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Nagoya Math. J.
Vol. 61 (1976), 117-125

HASSE PRINCIPLES AND THE u-INVARIANT OVER FORMALLY REAL FIELDS

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0. Introduction

In this paper we investigate the connection between the u-invariant, $u(F)$, of a formally real field F as defined by Elman and Lam [2] and certain Hasse Principles studied by Elman, Lam and Prestel in [3].

In section 2 the notion of an effective diagonalization of a quadratic form is introduced and in section 3 it is shown that if F is a field having at most a finite number of orderings such that every form over F has an effective diagonalization (which happens, for example, if F is any field having at most one ordering) then the finiteness of the u-invariant is equivalent to the Hasse Principle H_n holding for all n larger than some fixed integer m.

In section 4 we present two generalizations of a theorem of Kneser which states that if F is a non-formally real field then $u(F) \leq q$, where q denotes the number of distinct square classes of F. If F is a formally real field such that every form over F can be effectively diagonalized then it is shown that $u(F) \leq t$ where t is the number of distinct square classes of *totally positive* elements of F and H_n is satisfied for all $n > \frac{1}{2}q$.

I would like to express my thanks to T. Y. Lam who pointed out the improved bounds in the proof of Theorem 3.1 and the statement of Theorem 4.4.

1. Notations and terminology

The terminology and notations will primarily follow [2,3,6]. All fields F will have characteristic different from two, $\dot{F}$ denotes the multiplicative group of $F, \dot{F}^2$ the subgroup of non-zero squares, and $\Sigma\dot{F}^2$ the

Received September 10, 1975.

subgroup consisting of all sums of squares (= totally positive elements). Isometries of quadratic forms over F will be written as $\cong$, $\phi \perp \psi$ and $\phi \otimes \psi$ will denote, respectively, the orthogonal sum and tensor product of two forms ϕ and ψ, and for any natural number m the form $\phi \perp \phi \perp \cdots \perp \phi$ (m times) will be denoted by $m\phi$. We will write $\phi = \langle a_1, a_2, \cdots, a_n \rangle$ to mean ϕ has an orthogonal basis $e_1, e_2, \cdots, e_n$ with $\phi(e_i) = a_i \in \dot{F}$. The *Witt ring* of non-singular quadratic forms over F will be denoted by $W(F)$ and its torsion subgroup by $W_t(F)$. The *u-invariant* of F is defined to be $u(F) = \max \{\dim \phi\}$ where ϕ ranges over all anisotropic forms in $W_t(F)$ [2].

If F is a formally real field then any ordering $<$ on F induces a ring homomorphism $\sigma_<: W(F) \to Z$ via $\sigma_<(\phi) = \sum_i \sigma_<(a_i)$, where $\phi = \langle a_1, \cdots, a_n \rangle$ and $\sigma_<(a_i) = 1$ if $0 < a_i$, $\sigma_<(a_i) = -1$ if $a_i < 0$. If ϕ is a form over F, $\sigma_<(\phi)$ is called *the signature of ϕ relative to the ordering* $<$. From [7, Satz 22] it follows that $W_t(F)$ consists precisely of those forms which have signature zero relative to all orderings on F. A form ϕ is called *totally indefinite* (or locally isotropic) over F if $|\sigma_<(\phi)| < \dim \phi$ for all orderings $<$ on F. Thus a form ϕ is totally indefinite if and only if ϕ is isotropic over all real closures $F_<$ of F as $<$ runs through the orderings of F. The formally real field F satisfies the *Hasse Principle* H_n (for some $n \geq 2$) if every totally indefinite form of dimension n over F is isotropic [3].

We denote by $X = X(F)$ the topological space of orderings on F [1,5]. The space X is compact, Hausdorff, and totally disconnected with a subbase of the topology given by the sets $W(a) = \{< \text{ in } X \mid a < 0\}$, $a \in \dot{F}$. We say F (or X) satisfies the *Strong Approximation Property* (SAP) if given any two disjoint closed subsets U, V of X there exists an element a in F which is positive at the orderings in U and negative at the orderings in V.

