mathfrak{S},\nu} : \iota_{\mathfrak{S},\nu}(B^{(\nu)}) \cdot V)$, where χ_{U} is the restriction to* U *of χ.*

We have also

LEMMA 3. *The kernel of $\iota_{\mathfrak{S},\nu}$ consists of the numbers $\beta \in B^{(\nu)}$ such that β is, for every $\mathfrak{p} \in \mathfrak{S}$, an l^ν-th power in the $\mathfrak{p}$-completion $\Omega_{\mathfrak{p}}$ of Ω.*

2. Let $P_{2,\infty}$ be the field obtained by adjunction to the rational number field P of all 2^m-th roots of unity, where $m = 1, 2, \ldots$. Assume that the intersection $\Omega \cap P_{2,\infty}$ is real. Then there is an integer $T \geqq 2$ such that $\Omega \cap P_{2,\infty}$ is the largest real subfield of the field P_{2^T} obtained by adjunction to P of a primitive 2^T-th root of unity. In this case, we say that Ω is a *radical field* and, setting $\lambda_T = 4\cos^2 2\pi/2^{T+1}$, we call λ_T the *radical number* of Ω.[6] The rational number field P is a radical field with radical number $\lambda_2 = 2$. Numbers T and λ_T are uniquely determined whenever Ω is radical.

Denote now by l^ν a power of a prime number l and by $\Omega^{(\nu)}$ the group of

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5) We say that χ is ramified at $\mathfrak{p}$ if the corresponding cyclic extension of χ over Ω is ramified at $\mathfrak{p}$.

6) See Hasse [2], Einleitung.

the numbers α of $\Omega^\times$ such that α is an l^ν-th power in the field ΩP_{l^ν} obtained by adjunction to Ω of a primitive l^ν-th root of unity. Then a result[7] of Hasse yields

LEMMA 4. *We have in general* $\Omega^{(\nu)} = \Omega^{\times l^\nu}$. *Only in the special case where* $l=2$, Ω *is a radical field with radical number* λ_T *and* $\nu \geqq 2$, *the factor group* $\Omega^{(\nu)}/\Omega^{\times 2^\nu}$ *is of order* 2 *and its only one non-trivial coset is represented by* $-\lambda_\nu^{2^{\nu-1}}$ *or by* $\lambda_T^{2^{\nu-1}}$ *according as* $2 \leqq \nu \leqq T$ *or* $\nu > T$.

Still assuming that Ω is a radical field with radical number λ_T, it follows from this lemma that, for every prime ideal $\mathfrak{p}$ of Ω prime to 2, $\lambda_T^{2^{\nu-1}}$ ($\nu > T$) is a 2^ν-th power in the $\mathfrak{p}$-completion $\Omega_\mathfrak{p}$ of Ω. Now, letting $\mathfrak{l}_1, \mathfrak{l}_2, \ldots$ be all the prime factors of 2 in Ω and $\Omega_{\mathfrak{l}_i}$ be the $\mathfrak{l}_i$-completion of Ω, we say that Ω is a *strongly radical field* if we have $\lambda_T = \lambda_i^2 \zeta_i$ for every i, where λ_i is an element of $\Omega_{\mathfrak{l}_i}$ and ζ_i is a root of unity in $\Omega_{\mathfrak{l}_i}$. The meaning of this definition is explained by the following

LEMMA 5. *Assume that* Ω *is radical with the radical number* λ_T. *Then* Ω *is strongly radical if and only if* $\lambda_T^{2^{\nu-1}}$ *is a* 2^ν*-th power in every local completion of* Ω *for every* $\nu > T$, *or equivalently for* $\nu = T+1$.

Proof. Suppose that $\lambda_T = \lambda_i^2 \zeta_i$ and $\nu > T$; then we have $\lambda_T^{2^{\nu-1}} = \lambda_i^{2^\nu} \zeta_i^{2^{\nu-1}}$. If $\Omega_{\mathfrak{l}_i}$ contains no primitive 2^ν-th root of unity, then $\zeta_i^{2^{\nu-1}} = 1$ and $\lambda_T^{2^{\nu-1}}$ is a 2^ν-th power in $\Omega_{\mathfrak{l}_i}$. If $\Omega_{\mathfrak{l}_i}$ contains a primitive 2^ν-th root of unity, then $\Omega_{\mathfrak{l}_i}$ contains ΩP_{2^ν}, whence, by Lemma 4, $\lambda_T^{2^{\nu-1}}$ is a 2^ν-th power in ΩP_{2^ν} and ***a fortiori*** in $\Omega_{\mathfrak{l}_i}$. The converse is obvious.

