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THE ASYMPTOTIC DISTRIBUTION OF EIGENVALUES OF THE LAPLACE OPERATOR IN AN UNBOUNDED DOMAIN

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§ 1. Introduction.

This paper is devoted to the study of the asymptotic distribution of eigenvalues of the Laplace operator with zero boundary conditions in a quasi-bounded domain contained in Euclidean space $\boldsymbol{R}^2$. Let us consider the following eigenvalue problem:

$$-\left(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}\right)u=\lambda u\,, \qquad u\in H_0^1(\Omega)\,, \tag{1.1}$$

where

$$\Omega=\{(x_1,x_2)\,|\,-\infty<x_1<\infty, 0<x_2<q(x_1)\}\,, \tag{1.2}$$

and $q(x)$ is a smooth positive function defined on $(-\infty,\infty)$ satisfying $\lim_{|x|\to\infty} q(x)=0$.

It has been shown in [1] that the problem (1.1) has an infinite sequence of discrete eigenvalues approaching to ∞. We denote by $N(h)$ the number of eigenvalues less than h of the problem (1.1). We are concerned with the asymptotic behavior of $N(h)$ as $h\to\infty$.

The asymptotic distribution of eigenvalues of the Laplace operator with zero boundary conditions in a quasi-bounded domain has been studied by Clark [1], Hewgill [4], and Glazman and Skacek [2]. It seems to the author that any true asymptotic formula for $N(h)$ even in the case of such a simple domain as (1.2) has not been known.

We shall study the problem (1.1) under the formulation as an eigenvalue problem of a differential operator with operator-valued coefficients. On the other hand, Kostjuchenko and Levitan [5] studied the eigenvalue problem for the operator $-(d^2/dt^2)+Q(t)$ under the assumption that $Q(t)$ is

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a semi-bounded self-adjoint operator for each fixed $t \in \boldsymbol{R}^1$ with the common domain of definition $\mathscr{D}(Q(t)) = \mathscr{D}_0$ and other restrictions. Our method is different from that in [5] in some ways.

DEFINITION 1.1. We denote by $K(m)$ $(0 < m \leq 1)$ the set of smooth functions $q(x)$ defined on $\boldsymbol{R}^1$ satisfying the following conditions: There exist positive constants C_1, C_2 and C independent of x and y such that

(i) $C_1(1 + |x|)^{-m} \leq q(x) \leq C_2(1 + |x|)^{-m}$,

(ii) for $|x - y| \leq 1$, $|q(x) - q(y)| \leq Cq(x)\,|x - y|$,

(iii) $\left|\left(\dfrac{d}{dx}\right)^j q(x)\right| = |q^{(j)}(x)| \leq Cq(x)$ $(j = 1, 2)$.

Now we shall state our main theorem which will be proved in § 4.

THEOREM 1.1. *Let* $\{\zeta_j > 0\}_{j=1}^{\infty}$ *be eigenvalues of the operator* A_0 $\left(= -\dfrac{\partial^2}{\partial y^2}\right)$ *with the domain of definition* $\mathscr{D}(A_0) = H_0^1(0, 1) \cap H^2(0, 1)$. *Suppose that* $q(x)$ *belongs to* $K(m)$ $(0 < m \leq 1)$. *Then, as* $h \to \infty$

$$N(h) \sim \frac{1}{\pi} \sum_{j=1}^{\infty} \int_{\Omega_j(h)} (h - \zeta_j q(x)^{-2})^{1/2} dx \ ,$$

where $\Omega_j(h) = \{x \in \boldsymbol{R}^1 \,|\, hq(x)^2 > \zeta_j\}$, *while* $f(h)) \sim g(h)$ *means that* $\lim_{h \to \infty} f(h)^{-1} g(h) = 1$.

Remark. $\{\zeta_j q(x)^{-2}\}_{j=1}^{\infty}$ are regarded as eigenvalues of the operator $-\dfrac{\partial^2}{\partial y^2}$ with the domain of definition $H_0^1(0, q(x)) \cap H^2(0, q(x))$.

EXAMPLE. If $\lim_{|x| \to \infty} |x|^m q(x) = a \ (> 0)$, then

$$\begin{aligned} N(h) &\sim C(a) h^{-1/2 + 1/2m} \qquad (0 < m < 1) \ , \\ &\sim C(a) h \log h \qquad (m = 1) \ . \end{aligned}$$

Throughout this paper, we confine ourselves to such a simple problem as (1.1), but some generalizations will be discussed without proofs in § 7. Finally we note that in this paper we use one and the same symbol C in order to denote positive constants which may differ from each other. When we specify the dependence of such a constant on a parameter, say m, we denote it by $C(m)$ or C_m.

§ 2. Preliminaries.

Let us introduce some function spaces.

$H^j(0,1)$ and $H_0^j(0,1)$ $(j=1,2,\cdots)$ denote the usual Sobolev spaces on the interval $(0,1)$. For any real s, the Sobolev space $H^s(0,1)$ can be defined by interpolation methods ([6]). We denote by X the Hilbert space $L^2(0,1)$ with the usual scalar product $(u,v)_0=\int_0^1 u(x)\overline{v(x)}dx$, and the norm $\|u\|_0=\int_0^1 |u|^2\,dx$. $L^2(-\infty,\infty\,;X)$ denotes the Hilbert space of X-valued square integrable functions with the scalar product $\langle f,g\rangle=\int_{-\infty}^{\infty}(f(x),g(x))_0dx$.

Let $q(x)$ be a function belonging to $K(m)$ and let Ω be an open domain defined by $\Omega=\{(x_1,x_2)\,|\,-\infty<x_1<\infty, 0<x_2<q(x_1)\}$ and G be a cylinder domain defined by $G=\{(x,y)\,|\,-\infty<x<\infty, 0<y<1\}$. Then we define the unitary operator U from $L^2(\Omega)$ onto $L^2(G)$ by

$$(U\cdot u)(x,y)=q(x)^{1/2}u(x,q(x)y)\,,\qquad u\in L^2(\Omega)\,. \tag{2.1}$$

Similary we define the unitary operator $V(=U^*)$ from $L^2(G)$ onto $L^2(\Omega)$ by

$$(V\cdot v)(x_1,x_2)=q(x_1)^{-1/2}v(x_1,q(x_1)^{-1}x_2)\,,\qquad v\in L^2(G)\,. \tag{2.2}$$

Now consider the following eigenvalue problem in $L^2(\Omega)$:

$$Hu=-\Big(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}\Big)u=\lambda u\,,\qquad u\in H_0^1(\Omega)\,. \tag{1.1}$$

Here the operator H is a positive self-adjoint operator associated with the symmetric bilinear form

$$a(u,v)=\int_\Omega\Big(\frac{\partial}{\partial x_1}u\frac{\partial}{\partial x_1}\bar v+\frac{\partial}{\partial x_2}u\frac{\partial}{\partial x_2}\bar v\Big)dx_1dx_2\,,\qquad u,v\in H_0^1(\Omega)\,. \tag{2.3}$$

By using the operators U and V, we shall transform the above problem (1.1) into the problem in $L^2(G)$.

