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ON BOUNDARY VALUE PROBLEMS FOR ELLIPTIC EQUATIONS IN A SINGULAR DOMAIN

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1. Let Ω be a bounded domain in the plane and denotes its closure and boundary by $\bar{\Omega}$ and $\partial\Omega$, respectively. We shall say that the domain Ω is regular, if every point $P \in \partial\Omega$ has an 2-dimensional neighborhood U such that $\partial\Omega \cap U$ can be mapped in a one-to-one way onto a portion of the tangent line through P by a mapping T which together with its inverse is infinitely differentiable. Let L be an elliptic operator of order $2m$ defined in $\bar{\Omega}$ and let $\{B_j\}_{j=1}^m$ be a normal set of boundary operators of orders $m_j < 2m$. If f is a given function in Ω, the boundary value problem $\Pi(L, f, B_j)$ will be to find a solution u of

$$Lu = f \quad \text{in} \quad \Omega$$

satisfying

$$B_j u = 0 \quad \text{on} \quad \partial\Omega, \quad j = 1, \cdots, m.$$

Schechter [8] proved the following: If the set $\{B_j\}_{j=1}^m$ is normal and covers L, there is another normal set $\{B'_j\}_{j=1}^m$ such that a solution of the problem $\Pi(L, f, B_j)$ exists if and only if the only solution of $\Pi(L^*, 0, B'_j)$ is $u = 0$. Here L^* denotes the formal adjoint of L.

We consider the problem $\Pi(L, f, B_j)$ when Ω is not regular in our sense. When Ω is a domain in the plane, we shall call it singular if $\partial\Omega$ consists of a set $\{\Gamma_i\}_{i=1}^N$ of boundary portions which are sufficiently smooth and satisfy the following conditions.

(i) Each boundary portion Γ_i is a slit in $\bar{\Omega}$ or is contained in the outer boundary of Ω. When Γ_i is a slit, we distinguish between both sides.

(ii) If Γ_i and $\Gamma_{i'}$ are contained in the outer boundary and adjoining at

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S, they are tangent at S of infinite order from the interior. More precisely, some neighborhood of S in Ω can be mapped in a one-to-one C^∞ way into an open disk which has an incision.

In this note we consider general boundary value problems for elliptic partial differential equations when Ω is singular in our sense.

Let $\{B_{ij}\}_{j=1}^m$ be a set of partial differential operators on each Γ_i. The problem we consider is the following: Given a function f in Ω, find the solution u such that

$$Lu = f \quad \text{in} \quad \Omega,$$
$$B_{ij}u = 0 \quad \text{on} \quad \Gamma_i,$$
$$i = 1, \cdots, N, \quad j = 1, \cdots, m.$$

Our method employs coerceiveness inequalities specially adapted to the probelm. In neighborhood of points of the inner part of Γ_i, no new inequalities are needed (c.f. [1, 8]). For the endpoint of Γ_i we obtain special inequalities which are reduced to the mixed boundary value problems.

Mixed boundary value problems in a planar domain were studied quite extensively by Peetre [7] and Shamir [12]. They used some properties of the Hilbert transform on the half line which were given in [5], [11], and [15]. For arbitrary dimension, Schechter [9] treated the mixed boundary problems under a rather complicated compatibility condition. In this note our proof relies upon mainly the results of Schechter [9] and Shamir [12].

2. Let R^n be the n-dimensional Euclidean space. Throughout this note we consider only the case $n = 1$ or 2. Points in R^2 are denoted by $P = (x, t)$ and $|P|^2 = |x|^2 + |t|^2$. The half space $t > 0$ (<0) is denoted by $R^2_+(R^2_-)$. Let $\alpha = (\alpha_1, \alpha_2)$ be a multi-index of non-negative integers with length $|\alpha| = \alpha_1 + \alpha_2$. We shall write

$$D = (D_x, D_t), \quad D^\alpha = D_x^{\alpha_1} D_t^{\alpha_2} \qquad (D_x = \partial/\partial x, \ D_t = \partial/\partial t).$$

We consider an elliptic differential operator of the form

$$L(D) = \sum_{|\alpha|=2m} a_\alpha D^\alpha, \tag{2. 1}$$

where the coefficients a_α are complex numbers and $2m$ is the order of $L(D)$. The characteristic polynomial corresponding to $L(D)$ is

$$L(\xi, \eta) = \sum_{|\alpha|=2m} a_\alpha \xi^{\alpha_1} \eta^{\alpha_2}.$$

We set

(2. 2) $$L_1(\xi, \tau) = L(\xi, \tau), \quad L_2(\xi, \tau) = L(\xi, -\tau)$$

and

(2. 3) $$B_{1j}^-(D) = D_t^{j-1}, \quad B_{2j}^-(D) = (-1)^j D_t^{j-1}, \qquad j = 1, \cdots, 2m.$$

In this section we shall mainly describe Agmon-Douglis-Nirenberg's results for the boundary value problem of the elliptic system:

(2. 4) $$\begin{aligned} &L_1(D)u_1 = f_1, \quad L_2(D) = f_2, \quad t > 0, \\ &B_{1j}^- u_1 + B_{2j}^- u_2 = \varphi_j, \quad t = 0, \quad j = 1, \cdots, 2m. \end{aligned}$$

Denote by $\tau_{i,k}^+(\xi)$ (or $\tau_{i,k}^-(\xi)$), $k = 1, \cdots, m$ the roots of $L_i(\xi, \tau) = 0$ with positive (or negative) imaginary parts, and set

$$L_i^\pm(\xi, \tau) = \prod_1^m (\tau - \tau_{i,k}^\pm(\xi)),$$

$$M^+(\xi, \tau) = L_1^+(\xi, \tau) L_2^+(\xi, \tau), \quad \xi \neq 0.$$

Then we have

LEMMA 2. 1. *The boundary value problem* (2. 4) *satisfies the Complementing Condition in the sense of* [2]. *That is, for each real* $\xi \neq 0$ *the relations*