2. Effective diagonalization of quadratic forms

A form $\phi = \langle a_1, a_2, \cdots, a_n \rangle$ over a formally real field F is said to be *effectively diagonalized* if $W(a_i) \subset W(a_{i+1})$, $i = 1, 2, \cdots, n-1$. The field F is said to satisfy *ED* if every form over F can be effectively diagonalized.

LEMMA 2.1. *Suppose F is a formally real field and ϕ is a form which* *can be effectively diagonalized. Then*

(i) *If ϕ is totally indefinite then we can write $\phi = \beta \perp \phi'$ where $\beta = \langle a, b\rangle$ is a binary form with a totally positive and b totally negative,*

(ii) *If ϕ is totally indefinite then there exists an integer $m \geq 1$ such that $m\phi$ is isotropic (i.e. ϕ is weakly isotropic in the sense of* [3,8]).

(iii) *If $\phi \in W_t(F)$ then $\phi = \beta_1 \perp \cdots \perp \beta_n$ where $\beta_i = \langle a_i, b_i\rangle \in W_t(F)$ with a_i totally positive and b_i totally negative. In particular, ϕ is strongly balanced in the sense of* [7].

(iv) *If $\phi \in W_t(F)$ with $\dim \phi = 2n$ then $\phi = \phi_1 \perp \phi_2$ with $\dim \phi_i = n$, $i = 1, 2$, and where ϕ_1 has signature n and ϕ_2 has signature $-n$ relative to all orderings on F.*

Proof. (i) Write $\phi = \langle a_1, a_2, \cdots, a_k\rangle$ with $W(a_i) \subset W(a_{i+1})$ for all i. Since ϕ is totally indefinite $W(a_1)$ must be empty and $W(a_k) = X$. Thus a_1 is totally positive and a_k is totally negative so we can take $\beta = \langle a_1, a_k\rangle$.

(ii) Write $\phi = \beta \perp \phi'$ with $\beta = \langle a, b\rangle \in W_t(F)$. Choose $m \geq 1$ so that $m\beta = 0$ in $W(F)$. Then $m\phi$ is isotropic.

(iii) Write $\phi = \langle a_1, a_2, \cdots, a_k\rangle$ with $W(a_i) \subset W(a_{i+1})$ for all i. Since F is formally real and $\phi \in W_t(F)$ it follows that $k = 2n$ is even, $a_1, \cdots, a_n$ are totally positive and $a_{n+1}, \cdots, a_k$ are totally negative. Hence we can take $b_i = a_{n+i}$ for $i = 1, 2, \cdots, n$.

(iv) follows immediately from (iii).

COROLLARY 2.2. *If F is a formally real field satisfying ED then F satisfies SAP.*

Proof. This is a consequence of Lemma 2.1 (ii), [3, Th. C], and [8, Satz 3.1] (see also [9, Th. 3.1]).

EXAMPLES. (i) If F has a unique ordering then F satisfies ED.

(ii) Let $F = \boldsymbol{Q}((t))$ be the field of formal power series over $\boldsymbol{Q}$. As observed by Elman, Lam, and Prestel [3], the form $\langle t, -2t\rangle \in W_t(F)$ does not represent a totally negative element and consequently cannot be effectively diagonalized. Thus F does not satisfy ED. Since F has only two orderings, F does satisfy SAP. Thus SAP does not imply ED.

However, we do have the following

PROPOSITION 2.3. *A formally real field F satisfies SAP if and only* *if for any form ϕ over F there exists an effectively diagonalized form $\psi = \langle b_1, b_2, \cdots, b_n\rangle$, $n = \dim \phi$, such that $\phi - \psi \in W_t(F)$.*