Let U and V be the operators defined by (2.1) and (2.2) respectively. Then, we have

$$U\frac{\partial}{\partial x_1}V=\frac{d}{dx}-q(x)^{-1}q^{(1)}(x)y\frac{\partial}{\partial y}-\frac{1}{2}q(x)^{-1}q^{(1)}(x)=\frac{d}{dx}+P(x)\,, \tag{2.4}$$

$$U\frac{\partial}{\partial x_2}V = q(x)^{-1}\frac{\partial}{\partial y} = Q(x)\ , \tag{2.5}$$

where for each fixed $t \in \boldsymbol{R}^1, P(t)$ and $Q(t)$ are regarded as operators acting on X with the domain of definition $\mathscr{D}(P(t)) = \mathscr{D}(Q(t)) = H_0^1(0, 1)$. Here we remark that the coefficients of the operator $P(t)$ are uniformly bounded. By using (2.4) and (2.5), the symmetric bilinear form $a(u, v)$ is transformed into a bilinear form

$$\begin{aligned} b(u, v) &= \int_{-\infty}^{\infty}\left(\left(\frac{d}{dx} + P(x)\right)u, \left(\frac{d}{dx} + P(x)\right)v\right)_0 dx \\ &\quad + \int_{-\infty}^{\infty}(Q(x)u, Q(x)v)_0 dx\ , \end{aligned} \tag{2.6}$$

where $b(u, v)$ is defined on the set $\mathscr{D}(b) = \left\{u \in L^2(-\infty, \infty\,; X) \mid \frac{d}{dx}u \in L^2 \cdot(-\infty, \infty\,; X), Q(x)u \in L^2(-\infty, \infty\,; X)\right\}$. The symmetric bilinear form $b(u, v)$ induces a unique positive self-adjoint operator T in the sense of Friedrichs. T has the following expression:

$$\begin{aligned} T &= -\left(\frac{d}{dx}\right)^2 - \frac{d}{dx}P(x) + P^*(x)\frac{d}{dx} + P^*(x)P(x) + Q^*(x)Q(x)\ , \\ &= -\left(\frac{d}{dx}\right)^2 - \frac{d}{dx}P(x) + P^*(x)\frac{d}{dx} + A(x)\ , \end{aligned} \tag{2.7}$$

where for each fixed $t \in \boldsymbol{R}^1, P^*(t)$ and $Q^*(t)$ denote the adjoint operators in X and $A(t)$ (t; fixed) is a self-adjoint operator with the domain of definition $\mathscr{D}(A(t)) = H_0^1(0, 1) \cap H^2(0, 1)$. Thus we have transformed the eigenvalue problem (1.1) in $L^2(\Omega)$ into the following equivalent problem (2.8) in $L^2(-\infty, \infty\,; X)$:

$$Tu = -\left(\frac{d}{dx}\right)^2 u - \frac{d}{dx}P(x)u + P^*(x)\frac{d}{dx}u + A(x)u = \lambda u\ . \tag{2.8}$$

We denote by $\{\mu_j\}_{j=1}^{\infty}$ and $\{u_j\}_{j=1}^{\infty}$ eigenvalues of the problem (2.8) and the normalized eigenfunctions corresponding to $\{\mu_j\}_{j=1}^{\infty}$ respectively.

Let A_0 be the positive self-adjoint operator $-\frac{\partial^2}{\partial y^2}$ with the domain of definition $\mathscr{D}(A_0) = H_0^1(0, 1) \cap H^2(0, 1)$. Then, we have with a constant C independent of $t \in \boldsymbol{R}^1$,

$$\|A(t)u\|_0 \leq Cq(t)^{-2}\|A_0u\|_0 , \qquad \text{for any } u \in \mathscr{D}(A_0) ,$$

and

$$\|A(t)^{1/2}v\|_0 \geq Cq(t)^{-1}\|A_0^{1/2}v\|_0 , \qquad \text{for any } v \in \mathscr{D}(A_0^{1/2}) .$$

Hence, with the aid of the Heinz interpolation inequality, we have

LEMMA 2.1.

$$(2.9) \qquad \|A(t)^{\alpha}u\|_0 \leq Cq(t)^{-2\alpha}\|A_0^{\alpha}u\|_0 , \qquad \textit{for } u \in \mathscr{D}(A_0^{\alpha}),\ (0 \leq \alpha \leq 1) ,$$

$$(2.10) \qquad \|A(t)^{\beta}u\|_0 \geq Cq(t)^{-\beta}\|A_0^{\beta}u\|_0 , \qquad \textit{for } u \in \mathscr{D}(A_0^{\beta}),\ (0 \leq \beta \leq 1/2) ,$$

where C is a constant independent of $t \in \boldsymbol{R}^1$ and u.

The following lemma is obvious from the definitions of the operators $P(t)$ and $A(t)$.

LEMMA 2.2. *For any t and s such that $|t-s| \leq 1$, we have*

$$(2.11) \qquad \|(P(t)-P(s))u\|_0 \leq C|t-s|\|A(t)^{1/2}u\|_0 ,$$

$$(2.11') \qquad \|(P^*(t)-P^*(s))u\|_0 \leq C|t-s|\|A(t)^{1/2}u\|_0 ,$$

$$(2.12) \qquad \|(A(t)-A(s))v\|_0 \leq C|t-s|\|A(t)v\|_0 ,$$

where C is a constant independent of $t, s, u \in \mathscr{D}(A(t)^{1/2}) = H_0^1(0,1)$ and $v \in \mathscr{D}(A(t))$.

By Lemma 2.2, we see that the operators $A(t)^{-1/2}(P(t)-P(s))$, $A(t)^{-1/2}\cdot(P^*(t)-P^*(s))$ and $A(t)^{-1}(A(t)-A(s))$ can be extended to bounded operators in X and satisfy

$$(2.13) \qquad |||A(t)^{-1/2}(P(t)-P(s))|||_0 \leq C|t-s| ,$$

$$(2.13') \qquad |||A(t)^{-1/2}(P^*(t)-P^*(s))|||_0 \leq C|t-s| ,$$

$$(2.14) \qquad |||A(t)^{-1}(A(t)-A(s))|||_0 \leq C|t-s| , \qquad (|t-s| \leq 1)$$

where $|||\cdot|||_0$ stands for the usual operator norm for bounded operators in X.

LEMMA 2.3. *Let $b(u,v)$ be the symmetric bilinear form defined by (2.6). Then, we have for a constant C independent of $u \in \mathscr{D}(b)$,*

$$b(u,u) \geq C\int_{-\infty}^{\infty}\left\{\left(\frac{d}{dx}u, \frac{d}{dx}u\right)_0 + |x|^{2m}(A_0^{1/2}u, A_0^{1/2}u)_0\right\}dx .$$

Proof. We note that there exist constants C and β independent of x and $v \in \mathscr{D}(A_0^{1/2})$ such that $\|P(x)v\|_0^2 \leq \beta \|Q(x)v\|_0^2$ and $\|Q(x)v\|_0^2 \geq C|x|^{2m} \cdot \|A_0^{1/2}v\|_0^2$. For any $\varepsilon > 0$, we have

$$(2.15)\quad \begin{aligned} b(u,u) &\geq \int_{-\infty}^{\infty} \left(\left\|\frac{d}{dx}u\right\|_0^2 - 2\left\|\frac{d}{dx}u\right\|_0 \|P(x)u\|_0 + \|P(x)u\|_0^2 + \|Q(x)u\|_0^2\right)dx \\ &\geq \int_{-\infty}^{\infty} \left(\varepsilon\left\|\frac{d}{dx}u\right\|_0^2 - \varepsilon\beta/(1-\varepsilon)\|Q(x)u\|_0^2 + \|Q(x)u\|_0^2\right)dx\,. \end{aligned}$$

Hence, by choosing ε in (2.15) so that $\varepsilon(1+\beta) < 1$, we obtain the proof. Q.E.D.

Let us consider the following eigenvalue problem in $L^2(-\infty,\infty;X)$:

$$(2.16)\quad -\frac{d^2}{dx^2}u + |x|^{2m} A_0 u = \lambda u\,.$$

We denote by $\{\zeta_j\}_{j=1}^{\infty}$ eigenvalues of the operator A_0 and by $\{\lambda_k\}_{k=1}^{\infty}$ eigenvalues of the operator $-\frac{d^2}{dx^2} + |x|^{2m}$ considered in $L^2(-\infty,\infty)$. Then with the aid of separation of variables, we easily see that the eigenvalues $\{\nu_j\}_{j=1}^{\infty}$ of the problem (2.16) are given by $\{\zeta_j^{1/(1+m)}\lambda_k\}_{j,k=1}^{\infty}$.