(2. 5) $$\begin{aligned} &\sum_{j=1}^{2m} \lambda_j B_{1j}^-(\tau) L_2(\xi, \tau) = U_1(\tau) M^+(\xi, \tau) \\ &\sum_{j=1}^{2m} \lambda_j B_{2j}^-(\tau) L_1(\xi, \tau) = U_2(\tau) M^+(\xi, \tau) \end{aligned}$$

imply that $U_1(\tau)$, $U_2(\tau)$ *and the* λ_j *all vanish, where the* λ_j *are complex constants and the* $U_i(\tau)$ *are polynomials.*

Proof. We note that (2. 5) are equivalent to

(2. 6) $$\begin{aligned} &\sum_{j=1}^{2m} \lambda_j B_{1j}^-(\tau) = U_1'(\tau) L_1^+(\xi, \tau), \\ &\sum_{j=1}^{2m} \lambda_j B_{2j}^-(\tau) = U_2'(\tau) L_2^+(\xi, \tau), \end{aligned}$$

where $U_i'(\tau)$ are other polynomials. From (2. 2) we have $L_2^+(\xi, \tau) = L^+(-\xi, \tau)$. Hence the relations (2. 6) imply that

$$\sum_{j=1}^{2m} \lambda_j \tau^{j-1} = U_1'(\tau) L^+(\xi, \tau),$$

$$\sum_{j=1}^{2m} \lambda_j (-1)^j \tau^{j-1} = U_2'(\tau) L^+(-\xi, \tau).$$

Thus it follows that

$$-U_1'(-\tau)L^+(\xi,-\tau)=U_2'(\tau)L^+(-\xi,\tau).$$

Noting that $L^+(\xi,-\tau)=(-1)^m L^-(-\xi,\tau)$, we see

$$(-1)^{m-1}U_1'(-\tau)L^-(-\xi,\tau)=U_2'(\tau)L^+(-\xi,\tau). \tag{2.7}$$

Since $U_i'(\tau)$ are of degree at most $m-1$, the relation (2. 7) means that every $U_i'(\tau)$ vanishes. Hence all λ_j vanish. This completes the proof.

We first consider the problem (2. 4) in the case $f_1=f_2=0$ and $\varphi_1(x)$, $\varphi_2(x)\in C_0^\infty(R)$[1]. This problem can be solved by the formula

$$u_i(x,t)=\sum_{j=1}^{2m}\int K_{ij}(x-y,t)\varphi_j(y)dy,\quad i=1,2, \tag{2.8}$$

where $K_{ij}(x,t)$ are Poisson kernels of class C^∞ for $t>0$ except at the origin. We set

$$G(z,M)=-(2\pi iM)^{-1}z^M(\log(z/i)-\sum_{k=1}^{M}1/k).$$

Then we have for odd $q>0$

$$K_{ij}(x,t)=\left(\frac{\partial}{\partial x}\right)^{(1+q)/2}\sum_{\pm}\left(\pm\frac{\partial}{\partial x}\right)^{2m-1-m_j^-}R_{ij}(x,t;\pm 1) \tag{2.9}$$
$$(m_j^-=\deg.\ B_j^-=j-1)$$

and

$$\begin{aligned}R_{ij}(x,t,\pm 1)=(2\pi i)^{-1}\sum_{\pm}\int_\gamma &L_i(\pm 1,\tau)\times\\ &\times G(\pm x+t\tau,q+2m-1)\\ &\times\sum_{l=0}^{2m-1}c_{ilj}^{\pm}\frac{M_{2m-l-1}(\pm 1,\tau)}{M^+(\pm 1,\tau)}d\tau,\end{aligned} \tag{2.10}$$

where γ is a closed curve in $Im\ \tau>0$ enclosing all the zeros of $M^+(\pm 1,\tau)$ and $c_{ilj}^{\pm}$ are constants depending on L_1 and L_2.

The functions M_{2m-l-1} in (2. 10) are polynomials such that

$$(2\pi i)^{-1}\int_\gamma\frac{M_{2m-1-l}(\pm 1,\tau)}{M(\pm 1,\tau)}\tau^k d\tau=\delta_{lk},$$
$$0\leqq j,k\leqq 2m-1.$$

1) We denote R^1 by R.

It is seen that $K_{ij}(x,t)$ are of class C^∞ for $t \geqq 0$, except at the origin, and satisfy

$$|D^\alpha K_{ij}| \leqq C(x^2+t^2)^{(m_j^- - |\alpha|-1)/2}(1+|\log(x^2+t^2)|). \tag{2. 11}$$

We now consider the problem (2. 4) for $f_1, f_2 \in C_0^\infty(\bar{R}_+^2)$. For this purpose we extend f_i to the whole plane R^2 as functions with compact support of class C^N (see [1, p. 519]). Let $f_i^{(N)}(x,t)$ be the extended functions. Having chosen some large N, we set

$$v_i(P) = \int \Gamma_i(P-Q) f_i^{(N)}(Q)dQ, \tag{2. 12}$$

where $\Gamma_i(P)$ is a fundamental solution of the equation $L_i u = 0$. The function v_i satisfies $L_i v_i = f_i^{(N)}$ and it is known that

$$D^\alpha v_i(P) = O(|P|^{2m-2-|\alpha|}(1+|\log|P||)), \quad P \to \infty. \tag{2. 13}$$

In addition, we see that for β such that $|\beta| = 2m$

$$D^\beta v_i = \int D^\beta \Gamma(P-Q) f_i^{(N)}(Q)dQ \tag{2. 14}$$

and that $D^\beta \Gamma$ is a homogeneous kernel of degree -2 to which Calderon-Zygmund's results on singular integrals can be applied.