§ 2. Structural constants

3. We begin by a reformulation of the main theorem of Wang [4].

Assuming that Ω is a radical field with the radical number λ_T, we say that a prime factor $\mathfrak{l}$ of 2 in Ω is *even* if λ_T is of the form $\lambda^2\zeta$, where λ is an element of the $\mathfrak{l}$-completion $\Omega_\mathfrak{l}$ of Ω and ζ is a root of unity in $\Omega_\mathfrak{l}$. Otherwise we say that $\mathfrak{l}$ is odd. In Wang [4], $\mathfrak{l}$ is said to be odd if $\Omega_\mathfrak{l}$ does not contain any of three numbers $\sqrt{-1}$, $\cos 2\pi/2^{T+1}$, $\sqrt{-1}\cos 2\pi/2^{T+1}$; otherwise, to be even. We now show that our definition is equivalent with Wang's one. Suppose that $\mathfrak{l}$ is

[7] See Hasse [2], §1, Satz 1 and Satz 2.

even. Then since $\lambda_T = 4\cos^2 2\pi/2^{T+1}$, $\Omega_{\mathfrak{l}}$ must contain at least one of the three numbers above. Conversely, suppose that $\Omega_{\mathfrak{l}}$ contains $\sqrt{-1}$. Then since $\Omega_{\mathfrak{l}}$ contains a primitive 2^T-th root ζ_{2^T} of unity and since $-\lambda_T^{2^{T-1}}$ is, by Lemma 4, a 2^T-th power in $\Omega(\zeta_{2^T})$, we see that $\mathfrak{l}$ is even. Furthermore, if we have either $\cos 2\pi/2^{T+1} \in \Omega_{\mathfrak{l}}$ or $\sqrt{-1}\cos 2\pi/2^{T+1} \in \Omega_{\mathfrak{l}}$, then $\mathfrak{l}$ is obviously even.

After these preliminaries, it follows from the main theorem of Wang [4] that we have

THEOREM 1. *Let χ be a character of Ω whose order $l^{\nu-r}$ $(0 \leqq r \leqq \nu)$ is a power of a prime number l and let $\mathfrak{S}$ be a finite set of places of Ω containing all ramification places of χ. Furthermore, denoting by $\chi_{\mathfrak{p}}$ the $\mathfrak{p}$-component[8] of χ and by $\Omega_{\mathfrak{p}}$ the $\mathfrak{p}$-completion of Ω, let there be given for every $\mathfrak{p} \in \mathfrak{S}$ a character $\psi_{\Omega_{\mathfrak{p}}}$ of $\Omega_{\mathfrak{p}}^{\times}$ such that $\chi_{\mathfrak{p}} = \psi_{\Omega_{\mathfrak{p}}}^{l^r}$. In the case where $l=2$, Ω is radical with the radical number λ_T, $r > T$ and all odd prime factors of 2 in Ω are in $\mathfrak{S}$, suppose that $\mathfrak{S}$ contains all prime factors of 2 in Ω and that we have $\prod_{\mathfrak{p}\in\mathfrak{S}} \psi_{\Omega_{\mathfrak{p}}}(\lambda_T^{2^{r-1}}) = 1$. Then there is a character ψ of order l^{ν} of Ω such that we have $\chi = \psi^{l^r}$ and that the $\mathfrak{p}$-component $\psi_{\mathfrak{p}}$ of ψ coincides with $\psi_{\Omega_{\mathfrak{p}}}$ for every $\mathfrak{p} \in \mathfrak{S}$.*

4. Let l be a prime number and ζ_l be a primitive l-th root of unity. Denote by ν_l a natural number such that the field $\Omega(\zeta_l)$ contains a primitive l^{ν_l}-th root of unity but no primitive l^{ν_l+1}-th root of unity. Then we have

LEMMA 6. *Let χ be a character of order $l^{\nu_l - r}$ of Ω with $0 \leqq r \leqq \nu_l$. Then there is a character ψ of order l^{ν_l} of Ω such that we have $\chi = \psi^{l^r}$.*

Proof. If $l=2$, $\nu_l = 1$, then the lemma is obvious. We may therefore assume that $\sqrt{-1} \in \Omega$ whenever we have $l=2$. Let $\mathfrak{S}$ be the set of all ramification prime ideals of χ. Since then, for every $\mathfrak{p} \in \mathfrak{S}$, we have $N\mathfrak{p} - 1 \equiv 0 \pmod{l}$, the $\mathfrak{p}$-completion $\Omega_{\mathfrak{p}}$ contains ζ_l and we have consequently $\Omega_{\mathfrak{p}} \supset \Omega(\zeta_l) = \Omega(\zeta_{l^{\nu_l}})$. From this follows $N\mathfrak{p} - 1 \equiv 0 \pmod{l^{\nu_l}}$, whence there is a character $\psi_{\Omega_{\mathfrak{p}}}$ of $\Omega_{\mathfrak{p}}^{\times}$ such that $\chi_{\mathfrak{p}} = \psi_{\Omega_{\mathfrak{p}}}^{l^r}$. Hence, Theorem 1 assures that there is a character ψ of order l^{ν_l} of Ω such that we have $\chi = \psi^{l^r}$, which completes the proof.