Proof. ($\Rightarrow$) As in [9, Th. 3.1] we let $Y_k = \{< \text{ in } X \mid \sigma_<(\phi) = -n + 2k\}$, $k = 0, 1, \cdots, n$. Then the family $\{Y_k \mid k = 0, 1, \cdots, n\}$ is a partition of X and each Y_k is an open and closed subset of X. Since F satisfies SAP, there exist elements $b_1, b_2, \cdots, b_{n+1}$ in $\dot{F}$ such that $W(b_i) = Y_0 \cup Y_1 \cup \cdots \cup Y_{i-1}$, $i = 1, 2, \cdots, n + 1$. Then $W(b_i) \subset W(b_{i+1})$ for all i and one readily checks that $\sigma_<(\langle b_1, b_2, \cdots, b_n\rangle) = \sigma_<(\phi)$ for all orderings $<$ in X. Hence $\phi - \langle b_1, b_2, \cdots, b_n\rangle$ lies in $W_t(F)$.
($\Leftarrow$) By [3, Th. C] and [8, Satz 3.1] it is enough to show that if ϕ is totally indefinite then there exists $m \geq 1$ such that $m\phi$ is isotropic. Let $\psi = \langle b_1, b_2, \cdots, b_n\rangle$, $n = \dim \phi$, be an effectively diagonalized form with $\phi - \psi \in W_t(F)$. Then there exists an integer $r \geq 1$ such that $r\phi \cong r\psi$. Since ϕ is totally indefinite, this implies ψ is also totally indefinite so by Lemma 2.1 (ii) there exists an integer $s \geq 1$ such that $s\psi$ is isotropic. Hence if $m = rs$ then $m\phi$ is isotropic.

THEOREM 2.4. *For a formally real field F the following statements are equivalent:*

(i) *F satisfies ED.*

(ii) *If ϕ is a form over F which represents 1 over all real closures of F then ϕ represents a totally positive element of F.*

Proof. (i) $\Rightarrow$ (ii). Write $\phi = \langle a_1, a_2, \cdots, a_n\rangle$ with $W(a_i) \subset W(a_{i+1})$. Since ϕ represents 1 over all real closures it follows that $W(a_1) = \phi$, i.e. a_1 is totally positive.
(ii) $\Rightarrow$ (i). We first show that any totally indefinite form over F is weakly isotropic and hence, in view of [3,8], F satisfies SAP. If ϕ is totally indefinite then ϕ represents 1 over all real closures and hence we can write $\phi = \langle a\rangle \perp \phi_1$ where a is totally positive element of F. But then ϕ_1 represents -1 over all real closures so ϕ_1 represent a totally negative element b in $\dot{F}$. Since $\langle a, b\rangle \in W_t(F)$ it follows that $\phi = \langle a, b\rangle \perp \psi$ is weakly isotropic.

Now let ψ be any form over F. Since F satisfies SAP there exists b in $\dot{F}$ such that $W(b) = \{< \in X \mid \sigma_<(\psi) = -\dim \psi\}$. If $W(b)$ is empty then ψ represents 1 over all real closures and hence represents a totally positive element. In this case the proof is finished by induction on $\dim \psi$. Hence we can assume that $W(b)$ is non empty. Now $W(b) \subset W(c)$ for all elements $c \neq 0$ represented by ψ and $\psi \perp \langle -b \rangle$ represents 1 over all real closures. Thus $\psi \perp \langle -b \rangle$ represents a totally positive element d. Since $-b$ is not totally positive we can write $d = a - bx^2$ where $a \neq 0$ is represented by ψ. Then $W(a) \subset W(b)$ so that $W(a) \subset W(c)$ for all c in $\dot{F}$ represented by ψ. Thus induction on $\dim \psi$ completes the proof.

COROLLARY 2.5. *If F is a formally real field satisfying some Hasse Principle H_n with $n \geq 4$ then F satisfies ED.*

Proof. Let ϕ be a form over F which represents 1 over all real closure of F. Then $\phi \perp n\langle -1 \rangle$ is totally indefinite whence isotropic. Thus there exists $x_1, \cdots, x_n$ in F such that ϕ represents the totally positive element $x_1^2 + \cdots + x_n^2 \in \dot{F}$.

COROLLARY 2.6 (cf. [1, Th. 5.3]). *For a formally real pythagorean field F the following statements are equivalent:*

(i) *F satisfies SAP.*

(ii) *F satisfies ED.*

(iii) *F satisfies H_n for all $n \geq 2$.*

Proof. The equivalence of (i) and (ii) is a consequence of Proposition 2.3 and the equivalence of (ii) and (iii) follows from Lemma 2.1 (i) and Corollary 2.5.