Proposition 2. 1[1]. *Let u_i be C^∞ solutions with compact support in $t \geqq 0$ of the problem* (2. 4). *Then it holds*

$$D^\alpha u_i = D^\alpha v_i + \sum_{j=1}^{2m} \int D^\alpha K_{ij}(x-y,t)\cdot(\varphi_j(y) - \psi_j(y))dy, \tag{2. 15}$$
$$(|\alpha| \geqq 2m-1)$$

where $\psi_j(y) = B_{1j}^- v_1(y,0) + B_{2j}^- v_2(y,0)$.

This was proved in detail in [1] and [2] for $|\alpha| \geqq 2m$ and we easily verify it for $|\alpha| = 2m-1$.

For an integral $r \geqq 0$ we use the norm

$$\|u, \Omega\|_r = \sum_{|\alpha| \leqq r} \left(\int_\Omega |D^\alpha u|^2 dx \right)^{1/2},$$

where $\Omega = R^n$ or $R_+^n (n=1,2)$. For a real $s \geqq 0$ we define the seminorms

[1] For single equations this was verified in [12].

$$[u, \Omega]_s = \left(\int_\Omega\int_\Omega \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}dxdy\right)^{1/2}, \quad 0 < s < 1,$$

$$[u, \Omega]_s = \sum_{|\alpha|=s}[D^\alpha u, \Omega]_{s-[s]}, \quad 1 \leqq s.$$

Let $W^s(\Omega)$ be the completion of $C^\infty(\bar{\Omega})$ with respect to the norm

$$\|u, \Omega\|_s = \|u, \Omega\|_{[s]} + [u, \Omega]_s.$$

Then we have from (2. 14) (c.f. [1])

$$\|v_i, R^2_+\|_{2m} \leqq C\|f_i^{(N)}, R^2\|_0 \leqq C\|L_j u_i, R^2_+\|_0. \tag{2. 16}$$

Proposition 2. 2. (c.f. [2]) Assume that $u_i(x, t)$ belong to $C_0^\infty(\bar{R}^2_+)$ and $l \geqq 2m$. Then there is a constant C such that

$$\begin{aligned} \|u_1, R^2_+\|_l + \|u_2, R^2_+\|_l \leqq C(&\|L_1u_1, R^2_+\|_{l-2m} \\ &+ \|L_2u_2, R^2_+\|_{l-2m} + \sum_{j=1}^{2m}\|B^-_{1j}u_1 + B^-_{2j}u_2, R\|_{l-m^-_j-\frac{1}{2}} \\ &+ \|u_1, R^2_+\|_0 + \|u_2, R^2_+\|_0). \end{aligned} \tag{2. 17}$$

This was proved in [1], [2] under potential theoretic considerations.

3. Let $\{B^+_j\}_{j=1}^m$ be a set of boundary operators with constant coefficients. We assume that B^+_j is homogeneous of degree m^+_j $(< 2m)$ and that the Complementing Condition on $\{B^+_j\}$ is satisfied. In this section we shall give a proof of the following mixed a priori estimates for $u_i \in C_0^\infty(\bar{R}^2_+)$,

$$\begin{aligned} \|u_1, R^2_+\|_{2m} + \|u_2, R^2_+\|_{2m} \leqq C(&\|L_1u_1, R^2_+\|_0 + \|L_2u_2, R^2_+\|_0 \\ &+ \sum_{j=1}^{2m}\|B^-_{1j}u_1 + B^-_{2j}u_2, R_-\|_{2m-m^-_j} \\ &+ \sum_{j=1}^{2m}\|B^+_{1j}u_1 + B^+_{2j}u_2, R_+\|_{2m-m^+_j} \\ &+ \|u_1, R^2_+\|_0 + \|u_2, R^2_+\|_0). \end{aligned} \tag{3. 1}$$

The proof of (3. 1) is obtained in a similar manner to the method developed by Shamir for single equations (c.f. [11]).

We consider now the Hilbert transform on R defined by

$$(\mathscr{H}^{\pm}f)(x) = \lim_{\varepsilon\downarrow 0}(2\pi i)^{-1}\int_{-\infty}^{\infty}\frac{f(y)}{x+i\varepsilon-y}dy.$$

Put $\mathscr{A}\varphi = (C\mathscr{H}^+ + D\mathscr{H}^-)\varphi$, where φ is a $2m$ dimensional vector function and C and D are $2m \times 2m$ matrices with constant coefficients.

PROPOSITION 3. 1. *If C and D are non singular and if the eigenvalues of $C^{-1}D$ do not lie on the negative real axis, then for $\phi \in W^{\frac{1}{2}}(R)$*

$$[\phi, R]_{\frac{1}{2}} \leqq C([\phi, R_-]_{\frac{1}{2}} + [\mathscr{A}\phi, R_+]_{\frac{1}{2}})^{1)}. \tag{3. 2}$$

The inequality (3. 2) was established by several authors (c.f.e.g., Koppelman-Pincus [5], J. Schwartz [14], Widom [15], Shamir [11] and for any dimensional case Shamir [13]). Now we set $u_i - v_i = w_i$, $\varphi_j - \psi_j = B^-_{1j}w_1 + B^-_{2j}w_2|_{t=0} = \omega_j$ in the representation formulas (2. 15). Then it follows from Proposition 2. 1 that

$$D^\alpha w_i(x, t) = \sum_{j=1}^{2m}\int D^\alpha K_{ij}(x-y, t)\omega_j(y)dy, \quad |\alpha| \geqq 2m-1. \tag{3. 3}$$

Put $l^\pm_j = 2m - 1 - m^\pm_j$. Then we obtain from (3. 3) by integration by parts

$$\begin{aligned} &D_x^{l^+_k}(B^+_{1k}w_1 + B^+_{2k}w_2)(x, t) \\ &= \sum_{j=1}^{2m}\int_{-\infty}^{\infty} \{D_x^{l^+_k}[B^+_{1k}K_{1j} + B^+_{2k}K_{2j}](x-y, t)\} \cdot \\ &\qquad D_x^{l^-_j}(B^-_{1j}w_1 + B^-_{2j}w_2)(y, 0)dy. \end{aligned} \tag{3. 4}$$

Let t tend to zero in both sides of (3. 4). Then we have

$$\begin{aligned} &D_x^{l^+_k}(B^+_{1k}w_1 + B^+_{2k}w_2)(x, 0) \\ &= \int_{-\infty}^{\infty} \sum_{j=1}^{2m}(c_{kj}\mathscr{H}^+ + d_{kj}\mathscr{H}^-) \cdot \\ &\qquad D_x^{l^-_j}[B^-_{1j}w_1 + B^-_{2j}w_2](y, 0)dy, \end{aligned} \tag{3. 5}$$

where $\{c_{kj}\}$, $\{d_{kj}\}$ are two matrices with constant coefficients. Put $C = \{c_{kj}\}$ and $D = \{d_{kj}\}$. We make the following assumption.