Another meaning of ν_l as a structural constant of the maximal abelian extension over Ω is found in the following

LEMMA 7. *Let ν be a rational integer with $\nu_l \leqq \nu$. Then there is an infinite*

[8] See foot-note 4.

set $\mathfrak{M}$ of characters of Ω satisfying the following conditions: i) *every character $\chi \in \mathfrak{M}$ is of order l.* ii) *for every ramification prime ideal $\mathfrak{p}$ of $\chi \in \mathfrak{M}$, we have $N\mathfrak{p} - 1 \equiv 0 \pmod{l^\nu}$, $N\mathfrak{p} - 1 \not\equiv 0 \pmod{l^{\nu+1}}$.* iii) *none of characters of $\mathfrak{M}$ is unramified and every two different characters of $\mathfrak{M}$ have no common ramification prime ideal.* iv) *for every $\chi \in \mathfrak{M}$ there is a character ψ of Ω such that we have $\chi = \psi^{l^{\nu-1}}$.*

Proof. Using notations in § 1, **1**, set $B^{(\nu)} = \Omega^\times \cap \mathbf{I}^{l^\nu}\mathbf{U}$. Let $\mathfrak{S} = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_m\}$ be a set of prime ideals, prime to l, of Ω such that m is larger than the rank of $B^{(\nu)}/B^{(\nu)l}$ and that we have $N\mathfrak{p}_i - 1 \equiv 0 \pmod{l^\nu}$, $N\mathfrak{p}_i - 1 \not\equiv 0 \pmod{l^{\nu+1}}$ for every i. Moreover, choose for every i a character ψ_i of order l^ν of $\mathbf{U}$ with trivial $\mathfrak{q}$-component for every place $\mathfrak{q}$ of Ω different from $\mathfrak{p}_i$. Then since the group $U_{\mathfrak{S},\nu}$ defined in § 1, **1** is of type $(l^\nu, \ldots, l^\nu)$ and since the rank of $\iota_{\mathfrak{S},\nu}(B^{(\nu)})$ is smaller than m, $U_{\mathfrak{S},\nu}/\iota_{\mathfrak{S},\nu}(B^{(\nu)})$ contains an element of order l^ν. Therefore a suitable multiplicative combination $\psi_{\mathbf{U}} = \psi_1^{a_1} \ldots \psi_m^{a_m}$ is trivial on $\iota_{\mathfrak{S},\nu}(B^{(\nu)})$, while the order of $\psi_{\mathbf{U}}$ is l^ν. By Lemma 1, $\psi_{\mathbf{U}}$ is the restriction to $\mathbf{U}$ of a character ψ of order l^ν of Ω. Therefore, a required set $\mathfrak{M}$ can be constructed as a set of characters of the form $\chi = \psi^{l^{\nu-1}}$, which completes the proof.

5. We insert here a lemma concerning the structure of local fields.[9]

LEMMA 8. *Let $\mathfrak{l}$ be a prime factor in Ω of a prime number l and let $\Omega_\mathfrak{l}$ be the $\mathfrak{l}$-completion of Ω. Denote by $U_{\mathfrak{l},1}$ the group of units u of $\Omega_\mathfrak{l}$ with $u \equiv 1$ (mod. $\mathfrak{l}$) and by $N_\mathfrak{l}$ the degree of $\Omega_\mathfrak{l}$ over the l-completion P_l of the rational number field. Then $U_{\mathfrak{l},1}$ is, as a topological group, the direct product of $N_\mathfrak{l}$ groups all isomorphic to the additive group of integers of P_l by the finite cyclic group consisting of all roots of unity in $\Omega_\mathfrak{l}$ whose orders are powers of l.*

Now, let l^ν be a power of a prime number l and $\mathfrak{S} = \{\mathfrak{l}_1, \mathfrak{l}_2, \ldots\}$ be the set of all prime factors of l in Ω. Denote by $\Omega_{\mathfrak{l}_i}$ the $\mathfrak{l}_i$-completion of Ω and by $B_0^{(\nu)}$ the kernel of the homomorphism $\iota_{\mathfrak{S},\nu}$ of § 1, **1**. Then we have

LEMMA 9. *Let* $\mathbf{e}$ *be the unit group of Ω. Then the index $(B^{(\nu)} : \mathbf{e}B_0^{(\nu)})$ becomes constant for sufficiently large ν.*

Proof. It follows from the finiteness of the class number of Ω that, for sufficiently large ν, $B^{(\nu)}/\mathbf{e}\Omega^{\times l^\nu}$ is isomorphic to $B^{(\nu+1)}/\mathbf{e}\Omega^{\times l^{\nu+1}}$ and that the iso-