3. Hasse principles and the u-invariant

Any non-formally real field vacuously satisfies ED since $X = X(F)$ is empty but need not satisfy H_n for any n. In fact, for F non-formally real, F satisfies H_n for some $n \geq 2$ if and only if $u = u(F)$ is finite. For formally real fields we have

THEOREM 3.1. *Let F be a formally real field having at most a finite number of orderings. Then the following statements are equivalent:*

(i) *F satisfies H_n for some $n \geq 4$.*

(ii) *F satisfies ED and $u(F) < \infty$.*

Before proving Theorem 3.1 we introduce some terminology. A quadratic form ϕ over F will be called *totally positive* if every non zero element of F represented by ϕ is totally positive. Thus ϕ is totally positive if and only if $\phi = \langle a_1, \cdots, a_n \rangle$ with $a_i \in \Sigma \dot{F}^2, i = 1, \cdots, n$, if and only if $\sigma_<(\phi) = \dim \phi$ for all orderings $<$ of F. Denote by h the exponent of $W_t(F)$. h is called the *height of* F and (when finite) $h = 2^m$ where $m \geq 0$ is the smallest integer such that every totally positive element of F is a sum of 2^m squares in F [6, p. 311]. It follows immediately that if $u(F)$ is finite then h is finite and $h \leq u(F)$.

The proof of Theorem 3.1 will use the following lemma:

LEMMA 3.2. *Suppose F is a field with $u = u(F) < \infty$. If ϕ is a totally positive form over F with* $\dim \phi > 4^m(u+1)$ *for some* $m \geq 0$ *then there exists a in $\Sigma \dot{F}^2$ such that* $\phi = 2^{m+1}\langle a\rangle \perp \psi$.

Proof. We proceed by induction on m. If $m = 0$ then $\dim \phi > u + 1$ so there exists an integer n with $u + 1 \leq 2n \leq \dim \phi$. Write $\phi = \langle a_1, \cdots, a_n, b_1, \cdots, b_n\rangle \perp \phi'$. Then $\langle a_1, \cdots, a_n, -b_1, \cdots, -b_n\rangle \in W_t(F)$ and has dimension larger than u. Hence $\langle a_1, \cdots, a_n\rangle$ and $\langle b_1, \cdots, b_n\rangle$ represent a common element $a \in \Sigma \dot{F}^2$. Thus $\phi = 2\langle a\rangle \perp \psi$.

Now assume $m > 0$ and choose ϕ_1 of biggest dimension such that $\phi = 2\phi_1 \perp \phi_2$. Then the foregoing argument shows that $\dim \phi_2 \leq u + 1$. Hence $\dim \phi_1 > \frac{1}{2}(4^m - 1)(u + 1)$. But $m > 0$ implies that $\frac{1}{2}(4^m - 1) > 4^{m-1}$ so $\dim \phi_1 > 4^{m-1}(u + 1)$. Hence by the induction hypothesis there exists a in $\Sigma \dot{F}^2$ such that $\phi_1 = 2^m\langle a\rangle \perp \psi_1$. But then $\phi = 2^{m+1}\langle a\rangle \perp \psi$ where $\psi = 2\psi_1 \perp \phi_2$.

Proof of Theorem 3.1. (i) $\Rightarrow$ (ii). This follows from Corollary 2.5 and the fact that if H_n holds for some $n \geq 2$ then $u(F) < n$.
(ii) $\Rightarrow$ (i). Let $s < \infty$ be the number of orderings on F. Since $u = u(F)$ is finite the height h of F is also finite (with $h \leq u$) so we can write $h = 2^m$ for some integer $m \geq 0$. We now assert that if $n > (s + 1)\left(\frac{h}{2}\right)^2 \cdot (u + 1)$ then H_n holds. To see this let ϕ be a totally indefinite form over F with $\dim \phi > (s + 1)\left(\frac{h}{2}\right)^2(u + 1)$. Since F satisfies ED we can find elements a_{ij} in F, $1 \leq i \leq k$, $1 \leq j \leq n_i$, such that for each i, $W(a_{i1}) = \cdots = W(a_{in_i})$, $W(a_{i1}) \subsetneqq W(a_{i+1,1})$, and $\phi = \phi_1 \perp \phi_2 \perp \cdots \perp \phi_k$ where $\phi_i = \langle a_{i1}, a_{i2}, \cdots, a_{in_i}\rangle$. Then by choosing orderings in $W(a_{i+1,1}) - W(a_{i1})$, $i = 1, 2, \cdots, k - 1$ we see that $s \geq k - 1$. Hence $\dim\phi = n_1 + n_2 + \cdots + n_k > (s + 1)\left(\frac{h}{2}\right)^2(u + 1) \geq k\left(\frac{h}{2}\right)^2(u + 1)$. Thus there must exist some i