ASSUMPTION 3. 1. *Two matrices C, D are non singular and eigenvalues of $C^{-1}D$ do not lie on the negative real axis.*

Then we have

THEOREM 3. 1. *Under Assumption 3. 1, the mixed a priori estimates (3. 1) holds.*

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1) If $\phi = (\phi_1, \cdots, \phi_{2m})$, we set $\|\phi, \Omega\|_s = \sum\|\phi_i, \Omega\|_s$ and $[\phi, \Omega]_s = \sum[\phi_i, \Omega]_s$.

Proof. We set

$$\varphi_j(x) = D_x^{l_j^-}(B_{1j}^- w_1 + B_{2j}^- w_2)(x, 0),$$
$$\psi_k(x) = D_x^{l_k^+}(B_{1k}^+ w_1 + B_{2k}^+ w_2)(x, 0)$$

and

$$\varphi = (\varphi_1, \cdots, \varphi_{2m}), \quad \psi = (\psi_1, \cdots, \psi_{2m}).$$

We have by (3. 5)

$$\psi = (C\mathscr{H}^+ + D\mathscr{H}^-)\varphi. \tag{3. 6}$$

Since $\varphi \in W^{\frac{1}{2}}(R)$ from (2. 13), Proposition 3. 1 is applicable to the equation (3. 6). Hence it follows that

$$\begin{aligned}\sum_j \|B_{1j}^- w_1 + B_{2j}^- w_2, R\|_{2m-m_j^- -\frac{1}{2}} \\ \leqq C \sum_{j,\pm} \|B_{1j}^+ w_1 + B_{2j}^+ w_2, R_+\|_{2m-m_j^+ -\frac{1}{2}}.\end{aligned} \tag{3. 7}$$

Since $w_i = u_i - v_i$, we see

$$\begin{aligned}&\|B_{1j}^\pm w_1 + B_{2j}^\pm w_2, R_\pm\|_{2m-m_j^\pm -\frac{1}{2}} \\ &\leqq \|B_{1j}^\pm u_1 + B_{2j}^\pm u_2, R_\pm\|_{2m-m_j^\pm -\frac{1}{2}} \\ &\quad + \|B_{1j}^\pm v_1 + B_{2j}^\pm v_2, R_\pm\|_{2m-m_j^\pm -\frac{1}{2}}.\end{aligned} \tag{3. 8}$$

According to the well known result (c.f.e.g. [1], [8]) there exists a constant C depending only on k ($\geqq 0$) such that the following inequality holds:

$$\|f, R\|_k \leqq C\|f, R_+^2\|_{k+\frac{1}{2}} \tag{3. 9}$$

for all $f \in C^\infty(\bar{R}_+^2)$.
Thus we see from (3. 9)

$$\begin{aligned}&\|B_{1j}^\pm v_1 + B_{2j}^\pm v_2, R_\pm\|_{2m-m_j^\pm -\frac{1}{2}} \\ &\leqq \|B_{1j}^\pm v_1 + B_{2j}^\pm v_2, R\|_{2m-m_j^\pm -\frac{1}{2}} \\ &\leqq \|B_{1j}^\pm v_1 + B_{2j}^\pm v_2, R_\pm^2\|_{2m-m_j^\pm} \\ &\leqq C(\|v_1, R_+^2\|_{2m} + \|v_2, R_+^2\|_{2m}).\end{aligned}$$

Using the inequalities (2. 16) and (3. 9), we have

$$\begin{aligned}&\|B_{1j}^\pm v_1 + B_{2j}^\pm v_2, R_\pm\|_{2m-m_j^\pm -\frac{1}{2}} \\ &\leqq C(\|L_1 u_1, R_+^2\|_0 + \|L_2 u_2, R_+^2\|_0).\end{aligned} \tag{3. 10}$$

On the other hand it follows from Proposition 2. 2 that

$$
\begin{aligned}
(3.11) \qquad & \|u_1, R^2_+\|_{2m} + \|u_2, R^2_+\|_{2m} \leqq C(\|L_1u_1, R^2_+\|_0 \\
& + \|L_2u_2, R^2_+\|_0 \\
& + \sum_{j=1}^{2m} \|B^-_{1j}v_1 + B^-_{2j}v_2, R\|_{2m-m_j^-} \\
& + \sum_{j=1}^{2m} \|B^-_{1j}w_1 + B^-_{2j}w_2, R\|_{2m-m_j^-} \\
& + \|u_1, R^2_+\|_0 + \|u_2, R^2_+\|_0).
\end{aligned}
$$

Combining (3. 7), (3. 8), (3. 10) and (3. 11), we obtain the proof of the theorem.

4. In this section we shall prove coerceive inequalities for a singular domain. Let $\mathscr{D}$ be an open disk with the center O and radius r which has an incision along the positive x axis. We denote by Γ_1, Γ_2 the upper and lower boundary portions of the incision respectively. Let $\tilde{\mathscr{D}}$ be the closure of the subspace $\mathscr{D}$ in a manifold which distinguish between Γ_1 and Γ_2. Put $\tilde{C}_0^\infty(\mathscr{D}) = \{u \in C^\infty(\tilde{\mathscr{D}}) \mid u = 0$ in a neighborhood of $|x| = 0$ and $|x| = r\}$.