[9] See Hasse [1], § 15, p. 177.

morphism is given by $B^{(\nu)} \ni \beta^{(\nu)l} \to \beta^{(\nu)l} \in B^{(\nu+1)}$. Furthermore, by Lemma 3, the image of $\mathrm{e}B_0^{(\nu)}/\mathrm{e}$ by the isomorphism is in $\mathrm{e}B_0^{(\nu+1)}/\mathrm{e}$. This means that the index $(B^{(\nu)} : \mathrm{e}B_0^{(\nu)})$ is monotonously decreasing for such a ν, from which at once follows our assertion.

Still using same notations, we now prove

LEMMA 10. *Set* $\mathrm{e}_{l,\nu} = \mathrm{e} \cap B_0^{(\nu)}$. *Then the index* $(\mathrm{e}_{l,\nu} : \mathrm{e}_{l,\nu+1})$ *becomes constant for sufficiently large* ν.

Proof. It follows from Lemma 8 that, for sufficiently large ν, a unit ε of Ω is an l^ν-th power in $\Omega_{\mathfrak{l}_i}$ if and only if ε^l is an $l^{\nu+1}$-th power in $\Omega_{\mathfrak{l}_i}$. Therefore, for such a ν, the l-th power $\varepsilon^{(\nu)l} \in \mathrm{e}_{l,\nu+1}$ of an element $\varepsilon^{(\nu)} \in \mathrm{e}_{l,\nu}$ is not in $\mathrm{e}_{l,\nu+2}$ unless we have $\varepsilon^{(\nu)} \in \mathrm{e}_{l,\nu+1}$. This means that we have $(\mathrm{e}_{l,\nu} : \mathrm{e}_{l,\nu+1}) \leqq (\mathrm{e}_{l,\nu+1} : \mathrm{e}_{l,\nu+2})$. Since from the finiteness of the dimension of e follows the boundedness of the index $(\mathrm{e}_{l,\nu} : \mathrm{e}_{l,\nu+1})$, the lemma is proved.

By this lemma, we have a new constant μ_l with $(\mathrm{e}_{l,\nu} : \mathrm{e}_{l,\nu+1}) = l^{\mu_l}$ for sufficiently large ν. The meaning of μ_l as a structural constant of the maximal abelian extension over Ω lies in the following

LEMMA 11. *Let* l^ν *be a power of a prime number* l *and* $\mathfrak{S} = \{\mathfrak{l}_1, \mathfrak{l}_2, \dots\}$ *be the set of all prime factors of* l *in* Ω. *Denote by* $T_{l,\nu}$ *the group of the characters* χ *of* Ω *such that the order of* χ *divides* l^ν *and that every ramification place of* χ *is in* $\mathfrak{S}$. *Then we have* $(T_{l,\nu+1} : T_{l,\nu}) = l^{N-\mu_l}$ *for sufficiently large* ν, *where* N ***is the absolute degree of*** Ω.

Proof. Denote by N_i the degree of the $\mathfrak{l}_i$-completion $\Omega_{\mathfrak{l}_i}$ of Ω over the l-completion of the rational number field and denote by $U_{\mathfrak{l}_i}$ the unit group of $\Omega_{\mathfrak{l}_i}$. Moreover, let $w_{\nu,i}$ be the number of roots of unity in $\Omega_{\mathfrak{l}_i}$ whose orders divide l^ν and let $U_{\mathfrak{l}_i,1}$ be the group consisting of all $u \in U_{\mathfrak{l}_i}$ with $u \equiv 1 \pmod{\mathfrak{l}_i}$. Then the number of characters of $U_{\mathfrak{l}_i}$ whose orders divide l^ν is, by Lemma 8, equal to $l^{N_i\nu} w_{\nu,i}$. Therefore Lemma 2 shows that, if h_ν is the l^ν-class number of Ω, then we have

$$(T_{l,\nu} : 1) = h_\nu \cdot \prod_i (l^{N_i\nu} w_{\nu,i}) \cdot (\iota_{\mathfrak{S},\nu}(B^{(\nu)}) : 1)^{-1}.$$

Now, with notations in Lemma 9 and in Lemma 10, we have $(\iota_{\mathfrak{S},\nu}(B^{(\nu)}) : 1) = (B^{(\nu)} : B_0^{(\nu)}) = (B^{(\nu)} : \mathrm{e}B_0^{(\nu)})(\mathrm{e} : \mathrm{e}_{l,\nu})$. From this and from the relation $\sum_i N_i = N$ follows

$$(T_{l,\nu+1} : T_{l,\nu}) = \frac{h_{\nu+1}}{h_\nu} \cdot l^N \cdot \prod_i \left(\frac{w_{\nu+1,i}}{w_{\nu,i}}\right) \cdot \frac{(B^{(\nu)} : eB_0^{(\nu)})}{(B^{(\nu+1)} : eB_0^{(\nu+1)})} \cdot (e_{l,\nu} : e_{l,\nu+1})^{-1}.$$

Numbers h_ν, $w_{\nu,i}$ are constant for sufficiently large ν and, by Lemma 9, so is also $(B^{(\nu)} : eB_0^{(\nu)})$. Thus, by Lemma 10, we have $\lim_{\nu\to\infty} (T_{l,\nu+1} : T_{l,\nu}) = l^{N-\mu_l}$, which completes the proof.