LEMMA 2.4. *Let $\{\nu_j\}_{j=1}^{\infty}$ be eigenvalues of the problem* (2.16). *Then, for any $p > 1/2 + 1/2m$ and $h > 2$, we have*

$$\begin{aligned} \sum_{j=1}^{\infty} (\nu_j + h)^{-p} &\leq C(p)h^{1/2+1/2m-p} \qquad (0 < m < 1) \\ &\leq C(p)h^{1-p}\log h \qquad (m = 1)\,. \end{aligned}$$

Proof. It is known that there exists a constant C independent of k such that $\lambda_k \geq Ck^{2m/(m+1)}$ and it is clear that $\zeta_j \geq Cj^2$. ([7])
Hence, it follows that

$$\sum_{j=1}^{\infty} (\nu_j + h)^{-p} \leq C \sum_{j,k=1}^{\infty} (j^{2/(m+1)}k^{2m/(m+1)} + h)^{-p}\,.$$

Furthermore, by using the estimate with a constant C independet of h

$$\sum_{k=1}^{\infty} (k^{2m/(m+1)} + h)^{-p} \leq Ch^{1/2+1/2m-p}\,,$$

we have

$$\sum_{j,k=1}^{\infty} (j^{2/(m+1)}k^{2m/(m+1)} + h)^{-p} \leq C \sum_{j=1}^{\infty} j^{-1/m}h^{1/2+1/2m-p}\,.$$

This gives the proof in the case of $0 < m < 1$. When $m = 1$, it is easy to see that $\sum_{j,k=1}^{\infty} (jk + h)^{-p} \leq Ch^{1-p} \log h$, which completes the proof.

Q.E.D.

PROPOSITION 2.1. *Let* $\{\mu_j\}_{j=1}^{\infty}$ *be eigenvalues of the problem* (2.8). *Then, for any* $p > 1/2 + 1/2m$ *and* $h > 2$, *we have with a constant* C *independent of* h,

$$\sum_{j=1}^{\infty} (\mu_j + h)^{-p} \leq Ch^{1/2+1/2m-p} \qquad (0 < m < 1\,,$$
$$\leq Ch^{1-p} \log h \qquad (m = 1)\,.$$

Proof. By virture of Lemma 2.3, we see that there exists a constant C independent of j such that $\mu_j \geq C\nu_j$. Hence, by combining this fact with Lemma 2.4, we get our assertion. Q.E.D.

§ 3. Propositions.

In this section, we shall state fundamental propositions which will be used later.

Let us fix some notations.

Let Y be a separable Hilbert space with the scalar product $(\ ,\ )_Y$ and the norm $\|\ \|_Y$. $B(Y)$ stands for the Banach space of all bounded operators acting on Y with the operator norm $|||\cdot|||_0$, and $B_\alpha(Y)$ denotes the subspace of $B(Y)$ consisting of compact operators such that $|||K|||_\alpha = (\sum_{j=1}^{\infty} \beta_j^\alpha)^{1/\alpha} < \infty$, where $\{\beta_j > 0\}_{j=1}^{\infty}$ denote eigenvalues of $(K^*K)^{1/2}$.

PROPOSITION 3.1. *Let* $K(x)$ $(-\infty < x < \infty)$ *be a family of operators belonging to* $B_2(Y)$ *such that* $K(x)$ *is continuous under the norm* $|||\cdot|||_0$ *with respect to* x *and that* $\int_{-\infty}^{\infty} |||K(x)|||_2^2\, dx < +\infty$. *Let* $\{\psi_j(x\}_{j=1}^{\infty}$ *be a complete orthonormal system in* $L^2(-\infty, \infty\,; Y)$. *If we define* $\beta_j = \int_{-\infty}^{\infty} K(x)\psi_j(x)dx$, *then* $\sum_{j=1}^{\infty} \|\beta_j\|_Y^2 = \int_{-\infty}^{\infty} |||K(x)|||_2^2\, dx$.

Proof. Let $\{\theta_j\}_{j=1}^{\infty}$ be a complete orthonormal system in Y. Then, we set $k^{ji}(x) = (K(x)\theta_j, \theta_i)_Y$ and $t_{ji}(x) = (\psi_j(x), \theta_i)_Y$. We note that $\{t_j(x) = (t_{ji}(x))_{i=1}^{\infty}\}_{j=1}^{\infty}$ forms a complete orthonormal system in $L^2(-\infty, \infty\,; \ell^2)$, where ℓ^2 denotes the usual Hilbert space consisting of all complex-valued square summable series. By the above definitions of $k^{ji}(x)$ and $t_{ji}(x)$, we have

$$K(x)\psi_j(x) = \sum_{i,\ell=1}^{\infty} t_{ji}(x)k^{i\ell}(x)\theta_\ell \,. \tag{3.1}$$

By (3.1), we obtain

$$\begin{aligned}\sum_{j=1}^{\infty}\|\beta_j\|_Y^2 &= \sum_{j=1}^{\infty}\left(\int_{-\infty}^{\infty} K(x)\psi_j(x)dx, \int_{-\infty}^{\infty} K(x)\psi_j(x)dx\right)_Y \\ &= \sum_{j=1}^{\infty}\left(\sum_{i,\ell=1}^{\infty}\int_{-\infty}^{\infty} t_{ji}(x)k^{i\ell}(x)dx\theta_\ell, \sum_{n,m=1}^{\infty}\int_{-\infty}^{\infty} t_{jn}(x)k^{nm}(x)dx\theta_m\right)_X \\ &= \sum_{j=1}^{\infty}\sum_{\ell=1}^{\infty}\left|\sum_{i=1}^{\infty}\int_{-\infty}^{\infty} t_{ji}(x)k^{i\ell}(x)dx\right|^2 \\ &= \sum_{i=1}^{\infty}\sum_{\ell=1}^{\infty}\int_{-\infty}^{\infty} |k^{i\ell}(x)|^2\,dx = \int_{-\infty}^{\infty} |||K(x)|||_2^2\,dx \,,\end{aligned}$$

where we have used the fact that $\{t_j(x)\}_{j=1}^{\infty}$ forms a complete orthonormal system in $L^2(-\infty, \infty; \ell^2)$. This completes the proof.

Let $\mathscr{H}_\gamma(\gamma$: positive real number) be the Hilbert space with the scalar product $(u, v)_\gamma = (A_0^\gamma u, A_0^\gamma v)_0$ and the norm $\|u\|_\gamma^2 = \|A_0^\gamma u\|_0^2$. Let $\mathscr{H}_{-\gamma}(\gamma > 0)$ be the dual space of $\mathscr{H}_\gamma$. The norm in $\mathscr{H}_{-\gamma}$ is defined by

$$\|u\|_{-\gamma} = \sup_{v\in\mathscr{H}_\gamma}\frac{|(u, v)_0|}{\|v\|_\gamma} \,.$$

It has been shown in [3] that the space $\mathscr{H}_\gamma$ is characterized as follows:

$$\mathscr{H}_\gamma\begin{cases} = H_0^1(0,1) \cap H^{2\gamma}(0,1) \,, & (1/2 < \gamma \le 1) \\ = H_0^{2\gamma}(0,1) \,, & (1/4 < \gamma < 1/2) \\ = H^{2\gamma}(0,1) \,. & (-1/4 < \gamma < 1/4) \,. \end{cases} \tag{3.2}$$

Let $B(\alpha, \beta)$ be the Banach space of all bounded operators from $\mathscr{H}_\alpha$ to $\mathscr{H}_\beta$ with the usual operator norm $|||\ \ |||_{(\alpha,\beta)}$. When B belongs to $B(\alpha, \alpha)$, in particuler, we write $|||\cdot|||_\alpha$ instead of $|||\cdot|||_{(\alpha,\alpha)}$.

The following proposition is well-known.

PROPOSITION 3.2. (*Interpolation theorem*) ([6]). *Let B be a bounded operator belonging to $B(\alpha, 0) \cap B(\beta, 0)$ $(\alpha \le \beta)$. Then, B is the bounded operator belonging to $B(\gamma, 0)$ $(\alpha \le \gamma \le \beta)$ and satisfies the following estimate*

$$|||B|||_{(\gamma,0)} \le C(\alpha, \beta, \gamma)|||B|||_{(\alpha,0)}^{(\beta-\gamma)/(\beta-\alpha)}\,|||B|||_{(\beta,0)}^{(\gamma-\alpha)/(\beta-\alpha)} \,.$$

where a constant $C(\alpha, \beta, \gamma)$ is independent of B.

LEMMA 3.1. *If $-1/4 < \gamma \le 0$, then the operators $A(t)^{-1/2}P(t)$ and*

$A(t)^{-1/2}P^*(t)$ can be extended to bounded operators belonging to $B(\gamma,\gamma)$ with the operator norm independent of $t\in \boldsymbol{R}^1$.