Let us consider an elliptic differential operator $L(D)$ of the form (2. 1) and let $\{\tilde{B}_{ij}\}_{j=1}^m$ be a set of boundary operators on Γ_i such that $\tilde{B}_{ij}$ is homogeneous of degree m_j $(< 2m)$.
Set

$$
\begin{aligned}
& L_1(D) = L(D), \quad L_2(D) = L(D_x, -D_t), \\
(4.1) \qquad & B^+_{1j}(D) = \tilde{B}_{1j}(D), \quad B^+_{2j}(D) = \tilde{B}_{2j}(D_x, -D_t) \\
& B^-_{1j}(D) = D_t^{j-1}, \quad B^-_{2j}(D) = (-1)^j D_t^{j-1}, \\
& j = 1, \cdots, m.
\end{aligned}
$$

Then we can prove the following

THEOREM 4. 1. *If $\{L_i(D), B^\pm_{ij}(D)\}$ of type* (4. 1) *satisfies Assumption* 3. 1 *and if $\{L_i(D), B^+_{ij}(D)\}$ satisfies the Complementing Condition, then there exists a constant C such that*

$$
\begin{aligned}
(4.2) \qquad & \|u, \mathscr{D}\|_{2m} \leqq C(\|Lu, \mathscr{D}\|_0 + \sum_{j=1}^m \|\tilde{B}_{1j}u, \Gamma_1\|_{2m-m_j-\frac{1}{2}} \\
& + \sum_{j=1}^m \|\tilde{B}_{2j}u, \Gamma_2\|_{2m-m_j-\frac{1}{2}} \\
& + \|u, \mathscr{D}\|_0)
\end{aligned}
$$

for all $u \in \tilde{C}_0^\infty(\mathscr{D})$.

Proof. Put

$$u_1(x, t) = u(x, t), \quad u_2(x, t) = u(x, -t), \quad t > 0.$$

Then we easily see

$$B_{1j}^- u_1 + B_{2j}^- u_2 = 0, \quad t = 0,$$
$$\tilde{B}_{1j} u \,|\, \Gamma_1 = B_{1j}^+ u_1 \,|_{t=0}$$

and

$$\tilde{B}_{2j} u \,|\, \Gamma_2 = B_{2j}^+ u_2 \,|_{t=0}.$$

Thus it is sufficient to prove that

$$\begin{aligned} \|u_1, R_+^2\|_{2m} + \|u_2, R_+^2\|_{2m} \leqq C(&\|L_1 u_1, R_+^2\|_0 + \|L_2 u_2, R_+^2\|_0 \\ &+ \sum_{j=1}^{2m} \|B_{1j}^- u_1 + B_{2j}^- u, R_-\|_{2m-m_j^-} \\ &+ \sum_{j=1}^{2m} \|B_{1j}^+ u_1 + B_{2j}^+ u_2, R_+\|_{2m-m_j^+} \\ &+ \|u_1, R_+^2\|_0 + \|u_2, R_+^2\|_0). \end{aligned}$$

This inequality follows from Theorem 3. 1. So, the proof of Theorem 4. 1 is obtained.

Let Ω be a singular domain in our sense. Denote by $\tilde{C}^\infty(\bar{\Omega})$ a set of functions which are C^∞ in $\bar{\Omega}$ and vanish near the endpoints of each boundary portion. We consider an elliptic operator of order $2m$ in the form

$$L(P, D) = \sum_{|\alpha| \leqq 2m} a_\alpha(x, t) D_x^{\alpha_1} D_t^{\alpha_2}, \quad a_\alpha(x, t) \in C^\infty(\bar{\Omega}). \tag{4. 3}$$

On each boundary portion Γ_i there are defined m partial differential operators

$$B_{ij}(P, D) = \sum_{|\alpha| \leqq m_{ij}} b_{ij\alpha}(x, t) D_x^{\alpha_1} D_t^{\alpha_2}, \quad j = 1, \cdots, m, \tag{4. 4}$$

where $m_{ij} < 2m$ and the coefficients are in $C^\infty(\Gamma_i)$.

We make the following assumption.

ASSUMPTION 4. 1. *We assume that the boundary set* $\{B_{ij}(P, D)\}_{j=1}^m$ *is normal in the sense of* [8] *and satisfies the Complementing Condition.*

Let P_0 be an endpoint of a boundary portion Γ_i. For a real vector τ tangent to Γ_i at P_0 and a real vector ν normal to Γ_i at P_0, we rewrite the operators $L(P_0, D)$, $B_{ij}(P_0, D)$ of type (4. 3), (4. 4) in the form

$$
\begin{aligned}
L(P_0, D) &= L(P_0, D_x, D_t) \\
&= \tilde{L}(P_0, D_\tau, D_\nu) = \tilde{L}(P_0, \tilde{D}), \\
B_{ij}(P_0, D) &= B_{ij}(P_0, D_x, D_t) \\
&= \tilde{B}_{ij}(P_0, D_\tau, D_\nu) = \tilde{B}_{ij}(P_0, \tilde{D}), \\
& 1 \leqq j \leqq m,
\end{aligned}
\tag{4. 5}
$$

where $D_\tau = \frac{\partial}{\partial \tau}$ and $D_\nu = \frac{\partial}{\partial \nu}$. Then we have the following

THEOREM 4. 2. *Under Assumption* 4. 1, *consider operators* $L(P, D)$, $B_{ij}(P, D)$ *of type* (4. 3), (4. 4) *in a singular domain* Ω. *Suppose that* $\tilde{L}(P_0, \tilde{D})$, $\tilde{B}_{ij}(P_0, \tilde{D})$ *of the form* (4. 5) *satisfy Assumption* 3. 1 *for each endpoint* P_0 *of boundary portions. Then there is a constant* C *depending only on* $L(P, D)$, $B_{ij}(P, D)$ *and such that*

$$
\begin{aligned}
\|u, \Omega\|_{2m} \leqq C(\|L(P, D)u, \Omega\|_0 \\
+ \sum_{i,j} \|B_{ij}(P, D)u, \Gamma_i\|_{2m-m_j-\frac{1}{2}} \\
+ \|u, \Omega\|_0)
\end{aligned}
\tag{4. 6}
$$

for all $u \in \tilde{C}^\infty(\bar{\Omega})$.