§ 3. Divisible characters

6. A character χ of Ω whose order is a power of a prime number l is said to be *divisible* if, for an arbitrary power l^r of l, there is a character ψ of Ω such that we have $\chi = \psi^{l^r}$. On the other hand, if $\mathfrak{p}$ is a place of Ω and if $\Omega_\mathfrak{p}$ is the $\mathfrak{p}$-completion of Ω, then χ is said to be divisible at $\mathfrak{p}$ whenever, for every l^r, there is a character $\psi_{\Omega_\mathfrak{p}}$ of $\Omega_\mathfrak{p}^\times$ such that we have $\chi_\mathfrak{p} = \psi_{\Omega_\mathfrak{p}}^{l^r}$, where $\chi_\mathfrak{p}$ is the $\mathfrak{p}$-component of χ. If χ is divisible at every place of Ω, then we say that χ is *everywhere locally divisible*. A character χ is of course everywhere locally divisible if it is divisible.

Taking a character χ of Ω whose order is a power of l, suppose that, for any place $\mathfrak{p}$ of Ω which either is a prime ideal prime to l or is infinite, χ is unramified at $\mathfrak{p}$. Moreover, letting $\mathfrak{l}$ be any prime factor of l in Ω and $\Omega_\mathfrak{l}$ be the $\mathfrak{l}$-completion of Ω, suppose that the $\mathfrak{l}$-component $\chi_\mathfrak{l}$ is trivial on the group consisting of all roots of unity in $\Omega_\mathfrak{l}$. Then it follows from Lemma 8 that χ is everywhere locally divisible. We see that the converse also is true.

Now, let l^ν be a power of a prime number l, let $\mathfrak{S} = \{\mathfrak{l}_1, \mathfrak{l}_2, \dots\}$ be the set of all prime factors of l in Ω and let $U_{\mathfrak{l}_i}$ be the unit group of the $\mathfrak{l}_i$-completion $\Omega_{\mathfrak{l}_i}$ of Ω. Denote by $\mathfrak{w}_i$ the group of roots of unity in $\Omega_{\mathfrak{l}_i}$ and set $V_{\mathfrak{S},\nu} = \prod_i \mathfrak{w}_i U_{\mathfrak{l}_i}^{l^\nu} / U_{\mathfrak{l}_i}^{l^\nu}$. Furthermore, let N be the absolute degree of Ω and $U_{\mathfrak{S},\nu}$ be as in § 1, **1.** Then it follows from Lemma 8 that the factor group $U_{\mathfrak{S},\nu}/V_{\mathfrak{S},\nu}$ is isomorphic to the direct product of N cyclic groups of order l^ν. On the other hand, we see that, with notations in § 1, **1**, the index $(\iota_{\mathfrak{S},\nu}(B^{(\nu)}) \cdot V_{\mathfrak{S},\nu} : V_{\mathfrak{S},\nu})$ is equal to the index $(B^{(\nu)} : B_*^{(\nu)})$, where $B_*^{(\nu)}$ is the group of all $\beta \in B^{(\nu)}$ with $\beta \in \mathfrak{w}_i \Omega_{\mathfrak{l}_i}^{l^\nu}$ for every $\mathfrak{l}_i$. Furthermore, it follows from what is stated above that a character χ of Ω with order dividing l^ν and with trivial $\mathfrak{q}$-component for every place $\mathfrak{q}$ of Ω outside $\mathfrak{S}$ is everywhere locally divisible if and only if its restriction χ_U to the unit idèle group U of Ω is, as a homomorphism of $U_{\mathfrak{S},\nu}$, trivial on $V_{\mathfrak{S},\nu}$. Therefore, by Lemma 2, the number of all everywhere locally divisible characters of Ω whose orders divide l^ν is equal to $h_\nu \cdot l^{N_\nu} \cdot (B^{(\nu)} : B_*^{(\nu)})^{-1}$, where h_ν is the l^ν-class number of Ω.