Proof. We shall give the proof only for $A(t)^{-1/2}P(t)$. Since $C_0^\infty(0,1)$ is dense in $\mathscr{H}_\gamma$ under the norm $\|\cdot\|_\gamma$, it is sufficient to show that for any $u\in C_0^\infty(0,1)$,

$$\|A(t)^{-1/2}P(t)u\|_\gamma \le C\,\|u\|_\gamma\,. \tag{3.3}$$

Let $u\in C_0^\infty(0,1)$. Then, by the definition of $\mathscr{H}_\gamma$, we have

$$\begin{aligned}\|A(t)^{-1/2}P(t)u\|_\gamma &= \sup_{v\in\mathscr{H}_{-\gamma}}\frac{|(A(t)^{-1/2}P(t)u,v)_0|}{\|v\|_{-\gamma}}\\ &= \sup_{v\in\mathscr{H}_{-\gamma}}\frac{|(u,P^*(t)A(t)^{-1/2}v)_0|}{\|v\|_{-\gamma}}\\ &\le \|u\|_\gamma \sup_{v\in\mathscr{H}_{-\gamma}}\frac{\|P^*(t)A(t)^{-1/2}v\|_{-\gamma}}{\|v\|_{-\gamma}}\,.\end{aligned} \tag{3.4}$$

We note that $P^*(t)$ it a bounded operator from $\mathscr{H}_{1/2-\gamma}=H_0^1(0,1)\cap H^{1-2\gamma}(0,1)$ to $\mathscr{H}_{-\gamma}=H^{-2\gamma}(0,1)$ with the operator norm independent of t. Hence, we have

$$\|P^*(t)A(t)^{-1/2}v\|_{-\gamma}\le C\,\|A(t)^{-1/2}v\|_{1/2-\gamma}\le C\,\|v\|_{-\gamma}\,, \tag{3.5}$$

which together with (3.4) implies our assertion (3.3). Q.E.D.

§ 4. Main theorem.

In this section, our main theorem stated in § 1 will be proved by a series of lemmas.

Let $t\in\boldsymbol{R}^1$ be fixed. Then, we consider the following differential equation for given $f\in L^2(-\infty,\infty;X)$ and any $h>0$:

$$(T(t)+h)u=-\frac{d^2}{dx^2}u-\frac{d}{dx}P(t)u+P^*(t)\frac{d}{dx}u+A(t)u+hu=f\,. \tag{4.1}$$

The solution $u(x)$ is given by

$$u(x)=(T(t)+h)^{-1}f=R_t(h)f=\int_{-\infty}^{\infty}K_t(x-s;h)f(s)ds\,,$$

where

$$K_t(x-s;h)=(2\pi)^{-1}\int_{-\infty}^{\infty}e^{i(x-s)\xi}(\xi^2+h+E(t,\xi))^{-1}d\xi\,,\qquad (i=\sqrt{-1}) \tag{4.2}$$

$$E(t, \xi) = -i\xi P(t) + i\xi P^*(t) + A(t) . \tag{4.3}$$

We denote by $R_t^{(j)}(h)$ ($= 0, 1, 2, \cdots$) the operator $\left(\frac{d}{dh}\right)^j R_t^{(j)}(h)$, which is defined by

$$R_t^{(j)}(h)f = \int_{-\infty}^{\infty} K_t^{(j)}(x - s)f(s)ds , \tag{4.4}$$

where

$$K_t^{(j)}(x - s; h) = (-1)^j(j!)(2\pi)^{-1}\int_{-\infty}^{\infty} e^{i(x-s)\xi}(\xi^2 + h + E(t, \xi))^{-(j+1)}d\xi . \tag{4.5}$$

We often write $R_t^{(0)}(h)$ and $K_t^{(0)}(x; h)$ instead of $R_t(h)$ and $K_t(x; h)$ respectively.

Let us introduce real-valued C_0-functions $\varphi(x), \psi(x)$ and $\chi(x)$ defined on $\boldsymbol{R}^1$ such that $\varphi(x), \psi(x)$ and $\chi(x) = 1$ if $|x| \leq 1, = 0$ if $|x| \geq 2$, and that $\varphi(x)\psi(x) \equiv \varphi(x)$ and $\psi(x)\chi(x) \equiv \psi(x)$. For each fixed $t \in \boldsymbol{R}^1$ and $\varepsilon > 0$ ($\varepsilon < 1$), we denote by $\varphi_{t,\varepsilon}(x)$ the function $\varphi_{t,\varepsilon}(x) = \varphi((x - t)/\varepsilon)$. Similarly we define the functions $\psi_{t,\varepsilon}(x)$ and $\chi_{t,\varepsilon}(x)$. Then, putting $R(h) = (T + h)^{-1}$ and using the resolvent equation, we have

$$\begin{aligned} \varphi_{t,\varepsilon}R(h) &= \psi_{t,\varepsilon}R_t(h)\varphi_{t,\varepsilon} \\ &\quad + \psi_{t,\varepsilon}R_t(h)((T(t) + h)\varphi_{t,\varepsilon} - \varphi_{t,\varepsilon}(T + h))\chi_{t,\varepsilon}R(h) . \end{aligned} \tag{4.6}$$

(We have used that $\varphi_{t,\varepsilon}(T + h)\chi_{t,\varepsilon} = \varphi_{t,\varepsilon}(T + h)$.)

Let $\{u_j(x)\}_{j=1}^{\infty}$ be the normalized eigenfunctions corresponding to the eigenvalues $\{\mu_j\}_{j=1}^{\infty}$ of the operator T. Then, by letting (4.6) operate on each $u_j(x)$, we have

$$\begin{aligned} (\mu_j + h)^{-1}\varphi_{t,\varepsilon}u_j &= \psi_{t,\varepsilon}R_t(h)\varphi_{t,\varepsilon}u_j \\ &\quad + (\mu_j + h)^{-1}\psi_{t,\varepsilon}R_t(h)B(t, s, \varepsilon)u_j , \end{aligned} \tag{4.7}$$

where we have set

$$\begin{aligned} B(t, s, \varepsilon) &= (T(t)\varphi_{t,\varepsilon}(s) - \varphi_{t,\varepsilon}(s)T)\chi_{t,\varepsilon}(s) = (T(t)\varphi_{t,\varepsilon}(s) - \varphi_{t,\varepsilon}(s)T) \\ &= \left(\left(-\frac{d^2}{ds^2} - \frac{d}{ds}P(t) + P^*(t)\frac{d}{ds} + A(t)\right)\varphi_{t,\varepsilon}(s)\right. \\ &\quad \left. - \varphi_{t,\varepsilon}(s)\left(-\frac{d^2}{ds^2} - \frac{d}{ds}P(s) + P^*(s)\frac{d}{ds} + A(s)\right)\right) . \end{aligned} \tag{4.8}$$

By differentiating (4.7) n-times with respect to h in the sense of $L^2(-\infty, \infty; X)$, we have

$$
\begin{aligned}
&(-1)^n(n!)(\mu_j + h)^{-(n+1)}\varphi_{t,\varepsilon}u_j \\
(4.9)\qquad &= \psi_{t,\varepsilon}R_t^{(n)}(h)\varphi_{t,\varepsilon}u_j + \sum_{p=0}^{n} C_p(\mu_j + h)^{-(n-p+1)}\psi_{t,\varepsilon}R_t^{(p)}(h)B(t, s, \varepsilon)u_j \,.
\end{aligned}
$$

Furthermore, by rewriting (4.9) in the form of the integral equation,

$$
\begin{aligned}
&(-1)^n(n!)(\mu_j + h)^{-(n+1)}\varphi_{t,\varepsilon}(x)u_j(x) \\
&= \psi_{t,\varepsilon}(x)\int_{-\infty}^{\infty} K_t^{(n)}(x - s;\, h)\varphi_{t,\varepsilon}(s)u_j(s)ds \\
(4.10)\qquad &\quad + \sum_{p=0}^{n} C_p(\mu_j + h)^{-(n-p+1)}\psi_{t,\varepsilon}(x)\int_{-\infty}^{\infty} K_t^{(p)}(x - s;\, h)B(t, s, \varepsilon)u_j(s)ds \\
&= a_j(t, x, \varepsilon) + \sum_{p=0}^{n} C_p b_{j,p}(t, x, \varepsilon) \,.
\end{aligned}
$$