with $n_i > \left(\frac{h}{2}\right)^2(u+1) = 4^{m-1}(u+1)$. Now $W(a_{i1}) = \cdots = W(a_{in_i})$ so the form $\langle a_{i1}\rangle\phi_i = \langle a_{i1}\rangle \otimes \phi_i$ is totally positive and hence by Lemma 3.2, $\langle a_{i1}\rangle\phi_i = 2^m\langle a\rangle \perp \psi$ for some a in $\Sigma\dot{F}^2$. Hence $\langle a_{i1}\rangle\phi = 2^m\langle a\rangle \perp \phi'$ for some subform ϕ'. Let $\phi' = \langle b_1, b_2, \cdots, b_r\rangle$ be an effective diagonalization of ϕ'. Then $\langle a_{i1}\rangle\phi = 2^m\langle a\rangle \perp \langle b_1, b_2, \cdots, b_r\rangle$ is an effective diagonalization. Since ϕ is totally indefinite so is $\langle a_{i1}\rangle\phi$ so b_r must be totally negative. But $h = 2^m$ implies that $2^m\langle a\rangle$ represents all totally positive elements of F. Thus $\langle a_{i1}\rangle\phi$ is isotropic whence ϕ is also isotropic.

Remark. For many fields the bound $\left(n > (s+1)\left(\frac{h}{2}\right)^2(u+1)\right)$ obtained in the proof of Theorem 3.1 is not very precise. In the case that $F = Q$, the proof shows that H_n holds for all $n > 40$ while it is well known that $n \geq 5$ suffices. Moreover, there exist fields having an infinite number of orderings (for example, the pythagorean closure of Q) which satisfy the equivalent conditions of the theorem.

COROLLARY 3.3. *Let F be a field having a unique ordering. Then $u(F) < \infty$ if and only if F satisfies H_n for some $n \geq 2$. In this case, F satisfies H_n for all $n > \frac{1}{2}h^2(u+1)$.*

Proof. A field having a unique ordering satisfies ED.

EXAMPLE. If $F = Q((t))$ then F has exactly two orderings and $u(F) = 8$ but as observed in [3], F fails to satisfy H_n for any $n \geq 2$.

4. Kneser's Theorem

In this section we present two more generalizations (cf. [2, Th. 2.4, Cor. 2.5, and Th. 3.1]) of Kneser's Theorem which states that if F is a non-formally real field and $q = |\dot{F}/\dot{F}^2|$ then $u(F) \leq q$. For this purpose we introduce the following notation. For a form ϕ over F, let $D(\phi) = \{a \in \dot{F}/\dot{F}^2 \mid a \text{ is represented by } \phi\}$.

LEMMA 4.1. *Let F be a field and ϕ a totally positive form over F. If $D(\phi) \neq \Sigma\dot{F}^2/\dot{F}^2$ then for any a in $\Sigma\dot{F}^2, D(\phi \perp \langle a\rangle) \neq D(\phi)$.*

Proof. If $D(\phi \perp \langle a\rangle) = D(\phi)$ then for any integer $n \geq 1, D(\phi \perp n\langle a\rangle) = D(\phi)$. Now if $b \in \Sigma\dot{F}^2$ then ab is a sum of k squares in F for some $k \geq 1$ which implies that b is represented by the form $k\langle a\rangle$. Hence

$b \in D(\phi \perp k\langle a\rangle) = D(\phi)$, contrary to assumption.