7. We now prove two theorems which display characteristic properties of divisible characters.

THEOREM 2. *Let χ be an everywhere locally divisible character of Ω whose order is a power of a prime number l. Then, in general, the character χ is divisible. In the special case where $l=2$ and Ω is strongly radical with the radical number λ_T, the character χ is divisible if and only if the following condition is fulfilled: let $\mathfrak{S}=\{\mathfrak{l}_1, \mathfrak{l}_2, \ldots\}$ be the set of all prime factors of 2 in Ω and write, for every i, $\lambda_T = \lambda_i^2 \zeta_i$ with an element λ_i of the $\mathfrak{l}_i$-completion $\Omega_{\mathfrak{l}_i}$ of Ω and with a root of unity ζ_i in $\Omega_{\mathfrak{l}_i}$; then we have $\prod_i \chi_{\mathfrak{l}_i}(\lambda_i)=1$, where $\chi_{\mathfrak{l}_i}$ is the $\mathfrak{l}_i$-component of χ.*

Proof. Suppose that Ω is not radical whenever $l=2$. Then, since χ is everywhere locally divisible, the ramification places of χ are, by **6**, in $\mathfrak{S}$, and we can choose for any $\mathfrak{l}_i \in \mathfrak{S}$ and for any power l^r of l a character $\psi_{\Omega_{\mathfrak{l}_i}}$ of $\Omega_{\mathfrak{l}_i}^\times$ such that we have $\chi_{\mathfrak{l}_i} = \psi_{\Omega_{\mathfrak{l}_i}}^{l^r}$. Therefore, by Theorem 1, there is a character ψ of Ω with $\chi = \psi^{l^r}$.

Suppose next that $l=2$, and that Ω is radical with the radical number λ_T but not strongly radical. Then since we have $(\prod_i \psi_{\Omega_{\mathfrak{l}_i}}(\lambda_T^{2^{r-1}}))^2 = \prod_i \chi_{\mathfrak{l}_i}(\lambda_T) = \chi(\lambda_T) = 1$, the product $\prod_i \psi_{\Omega_{\mathfrak{l}_i}}(\lambda_T^{2^{r-1}})$ is ± 1. We may, however, assume that the product is 1, provided that we have $r > T$. For, since Ω is not strongly radical, we can choose a character η, say, of $\Omega_{\mathfrak{l}_1}^\times$ such that $\eta^2=1$, $\eta(\lambda_T) = -1$ and that η is trivial on the group of roots of unity in $\Omega_{\mathfrak{l}_1}$, whence, choosing a character η' of $\Omega_{\mathfrak{l}_1}^\times$ with $\eta'^{2^{r-1}} = \eta$ and using $\psi'_{\Omega_{\mathfrak{l}_1}} = \psi_{\Omega_{\mathfrak{l}_1}}\eta'$ instead of $\psi_{\Omega_{\mathfrak{l}_1}}$, the above product becomes 1. Therefore, again by Theorem 1, we find a character ψ of Ω with $\chi = \psi^{2^r}$.

Lastly considering the very special case in the theorem, suppose that χ is divisible. Then, for any power 2^r of 2, there is a character ψ of Ω with $\chi = \psi^{2^r}$. Therefore, if $\psi_{\mathfrak{l}_i}$ is the $\mathfrak{l}_i$-component of ψ, then we have $\prod_i \psi_{\mathfrak{l}_i}(\lambda_T^{2^{r-1}}) = 1$, because λ_T is prime to every prime ideal of Ω outside $\mathfrak{S}$.[10] Provided that, for every i, there is no root of unity whose order is higher than 2^{r-1}, we have $\psi_{\mathfrak{l}_i}(\lambda_T^{2^{r-1}}) = \psi_{\mathfrak{l}_i}(\lambda_i^{2^r}\zeta_i^{2^{r-1}}) = \psi_{\mathfrak{l}_i}^{2^r}(\lambda_i) = \chi_{\mathfrak{l}_i}(\lambda_i)$, whence $\prod_i \chi_{\mathfrak{l}_i}(\lambda_i) = 1$. Conversely, assume this

10) See foot-note 6

relation and take a character $\psi_{\Omega_{\mathfrak{l}_i}}$ of $\Omega_{\mathfrak{l}_i}^{\times}$ for every i such that we have $\chi_{\mathfrak{l}_i} = \psi_{\Omega_{\mathfrak{l}_i}}^{2^r}$. Then we have $\prod_i \psi_{\Omega_{\mathfrak{l}_i}}(\lambda_T^{2^{r-1}}) = \prod_i \psi_{\Omega_{\mathfrak{l}_i}}(\lambda_i^{2^r} \zeta_i^{2^{r-1}}) = \prod_i \chi_{\mathfrak{l}_i}(\lambda_i) = 1$ whenever r is so large that $\zeta_i^{2^{r-1}} = 1$. Hence, by Theorem 1, χ is divisible. The theorem is thus completely proved.