Proof. The passage from the equations with constant coefficients in a half space to the estimate (4. 6) is performed in a familiar method based on a partition of unity (c.f.e.g. [4, 8, 9, 10]). Thus we shall show (4. 6) only in a neighborhood of the endpoints of each Γ_i.

Let P_0 be an endpoint of Γ_i. From our definition of singular domains, we can take a sufficiently small neighborhood $U(P_0)$ of P_0 such that $U(P_0)$ can be mapped in a one-to-one C^∞ way into an open disk $\mathscr{D}$ which has an incision along the positive x axis. By applying Theorem 3. 1, it follows that

$$
\begin{aligned}
\|u, U(P_0) \cap \Omega\|_{2m} \leqq C(\|L(P_0, D)u, U(P_0) \cap \Omega\|_0 \\
+ \sum_j \|B_{i_1 j}(P_0, D)u, \Gamma_{i_1}\|_{2m-m_j-\frac{1}{2}} \\
+ \sum_j \|B_{i_2 j}(P_0, D)u, \Gamma_{i_2}\|_{2m-m_j-\frac{1}{2}} + \|u, U(P_0) \cap \Omega\|_0
\end{aligned}
\tag{4. 7}
$$

for all $u \in \tilde{C}_0^\infty(U(P_0) \cap \Omega)$. Here $\tilde{C}_0^\infty(U(P_0) \cap \Omega) = \{u \in C^\infty(U(P_0) \cap \Omega) \mid u = 0$ in a neighborhood of P_0 and $\partial U(P_0)\}$. We see from (4. 7)

$$
\begin{aligned}
\|u, U(P_0)\cap\Omega\|_{2m} \leqq C(&\|L(P, D)u, U(P_0)\cap\Omega\|_0 \\
&+\sum_j \|B_{i_1 j}(P, D)u, \Gamma_{i_1}\|_{2m-m_j-\frac{1}{2}} \\
&+\sum_j \|B_{i_2 j}(P, D)u, \Gamma_{i_2}\|_{2m-m_j-\frac{1}{2}} \\
&+\|(L(P_0, D)-L(P, D))u, U(P_0)\cap\Omega\|_0 \\
&+\sum_j \|(B_{i_1 j}(P_0, D)-B_{i_1 j}(P, D))u, \Gamma_{i_1}\|_{2m-m_j-\frac{1}{2}} \\
&+\sum_j \|(B_{i_2 j}(P_0, D)-B_{i_2 j}(P, D))u, \Gamma_{i_2}\|_{2m-m_j-\frac{1}{2}} \\
&+\|u, U(P_0)\cap\Omega\|_0).
\end{aligned}
\tag{4. 8}
$$

By the well known interpolation method, we find a neighborhood $U(P_0)$ for a given $\varepsilon>0$ such that

$$
\begin{aligned}
&\|(L(P_0, D)-L(P, D))u, U(P_0)\cap\Omega\|_0 \\
&\quad\leqq \varepsilon\|u, U(P_0)\cap\Omega\|_{2m} \\
&\qquad + C(\varepsilon)\|u, U(P_0)\cap\Omega\|_0, \\
&\sum_j \|(B_{i_k j}(P_0, D)-B_{i_k j}(P, D))u, \Gamma_{i_k}\|_{2m-m_j-\frac{1}{2}} \\
&\quad\leqq \varepsilon\|u, \Gamma_{i_k}\|_{2m-\frac{1}{2}} + C(\varepsilon)\|u, \Gamma_{i_k}\|_{-\frac{1}{2}} \\
&\qquad k=1, 2.
\end{aligned}
\tag{4. 9}
$$

By (3. 9) we see

$$
\begin{aligned}
&\sum_{k,j} \|(B_{i_k j}(P_0, D)-B_{i_k j}(P, D))u, \Gamma_{i_k}\|_{2m-m_j-\frac{1}{2}} \\
&\quad\leqq C(\varepsilon\|u, U(P_0)\cap\Omega\|_{2m} + C(\varepsilon)\|u, U(P_0)\cap\Omega\|_0).
\end{aligned}
\tag{4. 10}
$$

Combining (4. 8), (4. 9) and (4. 10), we can find $U(P_0)$ such that

$$
\begin{aligned}
\|u, U(P_0)\cap\Omega\|_{2m} \leqq C(&\|L(P, D)u, U(P_0)\cap\Omega\|_0 \\
&+\sum_j \|B_{i_1 j}(P, D)u, \Gamma_{i_1}\|_{2m-m_j-\frac{1}{2}} \\
&+\sum_j \|B_{i_2 j}(P, D)u, \Gamma_{i_2}\|_{2m-m_j-\frac{1}{2}} \\
&+\|u, U(P_0)\cap\Omega\|_0)
\end{aligned}
$$

for all $u\in\tilde{C}_0^\infty(U(P_0)\cap\Omega)$. This inequality means that (4. 6) holds in a neighborhood of the endpoints of Γ_i. The proof is thus complete.

5. Let us consider a set of partial differential operators $\{L(P, D), B_{ij}(P, D)\}$ of type (4. 3), (4. 4) in a singular domain Ω. Throughout this section we assume that the set of boundary operators $\{B_{ij}(P, D)\}$ satisfies Assumption 4. 1. In this section we shall prove the alternative theorem

for elliptic boundary value problems $\Pi(L, f, B_{ij})$ in a singular domain. Our method is essentially along the lines of Schechter [8, 9, 10]. We denote by $\{S\}$ a set of all endpoints of boundary portion Γ_i.