We remark that by the regularity theorem for elliptic operators, the eigenfunction $u_j(x)$ belongs to $C^\infty(-\infty, \infty; X)$ (the set of smooth functions with values in X). Hence, the equality (4.10) is well-defined for all x. By this fact, we can put $x = t$ in (4.10). Then, we have

$$
\begin{aligned}
&(-1)^n(n!)(\mu_j + h)^{-(n+1)}u_j(t) \\
(4.11)\qquad &= a_j(t, \varepsilon) + \sum_{p=0}^{n} C_p b_{j,p}(t, \varepsilon) \,,
\end{aligned}
$$

where we have set $a_j(t, \varepsilon) = a_j(t, t, \varepsilon)$ and $b_j(t, \varepsilon) = b_{j,p}(t, t, \varepsilon)$.
By taking the scalar products in X of both sides of (4.11) and the summation with respect to j, and integrating over $(-\infty, \infty)$, we have

$$
\begin{aligned}
(n!)^2 \sum_{j=1}^{\infty} (\mu_j + h)^{-2(n+1)} &= \sum_{j=1}^{\infty}\int_{-\infty}^{\infty} \|a_j(t, \varepsilon)\|_0^2\, dt \\
(4.12)\qquad &\quad + 2\sum_{j=1}^{\infty}\int_{-\infty}^{\infty} \mathrm{Re}\Big(a_j(t, \varepsilon), \sum_{p=0}^{n} C_p b_{j,p}(t, \varepsilon)\Big)_0 dt \\
&\quad + \sum_{j=1}^{\infty}\int_{-\infty}^{\infty} \Big\|\sum_{p=0}^{n} C_p b_{j,p}(t, \varepsilon)\Big\|_0^2\, dt \,.
\end{aligned}
$$

This is our basic equality in proving the main theorem.

From now on, we fix n (integer) such that

$$
(4.13)\qquad n > 1/2 + 1/2m - 1 \,.
$$

Let $\{\alpha_j(t, \xi)\}_{j=1}^{\infty}$ be eigenvalues of the operator $E(t, \xi)$. Then, we have

LEMMA 4.1. *For any $r > 1/2 + 1/2m$, there exist positive constants $C_1(r)$ and $C_2(r)$ independent of $h > 2$ such that*

$$\begin{matrix} C_1(r)h^{1/2+1/2m-r} \leq \\ C_1(r)h^{1-r}\log h \leq \end{matrix} \left\{\sum_{j=1}^{\infty}\int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} (\xi^2 + h + \alpha_j(t,\xi))^{-r} d\xi\right\} \begin{cases} \leq C_2(r)h^{1/2+1/2m-r} . \\ \qquad (0 < m < 1) \\ \leq C_2(r)h^{1-r}\log h . \\ \qquad (m = 1) \end{cases}$$

LEMMA 4.2.

$$\sum_{j=1}^{\infty}\int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} (\xi^2 + h + \alpha_j(t,\xi))^{-2(n+1)} d\xi$$
$$\sim \sum_{j=1}^{\infty}\int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} (\xi^2 + h + \zeta_j q(t)^{-2})^{-2(n+1)} d\xi , \qquad \text{as } h \to \infty .$$

The proofs of the above lemmas will be given in this section after the proof of Theorem 1.1. In what follows, we shall state two lemmas concerning the estimates for $\sum_{j=1}^{\infty}\int_{-\infty}^{\infty} \|a_j(t,\varepsilon)\|_0^2\, dt$ and $\int_{-\infty}^{\infty}\sum_{j=1}^{\infty}\|b_{j,p}(t,\varepsilon)\|_0^2\, dt$. These lemmas will be proved in the following two sections.

LEMMA 4.3. *For any $\varepsilon > 0$ and any sufficiently large r, there exists a constant $C(r,\varepsilon)$ such that*

$$\left|\sum_{j=1}^{\infty}\int_{-\infty}^{\infty} \|a_j(t,\varepsilon)\|_0^2\, dt - (n!)^2(2\pi)^{-1}\sum_{j=1}^{\infty}\int_{-\infty}^{\infty} dt\int_{-\infty}^{\infty}(\xi^2 + h + \alpha_j(t,\xi))^{-2(n+1)} d\xi\right|$$
$$\leq C(r,\varepsilon)h^{-r} .$$

LEMMA 4.4. *For any sufficiently small $\delta > 0$, we can take $\varepsilon(\delta)$ small enough and $h(\delta)$ large enough so that for any $h > h(\delta)$,*

$$\sum_{j=1}^{\infty}\int_{-\infty}^{\infty} \|b_{j,p}(t,\varepsilon(\delta))\|_0^2\, dt \leq \delta h^{1/2+1/2m-2(n+1)} \qquad (0 < m < 1) .$$
$$(\leq \delta h^{1-2(n+1)}\log h \ \ (m = 1).) \qquad (0 \leq p \leq n) .$$

Now we shall prove Theorem 1.1.

Proof of Theorem 1.1:

From (4.12) it follows that for any sufficiently small $\delta > 0$, there exists a constant $C(\delta)$ such that

$$\left|(n!)^2\sum_{j=1}^{\infty}(\mu_j + h)^{-2(n+1)} - \sum_{j=1}^{\infty}\int_{-\infty}^{\infty}\|a_j(t,\varepsilon)\|_0^2\, dt\right|$$
$$\leq \delta\sum_{j=1}^{\infty}\int_{-\infty}^{\infty}\|a_j(t,\varepsilon)\|_0^2\, dt + C(\delta)\sum_{j=1}^{\infty}\int_{-\infty}^{\infty}\sum_{p=0}^{n}\|b_{j,p}(t,\varepsilon)\|_0^2\, dt .$$

Hence, by virtue of Lemmas 4.1, 4.3 and 4.4, we can first choose $\varepsilon(\delta)$

small enough and next $h(\delta)$ large enough so that for any $h > h(\delta)$

$$\begin{aligned}(4.13)\quad &\left|\sum_{j=1}^{\infty}(\mu_j + h)^{-2(n+1)} - \sum_{j=1}^{\infty}(2\pi)^{-1}\int_{-\infty}^{\infty}dt\int_{-\infty}^{\infty}(\xi^2 + h + \alpha_j(t,\xi))^{-2(n+1)}d\xi\right| \\ &\leq \delta h^{1/2+1/2m-2(n+1)} \qquad (0 < m < 1)\ . \\ &\leq \delta h^{1-2(n+1)}\log h \qquad (m = 1)\ .\end{aligned}$$

Furthermore, by using Lemmas 4.1 and 4.2, we have

$$(4.14)\quad \sum_{j=1}^{\infty}(\mu_j + h)^{-2(n+1)} \sim (2\pi)^{-1}\int_{-\infty}^{\infty}dt\int_{-\infty}^{\infty}(\xi^2 + h + \zeta_j q(t)^{-2})^{-2(n+1)}d\xi \quad \text{as } h \to \infty\ .$$

Now we are in a position to apply the Tauberian theorem due to Keldysh (see [5]) to (4.14). Then, we have

$$N(h) \sim (\pi)^{-1}\sum_{j=1}^{\infty}\int_{\Omega_j(h)}(h - \zeta_j q(x)^{-2})^{1/2}dx \qquad \text{as } h \to \infty\ ,$$

which completes the proof. Q.E.D.

Next we shall prove Lemmas 4.1 and 4.2.