THEOREM 3. *Let Ω be a strongly radical field with the radical numer λ_T and let $\mathfrak{S} = \{\mathfrak{l}_1, \mathfrak{l}_2, \ldots\}$, λ_i and ζ_i be as in Theorem 2. Let $\mathbf{l}$ be the idèle of Ω whose $\mathfrak{l}_i$-component is λ_i for every i and whose $\mathfrak{q}$-component is 1 for every place $\mathfrak{q} \notin \mathfrak{S}$, and let 2^ν be a power of 2. Denote by $U_{\mathfrak{l}_i}$ the unit group of the $\mathfrak{l}_i$-completion $\Omega_{\mathfrak{l}_i}$ of Ω, by $\mathfrak{w}_i$ the group of roots of unity in $\Omega_{\mathfrak{l}_i}$ and by $\mathbf{V}_{\mathfrak{S},\nu}$ the group of unit idèles $\mathbf{u}$ of Ω such that the $\mathfrak{l}_i$-component of $\mathbf{u}$ is in $\mathfrak{w}_i U_{\mathfrak{l}_i}^{2^\nu}$ for every i. Furthermore, let $\mathbf{I}$, $\Omega^\times$ be the idèle group and the principal idèle group of Ω, respectively. Then the group of the everywhere locally divisible characters of Ω whose orders divide 2^ν coincides with the group of the divisible characters of Ω whose orders divide 2^ν whenever we have $\mathbf{l} \in \Omega^\times \mathbf{I}^{2^\nu} \mathbf{V}_{\mathfrak{S},\nu}$. Otherwise, the latter group is a subgroup of index 2 of the former one.*

Proof. In order that a character χ of Ω is everywhere locally divisible and that the order of χ divides 2^ν, it is, by **6**, necessary and sufficient that we have $\chi(\Omega^\times \mathbf{I}^{2^\nu} \mathbf{V}_{\mathfrak{S},\nu}) = 1$. On the other hand, Theorem 2 shows that such a χ is divisible if and only if we have $\chi(\mathbf{l}) = 1$. This, together with the fact that $\mathbf{l}^2$ is in $\Omega^\times \mathbf{I}^{2^\nu} \mathbf{V}_{\mathfrak{S},\nu}$, proves the theorem.

§ 4. Main results

8. We arrange preliminary results about infinite abelian groups which are for the most part obtained in Kaplansky [3].

An abelian group A is said to be a *torsion abelian group* if every element of A is of finite order, and A is said to be a *torsion abelian l-group* if the orders of all the elements of A are powers of a prime number l. Every torsion abelian group A has the unique largest torsion abelian l-group A_l for every prime number l and A is the direct product[11] of all the A_l. We call A_l the *l-component* of A.

Let A be a torsion abelian l-group. Then an element a of A is said to be *divisible* if, for any power l^r of l, there is an element b of A with $a = b^{l^r}$. If

[11] This means so called "weak" direct product arising most commonly in abstract algebra.

every element of A is divisible, then we say that A is divisible. Every torsion abelian l-group A has the unique *largest divisible subgroup* A_∞ and, if $Z(l, \infty)$ is the group of roots of unity whose orders are powers of l, then A_∞ is isomorphic to the direct product of finite or infinite number of groups all isomorphic to $Z(l, \infty)$. Moreover A_∞ is contained in the group A'_∞ consisting of all divisible elements of A.

Let again A be a torsion abelian l-group and L be the subgroup of A consisting of $a \in A$ with $a^l = 1$. We call the number of finite or infinite independent elements of L the rank of A. Furthermore, setting $L_\nu = L \cap A^{l^\nu}$, we call the rank υ_ν of $L_{\nu-1}/L_\nu$ the ν-th *Ulm invariant* of A, where $\nu = 1, 2, \ldots$.

9. Let now A be a countable torsion abelian l-group such that the group A'_∞ of all divisible elements of A is of finite rank; denote by $\upsilon_{\infty,\nu}$ the ν-th Ulm invariant of A'_∞. Then, except a finite number of ν, $\upsilon_{\infty,\nu}$ is equal to 0. In this case, we call $\upsilon_{\infty,\nu}$ the ν-th ***infinite Ulm invariant*** of A and, accordingly, call the ν-th Ulm invariant of A itself the ν-th *finite Ulm invariant* of A. Moreover, if A_∞ is the largest divisible subgroup of A, then we call the rank of A_∞ the *dimension* of A. Under this terminology, the theorem of Ulm[12] shows that the structure of A is determined whenever the finite and the infinite Ulm invariants of A as well as the dimension of A are known. The theorem also implies that A'_∞/A_∞ is a finite group because A'_∞/A_∞ contains no non-trivial divisible subgroup and its system of Ulm invariants coincides with that of a finite group.