LEMMA 5. 1. *There exists another boundary set* $\{B'_{ij}(P, D)\}$ *satisfying Assumption* 4. 1 *such that if* $u \in C^\infty(\bar{\Omega} - \{S\})$ *and if*

$$(u, L^*v) = (Lu, v)$$

for all $v \in \tilde{C}^\infty(\bar{\Omega})$ *satisfying* $B'_{ij}v = 0$ *on* Γ_i, *then* $B_{ij}u = 0$ *on* Γ_i.

The set $\{B'_{ij}\}$ is called adjoint to $\{B_{ij}\}$ relative to L. The proof of Lemma 5. 1 can be obtained in a quite similar manner to the proof developed by Aronszajn-Milgram [3] and Schechter [8] for regular domains. By a solution of the problem $\Pi(L, f, B_{ij})$ we shall mean a function u such that $u \in C^\infty(\bar{\Omega} - \{S\}) \cap L^2(\Omega)$ and such that

$$Lu = f \text{ in } \Omega, \quad B_{ij}u = 0 \text{ on } \Gamma_i, \quad j = 1, \cdots, m_{ij}.$$

THEOREM 5. 1. *Let* $\{L(P, D),\ B_{ij}(P, D)\}$ *be a set of operators of type* (4. 3), (4. 4) *in a singular domain* Ω. *Assume that the set of adjoint operators* $\{L^*(P_0, D),$ $B'_{ij}(P_0, D)\}$ *satisfies Assumption* 3. 1 *for each endpoint* P_0 *of boundary portions. Then the boundary value problem* $\Pi(L, f, B_{ij})$ *has a solution if the only solution of* $\Pi(L^*, 0, B'_{ij})$ *is* $u = 0$.

In the last section we shall give some example for Theorem 5. 1.

Proof. We proceed essentially the lines of Schechter [9, 10]. Let $\tilde{H}(\Omega)$ be the completion of $\tilde{C}^\infty(\bar{\Omega})$ with respect to the norm

$$|||\, u \,|||^2 = \|u, \Omega\|^2_{2m} + \sum_{i,j} \|B_{ij}u, \Gamma_i\|^2_{2m-m}.$$

It is easily verified that $\tilde{H}(\Omega)$ is a Hilbert space and is a subspace of $W^{2m}(\Omega)$. We also set

$$[u, v] = \iint_\Omega L^*u\overline{L^*v}\, dxdt + \sum_{i,j} (B'_{ij}u, B'_{ij}v)_{2m-m_{ij}, \Gamma_i}$$

for all $u, v \in \tilde{C}^\infty(\Omega)$[1]. Then we can see from Theorem 4. 2 that $[u, v]$ is defined for $u, v \in \tilde{H}(\Omega)$ and that there is a positive constant c such that

$$c^{-1}\|u\|^2_{2m} \leq [u, u] + \|u\|^2_0 \leq c\|u\|^2_{2m} \tag{5. 1}$$

for all $u \in \tilde{H}(\Omega)$. For simplicity we denote $\|u, \Omega\|_k$ by $\|u\|_k$.

[1] Boundary inner products are defined by a partition of unity and Fourier transformation (see e.g. [8]).

Now we can prove that there is a positive constant c such that

$$c^{-1}\|u\|_{2m}^2 \leqq [u, u] \leqq c\|u\|_{2m}^2 \tag{5. 2}$$

for all $u \in \tilde{H}(\Omega)$. Assume that the estimate (5. 2) does not hold. Then there is a sequence $\{u_n\}$ belonging to $\tilde{H}(\Omega)$ such that $n^{-1}\|u_n\|_{2m}^2 \geqq [u_n, u_n]$.

If we put $v_n = u_n/\|u_n\|_{2m}$, it follows that

$$\|v_n\|_{2m} = 1, \quad v_n \in \tilde{H}(\Omega) \tag{5. 3}$$

and

$$[v_n, v_n] \to 0 \qquad (n \to \infty). \tag{5. 4}$$

Applying Rellich's lemma to (5. 3), we have a subsequence (which is also denoted by $\{v_n\}$ for the brevity) such that

$$\|v_n - v\|_0 \to 0 \qquad (n \to \infty). \tag{5. 5}$$

Now it follows from (5. 1) that

$$\begin{aligned} c^{-1}\|v_n - v_{n'}\|_{2m}^2 &\leqq [v_n - v_{n'}, v_n - v_{n'}] + \|v_n - v_{n'}\|_0^2 \\ &\leqq [v_n, v_n] + [v_{n'}, v_{n'}] - [v_n, v_{n'}] \\ &\quad - [v_{n'}, v_n] + \|v_n - v_{n'}\|_0^2. \end{aligned} \tag{5. 6}$$

By Schwarz inequality

$$[v_{n'}, v_n] \leqq [v_{n'}, v_{n'}]^{\frac{1}{2}}[v_n, v_n]^{\frac{1}{2}}. \tag{5. 7}$$

Combining (5. 4)~(5. 7), we see

$$v_n \to v \quad \text{in} \quad W^{2m}(\Omega).$$

Hence $[v, v] = \lim [v_n, v_n] = 0$. This implies that $L^*v = 0$ in Ω and $B'_{ij}v = 0$ on Γ_i in the weak sense. Applying the regularity theorem, we see that $v \in C^\infty(\bar{\Omega} - \{S\}) \cap L^2(\Omega)$. From our assumptions this means that $v = 0$ in Ω. On the other hand $\|v\|_0 = \lim_{n\to\infty} \|v_n\| = 1$. It is a contradiction. Thus (5. 2) holds. That is, there is a constant $c > 0$ such that