Proof of Lemma 4.1:

An argument similar to the proof of Lemma 2.3 shows that for any $u \in \mathscr{D}(A_0)$,

$$\begin{aligned}C((\xi^2 u, u)_0 &+ (1 + |t|^{2m})(A_0 u, u)_0) \\ &\leq (\xi^2 u, u)_0 + (E(t,\xi)u, u)_0 \\ &\leq C((\xi^2 u, u)_0 + (1 + |t|^{2m})(A_0 u, u)_0)\ .\end{aligned}$$

Hence, we have

$$(4.15)\quad C(\xi^2 + \zeta_j(1 + |t|^{2m})) \leq \xi^2 + \alpha_j(t,\xi) \leq C(\xi^2 + \zeta_j(1 + |t|^{2m}))\ .$$

By using (4.15), we have

$$\begin{aligned}&\sum_{j=1}^{\infty}\int_{-\infty}^{\infty}dt\int_{-\infty}^{\infty}(\xi^2 + h + \alpha_j(t,\xi))^{-r}d\xi \\ &\qquad \leq C\sum_{j=1}^{\infty}\zeta_j^{-1/2m}(\zeta_j + h)^{1/2+1/2m-r} \leq Ch^{1/2+1/2m-r} \qquad (0 < m < 1)\ . \\ &\qquad\qquad (\leq Ch^{1-r}\log h\ \ (m = 1).)\end{aligned}$$

Thus we have obtained the estimate from above. Similarly we can get the estimate from below. Q.E.D.

Proof of Lemma 4.2:

We shall give only an outline. As was remarked in §2, coefficients of $P(t)$ are uniformly bounded. Hence, for any sufficiently small $\varepsilon > 0$, there exists $R(\varepsilon)$ such that for $|t| \geq R(\varepsilon)$ and $u \in \mathscr{D}(A_0^{1/2})$,

$$\|P(t)u\|_0 \leq \varepsilon q(t)^{-1} \|A_0^{1/2}u\|_0 . \tag{4.16}$$

From (4.16) it readily follows that for $|t| \geq R(\varepsilon)$,

$$\begin{aligned}(1 - C\varepsilon)(\xi^2 + h + \zeta_j q(t)^{-2}) &\leq \xi^2 + h + \alpha_j(t, \xi) \\ &\leq (1 + C\varepsilon)(\xi^2 + h + \zeta_j q(t)^{-2}) .\end{aligned} \tag{4.17}$$

On the other hand, for any bounded interval I, we have

$$\sum_{j=1}^{\infty} \int_I dt \int_{-\infty}^{\infty} (\xi^2 + h + \alpha_j(t, \xi))^{-2(n+1)} d\xi \leq C(I) h^{1-2(n+1)} . \tag{4.18}$$

By combining (4.17) and (4.18), we obtain the proof. Q.E.D.

§ 5. Proof of Lemma 4.3.

Lemma 4.3 is proved with the aid of the following lemma.

LEMMA 5.1. *Let α be any non-negative integer and let n be the fixed integer by* (4.13). *Then, the operator* $\left(\frac{d}{d\xi}\right)^{\alpha}(\xi^2 + h + E(t, \xi))^{-(n+1)}$ *belongs to $B_2(X)$ and satisfies the following estimate*:

$$\begin{aligned}&\left|\left|\left|\left(\frac{d}{d\xi}\right)^{\alpha}(\xi^2 + h + E(t, \xi))^{-(n+1)}\right|\right|\right|_2 \\ &\qquad \leq C(\alpha, n)(\xi^2 + h)^{-\alpha/2}\left(\sum_{j=1}^{\infty} (\xi^2 + h + \alpha_j(t, \xi))^{-2(n+1)}\right)^{1/2} .\end{aligned}$$

The proof of this lemma will be given at the end of this section

Proof of Lemma 4.3:

By virtue of Proposition 3.1, we have

$$\begin{aligned}\sum_{j=1}^{\infty} \|a_j(t, \varepsilon)\|_0^2 &= \int_{-\infty}^{\infty} |||K_t^{(n)}(t - s; h)\varphi_{t,\varepsilon}(s)|||_2^2 \, ds \\ &= \int_{-\infty}^{\infty} |||K_t^{(n)}(t - s; h)|||_2^2 \, ds \\ &\quad + \int_{-\infty}^{\infty} |||K_t^{(n)}(t - s; h)|||_2^2 \, (\varphi_{t,\varepsilon}(s)^2 - 1) ds , \\ &= I(t) + II(t) .\end{aligned} \tag{5.1}$$

We first investigate the term $I(t)$. By means of the Parseval equality we see that

$$
\begin{aligned}
I(t) &= (n!)^2(2\pi)^{-1}\int_{-\infty}^{\infty} |||(\xi^2+h+E(t,\xi))^{-2(n+1)}|||_2^2\, d\xi \\
&= (n!)^2(2\pi)^{-1}\sum_{j=1}^{\infty}\int_{-\infty}^{\infty}(\xi^2+h+\alpha_j(t,\xi))^{-2(n+1)}d\xi\ .
\end{aligned} \tag{5.2}
$$

Next we shall deal with the term $II(t)$. We note that

$$(t-s)^\alpha K_t^{(n)}(t-s;h) = C\int_{-\infty}^{\infty} e^{i(t-s)\xi}\left(\frac{d}{d\xi}\right)^\alpha(\xi^2+h+E(t,\xi))^{-2(n+1)}d\xi\ .$$

By using Lemma 5.1 and this equality, we calculate as follows:

$$
\begin{aligned}
&|||K_t^{(n)}(t-s;h)|||_2 \\
&\le C|t-s|^{-\alpha}\int_{-\infty}^{\infty}(\xi^2+h)^{-\alpha/2}\left(\sum_{j=1}^{\infty}(\xi^2+h+\alpha_j(t,\xi))^{-2(n+1)}\right)^{1/2}d\xi\ , \\
&\le C|t-s|^{-\alpha}h^{-\alpha/2+1/4}\left(\sum_{j=1}^{\infty}\int(\xi^2+h+\alpha_j(t,\xi))^{-2(n+1)}d\xi\right)^{1/2}\ .
\end{aligned} \tag{5.3}
$$

(Schwarz' inequality)

Furthermore, by virtue of Lemma 4.1 and (5.3), we have

$$
\begin{aligned}
\int_{-\infty}^{\infty}|II(t)|\,dt &\le C(\varepsilon)h^{-\alpha+1/2}\sum_{j=1}^{\infty}\int_{-\infty}^{\infty}dt\int_{-\infty}^{\infty}(\xi^2+h+\alpha_j(t,\xi))^{-2(n+1)}d\xi \\
&\le C(\varepsilon)h^{-\alpha+1+1/2m-2(n+1)} \qquad (0<m<1)\ . \\
&(\le C(\varepsilon)h^{-\alpha+2-2(n+1)}\log h \qquad (m=1).)
\end{aligned} \tag{5.4}
$$

$\left(\text{We have used the inequality } \int_{-\infty}^{\infty}|t-s|^{-2\alpha}|\varphi_{t,\varepsilon}(s)^2-1|\,ds \le C(\varepsilon).\right)$

By combining (5.2) and (5.4), we get the proof since α is arbitrary.

In order to prove Lemma 5.1, we have to prepare the following two lemmas

LEMMA 5.2. *For any non-negative integer α, the following estimates hold*:

$$\left|\left|\left|\left(\frac{d}{d\xi}\right)^\alpha(\xi^2+h+E(t,\xi))^{-1}\right|\right|\right|_0 \le C(\alpha)(\xi^2+h)^{-(\alpha/2+1)}\ ; \tag{5.5}$$

$$\left|\left|\left|P(t)\left(\frac{d}{d\xi}\right)^\alpha(\xi^2+h+E(t,\xi))^{-1}\right|\right|\right|_0 \le C(\alpha)(\xi^2+h)^{-(\alpha+1)/2}\ . \tag{5.6}$$

Replacing $P(t)$ by $P^*(t)$ in (5.6), *we have the same estimate as* (5.6).