THEOREM 4.2. *If F is a formally real field satisfying ED then $u(F) \leq |\Sigma\dot{F}^2/\dot{F}^2|$.*

Proof. Let $t = |\Sigma\dot{F}^2/\dot{F}^2|$. It is enough to show that if $\phi \in W_t(F)$ with $\dim \phi \geq t + 2$ then ϕ is isotropic. Since F is formally real and satisfies ED we can write $\phi = \langle a_1, \cdots, a_m, b_1, \cdots, b_m\rangle$ where $a_i \in \Sigma\dot{F}^2$, $b_i \in -\Sigma\dot{F}^2$, $i = 1, \cdots, m$, and $m \geq \frac{t+2}{2}$. Then by Lemma 4.1, $|D(\langle a_1, \cdots, a_m\rangle)| > \frac{t}{2}$ and $|D(\langle -b_1, \cdots, -b_m\rangle)| > \frac{t}{2}$. Thus there exists $a \in D(\langle a_1, \cdots, a_m\rangle) \cap D(\langle -b_1, \cdots, -b_m\rangle)$. But then $-a \in D(\langle b_1, \cdots, b_m\rangle)$, whence ϕ is isotropic.

EXAMPLE. The hypothesis that F satisfies ED is needed here since if we let F_0 be a formally real field having square classes $\{\pm 1, \pm 2\}$ (such fields exist by [4, p. 302]) and let $F = F_0((t))$ then $u(F) = 4$ but $t = |\Sigma\dot{F}^2/\dot{F}^2| = 2$.

COROLLARY 4.3. *Let F be a formally real field satisfying ED. If $q = |\dot{F}/\dot{F}^2| < \infty$ then $u(F) \leq 2^{-s}q$ where s is the number of distinct orderings of F.*

Proof. Since F satisfies ED, F also satisfies SAP so it follows from (the proof of) Example 4.10 (iii) in [5] that $|\dot{F}/\Sigma\dot{F}^2| = 2^s$. Hence $q = |\dot{F}/\dot{F}^2| = |\dot{F}/\Sigma\dot{F}^2|\,|\Sigma\dot{F}^2/\dot{F}^2| = 2^s\,|\Sigma\dot{F}^2/\dot{F}^2|$.

THEOREM 4.4. *Let F be a formally real field which satisfies ED and suppose $q < \infty$. Write $q = 2^s t$ where $t = |\Sigma\dot{F}^2/\dot{F}^2|$ and s is the number of orderings on F. Then F satisfies H_n for all $n > s(t-1) + 1$. In particular, H_n holds for all $n \geq \frac{q}{2} + 1$.*

Proof. Let ϕ be a totally indefinite form over F and write $\phi = \langle a_{11}, \cdots, a_{1n_1}, a_{21}, \cdots, a_{2n_2}, \cdots, a_{k1}, \cdots, a_{kn_k}\rangle$ where, for $i = 1, 2, \cdots, k$, $W(a_{i1}) = \cdots = W(a_{in_i})$ and $W(a_{i1}) \subsetneq W(a_{i+1,1})$. Then $n_1 + n_2 + \cdots + n_k = \dim \phi$ and $k \leq s + 1$. If ϕ is anisotropic then by Lemma 4.1, $n_1 + n_k \leq t$ since otherwise $D(\langle a_{11}, \cdots, a_{1n_1}\rangle)$ and $D(\langle -a_{k1}, \cdots, -a_{kn_k}\rangle)$ would have an element in common. Moreover, by replacing ϕ by $\langle a_{i1}\rangle\phi$ and using effective diagonalization (as in the proof of Theorem 3.1) we see that

$n_i \leq t-1$ for $i = 2, \cdots, k-1$. Hence $\dim \phi = n_1 + n_2 + \cdots + n_k \leq t + (k-2)(t-1) \leq t + (s-1)(t-1) = s(t-1)+1$. Thus if $\dim \phi > s(t-1)+1$ then ϕ is isotropic. For the last statement, note that $\frac{q}{2} + 1 = 2^{s-1}t + 1 > s(t-1)+1$.

COROLLARY 4.5. *Let F be a field having a unique ordering. If $q < \infty$ then H_n holds for all $n > \frac{q}{2}$.*

COROLLARY 4.6. *Let F be a formally real field satisfying ED. If F has more than one ordering then H_n holds for all $n \geq \frac{q}{2}$.*

Proof. If $s \geq 2$ then $\frac{q}{2} = 2^{s-1}t > s(t-1)+1$.

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