Let l^{c_ν} be the number of elements of A'_∞ whose orders divide l^ν. Then since A'_∞ is isomorphic to the direct product A_∞ by the finite group A'_∞/A_∞, it follows from elementary properties of finite abelian groups that we have $\upsilon_{\infty,\nu} = 2c_\nu - c_{\nu-1} - c_{\nu+1}$. On the other hand, if T is a subgroup of finite rank of A containing A_∞, then we see, as in the case of $T = A'_\infty$ above, that T is isomorphic to the direct product of A_∞ by the finite group T/A_∞. Therefore, denoting by T_ν the group of elements of T whose orders divide l^ν, we can determine the dimension $\dim A$ of A by $l^{\dim A} = \lim\limits_{\nu \to \infty} (T_{\nu+1} : T_\nu)$.

10. We are now able to expose the structure of the group X_l which is the l-component of the countable torsion abelian group X consisting of all the characters of Ω, where l is a prime number. Denote by $X'_{l,\infty}$ the group of all

12) See Kaplansky [3], §11.

divisible elements of X_l. Then, by **6**, $X'_{l,\infty}$ is contained in the group T of characters $\chi \in X_l$ such that χ is unramified at any place $\mathfrak{q}$ of Ω coinciding with none of the prime factors of l in Ω. Since T is of finite rank, so is also $X'_{l,\infty}$. Therefore, the results of **9** show that the structure of X_l is determined whenever the finite and the infinite Ulm invariants and the dimension of X_l are known. By Lemma 6 and Lemma 7, we have

THEOREM 4. *Let l be a prime number and ζ_l be a primitive l-th root of unity. Denote by ν_l a natural number such that the field $\Omega(\zeta_l)$ contains a primitive l^{ν_l}-th root of unity but no primitive l^{ν_l+1}-th root of unity. Then the ν-th finite Ulm invariant of X_l is 0 for $\nu < \nu_l$ and is ∞ for $\nu \geqq \nu_l$.*

The largest divisible subgroup $X_{l,\infty}$ of X_l is contained in the group T defined above. Therefore, by **9** and by Lemma 11, we have

THEOREM 5. *Let l be a prime number, $\mathfrak{S} = \{\mathfrak{l}_1, \mathfrak{l}_2, \ldots\}$ be the set of all prime factors of l in Ω and $\Omega_{\mathfrak{l}_i}$ be the $\mathfrak{l}_i$-completion of Ω. Denote by $\mathfrak{e}$ the unit group of Ω and by $\mathfrak{e}_{l,\nu}$ the group of $\varepsilon \in \mathfrak{e}$ such that ε is an l^ν-th power in every $\Omega_{\mathfrak{l}_i}$. Then there is a constant μ_l such that we have $l^{\mu_l} = (\mathfrak{e}_{l,\nu} : \mathfrak{e}_{l,\nu+1})$ for every sufficiently large ν and the dimension of X_l is equal to $N - \mu_l$, where N is the absolute degree of Ω.*

11. There is thus remained only the determination of infinite Ulm invariants of X_l. But this is substantially done in § 3. For we obtained there a method of finding the number l^{c_ν} of elements in X_l whose orders divide a power l^ν of l. We add here a few remarks.

Let l^ν be a power of an add prime number l and $B^{(\nu)}$ be the group of $\beta \in \Omega^\times$ such that the principal ideal (β) is the l^ν-th power of an ideal of Ω. Let $\mathfrak{S}$ and $\Omega_{\mathfrak{l}_i}$ be as in Theorem 5, let $\mathfrak{w}_i$ be the group of roots of unity in $\Omega_{\mathfrak{l}_i}$ and let $B^{(\nu)}_*$ be the group of $\beta \in B^{(\nu)}$ such that β is in $\mathfrak{w}_i \Omega_{\mathfrak{l}_i}^{l^\nu}$ for every i. Then, by **6** and by Theorem 2, we have $l^{c_\nu} = h_\nu \cdot l^{N\nu} \cdot (B^{(\nu)} : B^{(\nu)}_*)^{-1}$. Therefore, by **9**, the ν-th infinite Ulm invariant $\upsilon_{\infty,\nu}$ of X_l is given by

$$l^{\upsilon_{\infty,\nu}} = \frac{h_\nu^2}{h_{\nu-1} h_{\nu+1}} \quad \frac{(B^{(\nu-1)} : B_*^{(\nu-1)})(B^{(\nu+1)} : B_*^{(\nu+1)})}{(B^{(\nu)} : B_*^{(\nu)})^2},$$

where h_ν is the l^ν-class number of Ω. Let the first factor of the right side of this formula be equal to l^{b_ν}. Then b_ν is the number of direct factors of order

l^{ν} in the direct decomposition of the ideal class group of Ω into indecomposable cyclic groups.

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