Proof. We shall make induction on α. It is clear that (5.5) and (5.6) hold when $\alpha=0$. Assuming that (5.5) and (5.6) are valid for $\alpha \leq k$, we shall prove (5.5) when $\alpha = k+1$. For the sake of simplity, we set $(\xi^2 + h + E(t,\xi))^{-1} = F(t,\xi)$. Then, a direct calculation yields

$$
\begin{aligned}
\left(\frac{d}{d\xi}\right)^{(k+1)} F(t,\xi) &= -\left(\frac{d}{d\xi}\right)^{k} \{F(t,\xi)(2\xi - iP(t) + iP^*(t))F(t,\xi)\} \\
&= 2\xi \sum_{p=0}^{k} C(p)\left(\frac{d}{d\xi}\right)^{p} F(t,\xi)\left(\frac{d}{d\xi}\right)^{(k-p)} F(t,\xi) \\
&\quad + \sum_{q=0}^{k-1} C(q)\left(\frac{d}{d\xi}\right)^{q} F(t,\xi)\left(\frac{d}{d\xi}\right)^{(k-1-q)} F(t,\xi) \\
&\quad + \sum_{r=0}^{k} C(r)\left(\frac{d}{d\xi}\right)^{r} F(t,\xi)(-iP(t) + iP^*(t)) \\
&\quad \cdot \left(\frac{d}{d\xi}\right)^{(k-r)} F(t,\xi) .
\end{aligned}
\tag{5.7}
$$

By the assumption of induction, (5.7) implies (5.5) with $\alpha = k+1$. By multipying (5.7) by $P(t)$ from the left, we obtain (5.6) with $\alpha = k+1$. Q.E.D.

LEMMA 5.3. *For any non-negative integer* α, $\left(\frac{d}{d\xi}\right)^{\alpha} (\xi^2 + h + E(t,\xi))^{-1}$ *belongs to* $B_\gamma(X)$ $(\gamma \geq 2)$ *with the estimate*

$$
\begin{aligned}
&\left|\left|\left| \left(\frac{d}{d\xi}\right)^{\alpha} (\xi^2 + h + E(t,\xi))^{-1} \right|\right|\right|_\gamma \\
&\leq C(\alpha,\gamma)(\xi^2 + h)^{-\alpha/2} \left(\sum_{j=1}^{\infty} (\xi^2 + h + \alpha_j(t,\xi))^{-\gamma}\right)^{1/\gamma} .
\end{aligned}
\tag{5.8}
$$

Proof. As in the proof of the above lemma, we shall give the proof by induction on α. It is clear that (5.8) is valid for $\alpha = 0$. Under the assumption that (5.8) is true for $\alpha \leq k$, we shall show that (5.8) holds also when $\alpha = k+1$. Noting that if $A \in B_\gamma(X)$ and $B \in B(X)$, then $A \cdot B$ belongs to $B_\gamma(X)$ with the estimate $|||A \cdot B|||_\gamma \leq |||A|||_\gamma |||B|||_0$, we have, by means of Lemma 5.2 and (5.7), the desired result. Q.E.D.

Now we shall prove Lemma 5.1.

Proof of Lemma 5.1. As in the proof of Lemma 5.2, we set $F(t,\xi) = (\xi^2 + h + E(t,\xi))^{-1}$. Then, a simple calculation gives

$$\left(\frac{d}{d\xi}\right)^{\alpha} F(t,\xi)^{(n+1)}$$
$$= \sum_{\beta_1+\beta_2\cdots\beta_{n+1}=\alpha} C(\beta_1,\cdots,\beta_{n+1})\left(\frac{d}{d\xi}\right)^{\beta_1} F(t,\xi)\cdots\left(\frac{d}{d\xi}\right)^{\beta_{n+1}} F(t,\xi)\ .$$

Since each $\left(\frac{d}{d\xi}\right)^{\beta_k} F(t,\xi)$ $(k=1,2,\cdots,n+1)$ belongs to $B_{2(n+1)}(X)$, we have, by means of Lemma 5.3,

$$\left|\left|\left|\left(\frac{d}{d\xi}\right)^{\alpha} F(t,\xi)^{-(n+1)}\right|\right|\right|_2$$
$$\le C(\alpha)\sum_{\beta_1+\cdots+\beta_{n+1}=\alpha}\left|\left|\left|\left(\frac{d}{d\xi}\right)^{\beta_1} F(t,\xi)\right|\right|\right|_{2(n+1)}\cdots\left|\left|\left|\left(\frac{d}{d\xi}\right)^{\beta_{n+1}} F(t,\xi)\right|\right|\right|_{2(n+1)}$$
$$\le C(\alpha)(\xi^2+h)^{-\alpha/2}\left(\sum_{j=1}^{\infty}(\xi^2+h+\alpha_j(t,\xi))^{-2(n+1)}\right)^{1/2}.$$

(We have used the well-known fact that if $A\in B_p(X)$ and $B\in B_q(X)$, then $A\cdot B\in B_r(X)$ $(1/p+1/q=1/r)$ and satisfies $|||A\cdot B|||_r\le C(p,q)|||A|||_p\,|||B|||_q$.)

Q.E.D.

§ 6. Proof of Lemma 4.4.

In this section we shall prove Lemma 4.4. For this purpose, recalling the definition of the operator $B(t,s,\varepsilon)$ given by (4.8), we rewrite $B(t,s,\varepsilon)$ as follows:

$$\begin{aligned} B(t,s,\varepsilon) &= B_1(t,s,\varepsilon) - \frac{d}{ds}(P(t)-P(s))\varphi_{t,\varepsilon}(s) \\ &\quad + \frac{d}{ds}(P^*(t)-P^*(s))\varphi_{t,\varepsilon}(s) + (A(t)-A(s))\varphi_{t,\varepsilon}(s) \qquad (6.1) \\ &= B_1(t,s,\varepsilon) + \sum_{k=0}^{2} H_k(t,s,\varepsilon)\ , \end{aligned}$$

where

$$B_1(t,s,\varepsilon) = \left[-\left(\frac{d}{ds}\right)^2,\varphi_{t,\varepsilon}\right] + \left[\varphi_{t,\varepsilon},\frac{d}{ds}\right]P(s) + \left[\frac{d}{ds},P^*(s)\varphi_{t,\varepsilon}\right].$$

([,] stands for a commutator between two operators.)

Then, $b_{j,p}(t,\varepsilon)$ is rewritten as follows:

$$b_{j,p}(t,\varepsilon) = (\mu_j+h)^{-(n-p+1)}\int_{-\infty}^{\infty} K_t^{(p)}(t-s\,;h)B_1(t,s,\varepsilon)u_j(s)ds$$

$$+ (\mu_j + h)^{-(n-p+1)} \int_{-\infty}^{\infty} K_t^{(p)}(t-s; h) \sum_{k=0}^{2} H_k(t, s, \varepsilon) u_j(s) ds \tag{6.2}$$
$$= d_{j,p}(t, \varepsilon) + e_{j,p}(t, \varepsilon) + f_{j,p}(t, \varepsilon) + g_{j,p}(t, \varepsilon) .$$

We note that for $v(s) \in C_0^\infty(-\infty, \infty; X)$ (the set of all X-valued smooth functions with compact support) and $p \geq 1$,

$$\int_{-\infty}^{\infty} K_t^{(p)}(t-s; h) \frac{d}{ds} v(s) ds = \int_{-\infty}^{\infty} F_t^{(p)}(t-s; h) v(s) dt , \tag{6.3}$$

where

$$F_t^{(p)}(t-s; h) = C \int_{-\infty}^{\infty} e^{i(t-s)\xi} \xi(\xi^2 + h + E(t, \xi))^{-(p+1)} d\xi . \tag{6.4}$$

The relation (6.3) is easily obtained with the aid of the vector-valued Fourier transform. The integral (6.4) is valid also for $p = 0$. In this case, the integration must be taken in the weak sense. But we don't use this fact below. By virtue of (6.3), we can rewrite $e_{j,p}(t, \varepsilon)$ $(p \geq 1)$ as follows:

$$e_{j,p}(t, \varepsilon) = (\mu_j + h)^{-(n-p+1)} \int_{-\infty}^{\infty} F_t^{(p)}(t-s; h)(P(s) - P(t)) \varphi_{t,\varepsilon}(s) u_j(s) ds .$$

The following lemma plays an important role in the proof of Lemma 4.4.