$$|[u, v]| \leqq c\|u\|_{2m}\|v\|_{2m},$$
$$|[u, u]| \geqq c^{-1}\|u\|_{2m}^2$$

for all $u, v \in \tilde{H}(\Omega)$. For a given function $f \in C^\infty(\bar{\Omega})$, the L^2 inner product (f, v) is a bounded linear functional in $W^{2m}(\Omega)$. Hence there is a function $g \in \tilde{H}(\Omega)$ such that

$$[g, v] = (f, v) \tag{5. 8}$$

for all $v \in \tilde{H}(\Omega)$ (c.f. [6]). If $v \in C_0^\infty(\Omega)$, (5. 8) implies

$$(L^*g, L^*v) = (f, v).$$

Putting $L^*g = u$, we see

$$(u, L^*v) = (f, v), \quad v \in C_0^\infty(\Omega).$$

Hence, $Lu = f$ in Ω and $u \in C^\infty(\Omega)$. If we choose v such as $v \in \tilde{C}^\infty(\bar{\Omega})$ and $B'_{ij}v = 0$ on Γ_i, then we see $u \in C^\infty(\bar{\Omega} - \{S\})$ by the regularity theorem. Thus we obtain the proof by Lemma 5. 1

REMARK. *When each Γ_i is a closed smooth curve, N. Ikebe* [4] *has given the existence of solutions* $C^{2m+\alpha}(\bar{\Omega})$ $(\alpha > 0)$.

6. In this section we shall give some example for Theorem 5. 1. It is sufficient to give some example such that Assumption 3. 1 holds. Let $\mathscr{D}$ be the disk defined in the beginning of section 4. We consider the Laplace operator $L(D) = \Delta$. Then the operators defined in (4. 1) are of the form

$$L_1(D) = L_2(D) = \Delta$$

$$\begin{pmatrix} B_{11}^-(D) & B_{21}^-(D) \\ B_{12}^-(D) & B_{22}^-(D) \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ \dfrac{\partial}{\partial t} & \dfrac{\partial}{\partial t} \end{pmatrix}. \tag{6. 1}$$

Let us consider the boundary value problem (2. 4) in $t \geqq 0$. That is,

$$\begin{aligned} &\Delta u_1 = 0, \quad \Delta u_2 = 0, \quad t \geqq 0, \\ &B_{11}^- u_1 + B_{21}^- u_2 = \varphi_1, \\ &B_{12}^- u_1 + B_{22}^- u_2 = \varphi_2, \quad t = 0. \end{aligned}$$

Then we see by direct calculation that the kernels in (2. 10) are of the form

$$\begin{aligned} &\begin{pmatrix} R_{11}^-(x, t, \pm 1) & R_{21}^-(x, t, \pm 1) \\ R_{12}^-(x, t, \pm 1) & R_{22}^-(x, t, \pm 1) \end{pmatrix} \\ &\qquad = \begin{pmatrix} -2G^{(2)}(\pm x + it) & 2G^{(2)}(\pm x + it) \\ 2iG^{(2)}(\pm x + it) & 2iG^{(2)}(\pm x + it) \end{pmatrix}. \end{aligned}$$

Hence by (2. 9), the Poisson kernels for the problem (6. 1) are of the following form:

$$\begin{pmatrix} K_{11} & K_{21} \\ K_{12} & K_{22} \end{pmatrix} = \begin{pmatrix} c_1(z^{-1} - \bar{z}^{-1}) & c_1'(z^{-1} - \bar{z}^{-1}) \\ c_2(\log z^{-1} + \log(-\bar{z}^{-1})) & c_2(\log z + \log(-\bar{z}^{-1})) \end{pmatrix}$$

where $z = x + iy$ and c_i are constants.

(I) Consider the boundary operators on the incision of $\mathscr{D}$ such as

$$B_1(D) \equiv 1 \text{ on } \Gamma_1, \quad B_2(D) \equiv - D_t + aD_x \text{ on } \Gamma_2. \tag{6. 2}$$

Then from (4. 1)

$$\begin{pmatrix} B_{11}^{+}(D) & B_{21}^{+}(D) \\ B_{12}^{+}(D) & B_{22}^{+}(D) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \tau + a\xi \end{pmatrix}.$$

Thus by calculation of (3. 4), the integral equation (3. 6) is of the form

$$\begin{pmatrix} \psi_1 \\ \psi_1 \end{pmatrix} = \begin{pmatrix} -H^{+} + H^{-} & iH^{+} + iH^{-} \\ (i + a)H^{+} + (i - a)H^{-} & (ai - 1)H^{+} + (ai + 1)H^{-} \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}$$

that is, the matrices C, D in Assumption 3. 1 are

$$C = \begin{pmatrix} -1 & i \\ i + a & ai - 1 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & i \\ i - a & ai + 1 \end{pmatrix}.$$

Hence we see

$$C^{-1}D = \frac{i}{1 - ai} \begin{pmatrix} ai - 1 & -1 \\ -(i + a) & -1 \end{pmatrix}.$$

Thus we conclude that if a is real, the boundary operators (6. 2) satisfies Assumption 3. 1.

(II) Secondly we consider the boundary operators

$$B_1(D) \equiv D_t + aD_x \text{ on } \Gamma_1,$$
$$B_2(D) \equiv - D_t + aD_x \text{ on } \Gamma_2.$$

Then proceeding similarly as in I), we see

$$C^{-1}D = \begin{pmatrix} i + a & 1 - ai \\ -(i + a) & 1 - ai \end{pmatrix}^{-1} \begin{pmatrix} a - i & ai + 1 \\ i - a & ai + 1 \end{pmatrix} = \frac{i - a}{i + a} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

If $a \neq 0$ and a is not pure imaginary, we see that Assumption 3. 1 is satisfied.

When $a=0$, our assumption is not satisfied. But it is seen that the mixed a priori estimates (3. 1) hold from the relations

$$I=\mathscr{H}^{+}-\mathscr{H}^{-},\quad 2\mathscr{H}=\mathscr{H}^{+}+\mathscr{H}^{-},$$

where $\mathscr{H}$ denotes Hilbert transform on the whole real line.

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