LEMMA 6.1. *Let $K_t^{(p)}(t-s; h)$ and $F_t^{(p)}(t-s; h)$ be operators defined by* (4.5) *and* (6.4) *respectively. Then, $K_t^{(p)}(t-s; h)A(t)$, $K_t^{(p)}(t-s; h)A(t)^{1/2}$ and $F_t^{(p)}(t-s; h)A(t)^{1/2}$ can be extended to bounded operators in X $(t \neq s)$ and satisfy the following estimates*:

$$|||K_t^{(p)}(t-s; h)A(t)|||_0 \leq C(p, \alpha) h^{-p} |t-s|^{-\alpha} , \tag{6.5}$$

$$|||K_t^{(p)}(t-s; h)A(t)^{1/2}|||_0 \leq C(p, \alpha) h^{-p-1/2} |t-s|^{-\alpha} , \tag{6.6}$$
$$\leq C(p, \beta) h^{-p} |t-s|^{-\beta} .$$

$$|||F_t^{(p)}(t-s; h)A(t)^{1/2}|||_0 \leq C(p, \alpha) h^{-p} |t-s|^{-\alpha} , \qquad (p \geq 1) \tag{6.7}$$

where constants $C(p, \alpha)$ and $C(p, \beta)$ are independent of t, s and h, and α and β are some constants satisfying $0 < \alpha < 2$ and $0 < \beta < 1$ respectively.

Proof. We shall give the proof only for (6.5) with $p = 0$, because (6.5) with general p, (6.6), (6.6′) and (6.7) can be proved in the same manner. Let δ be a fixed number such that $0 < \delta < 1/8$. Then, we shall establish the following two assertions:

$$\|K_t^{(0)}(t-s;h)A(t)u\|_0 \leq C(\delta)q(t)^{-(1+2\delta)}\|u\|_{1/2+\delta}, \tag{6.8}$$

$$\|K_t^{(0)}(t-s;h)A(t)u\|_0 \leq C(\delta)q(t)^{2\delta}|t-s|^{-2}\|u\|_{-\delta}, \qquad \text{for } u \in \mathscr{D}(A_0). \tag{6.9}$$

All constants C appearing throughout the proof of this lemma may depend only on δ. If we can prove (6.8) and (6.9), the operator $K_t^{(0)}(t-s;h)A(t)$ can be extended to a bounded operator from $\mathscr{H}_{1/2+\delta}$ to X and from $\mathscr{H}_{-\delta}$ to X since $\mathscr{D}(A_0)$ is dense in both spaces $\mathscr{H}_{1/2+\delta}$ and $\mathscr{H}_{-\delta}$. Hence, the application of the well-known interpolation theorem (Proposition 3.2) shows that

$$|||K_t^{(0)}(t-s;h)A(t)|||_0 \leq C|t-s|^{-\alpha}, \qquad (0<\alpha<2).$$

Proof of (6.8): Since $\|A(t)^{1/2+\delta}u\|_0 \leq Cq(t)^{-(1+2\delta)}\|A_0^{1/2+\delta}u\|_0$ for $u \in \mathscr{D}(A_0)$ by (2.9), it is sufficient to prove that

$$\|K_t^{(0)}(t-s;h)A(t)^{1/2-\delta}v\|_0 \leq C\|v\|_0, \qquad \text{for } v \in \mathscr{D}(A_0^{1/2-\delta}). \tag{6.10}$$

By the definition of $K_t^{(0)}(t-s;h)$, we have

$$K_t^{(0)}(t-s;h)A(t)^{1/2-\delta} = C\int_{-\infty}^{\infty} e^{i(t-s)\xi}(\xi^2+h+E(t,\xi))^{-1}A(t)^{1/2-\delta}d\xi. \tag{6.11}$$

On the other hand, we easily see that

$$\|(\xi^2+h+E(t,\xi))^{\delta-1/2}A(t)^{1/2-\delta}v\|_0 \leq C\|v\|_0, \tag{6.12}$$

$$|||(\xi^2+h+E(t,\xi)^{-(1/2+\delta)}|||_0 \leq C(\xi^2+1)^{-(1/2+\delta)}. \tag{6.13}$$

Hence, in view of (6.11), (6.12) and (6.13), we have (6.10).

Proof of (6.9): Since $\|A(t)^{-\delta}u\|_0 \leq Cq(t)^{2\delta}\|A_0^{-\delta}u\|_0$ by (2.10), it suffices to show that for $w \in \mathscr{D}(A(t)^{1+\delta})$,

$$\|K_t^{(0)}(t-s;h)A(t)^{1+\delta}w\|_0 \leq C|t-s|^{-2}\|w\|_0. \tag{6.14}$$

The operator $K_t^{(0)}(t-s;h)A(t)^{1+\delta}$ is represented as

$$\begin{aligned} &K_t^{(0)}(t-s;h)A(t)^{1+\delta} \\ &\quad = C(t-s)^{-2}\int_{-\infty}^{\infty} e^{i(t-s)\xi}\left(\frac{d}{d\xi}\right)^2(\xi^2+h+E(t,\xi))^{-1}A(t)^{1+\delta}d\xi. \end{aligned} \tag{6.15}$$

For brevity, we again put $F(t,\xi) = (\xi^2+h+E(t,\xi))^{-1}$.
By the resolvent equation, we have

$$
\begin{aligned}
\left(\frac{d}{d\xi}\right)^2 F(t,\xi) &= \left(\frac{d}{d\xi}\right)^2 (\xi^2+h+A(t))^{-1} \\
&\qquad + i\left(\frac{d}{d\xi}\right)^2 \{F(t,\xi)\xi P(t)(\xi^2+h+A(t))^{-1}\} \\
&\qquad - i\left(\frac{d}{d\xi}\right)^2 \{F(t,\xi)\xi P^*(t)(\xi^2+h+A(t))^{-1}\} \\
&= \mathrm{I}\,(t,\xi) + i\,\mathrm{II}\,(t,\xi) - i\,\mathrm{III}\,(t,\xi)\,.
\end{aligned}
\tag{6.16}
$$

By inserting (6.16) into (6.15), we have

$$
\begin{aligned}
&K_t^{(0)}(t-s;h)A(t)^{1+\delta} \\
&\quad = C(t-s)^{-2}\int_{-\infty}^{\infty} e^{i(t-s)\xi}(\mathrm{I}\,(t,\xi)+i\,\mathrm{II}\,(t,\xi)) - i\,\mathrm{III}\,(t,\xi))A(t)^{1+\delta}d\xi \\
&\quad = C(t-s)^{-2}\sum_{k=1}^{2} K_{t,k}^{(0)}(t-s;h)A(t)^{1+\delta}
\end{aligned}
\tag{6.17}
$$

It is easy to see that for $w\in\mathscr{D}(A(t)^{1+\delta})$,

$$
\|\mathrm{I}\,(t,\xi)A(t)^{1+\delta}w\|_0 \le C(\xi^2+1)^{-(1-\delta)}\|w\|_0\,.
$$

This implies that

$$
\|K_{t,1}^{(0)}(t-s;h)A(t)^{1+\delta}w\|_0 \le \|w\|_0\,. \tag{6.18}
$$

Next we shall investigate the term $\mathrm{II}\,(t,\xi)A(t)^{1+\delta}$. A simple calculation yields

$$
\begin{aligned}
\mathrm{II}\,(t,\xi) &= \sum_{\beta_1+\beta_2=2}\left(\frac{d}{d\xi}\right)^{\beta_1}\{F(t,\xi)\}\xi P(t)\left(\frac{d}{d\xi}\right)^{\beta_2}\{(\xi^2+h+A(t))^{-1}\} \\
&\quad + \sum_{\alpha_1+\alpha_2=1}\left(\frac{d}{d\xi}\right)^{\alpha_1}\{F(t,\xi)\}P(t)\left(\frac{d}{d\xi}\right)^{\alpha_2}\{(\xi^2+h+A(t))^{-1}\}\,.
\end{aligned}
\tag{6.19}
$$

We shall consider only the term $\left(\frac{d}{d\xi}\right)\{F(t,\xi)\}P(t)(\xi^2+h+A(t))^{-1}$, because the other terms can be dealt with in the same way. Putting $2\xi - iP(t) + iP^*(t) = T(t,\xi)$, we have,

$$
\begin{aligned}
&\left(\frac{d}{d\xi}\right)\{F(t,\xi)\}P(t)(\xi^2+h+A(t))^{-1} \\
&\qquad = -F(t,\xi)T(t,\xi)F(t,\xi)P(t)(\xi^2+h+A(t))^{-1}\,.
\end{aligned}
$$

We shall show that for $w\in\mathscr{D}(A(t)^{1+\delta})$,

$$
\|F(t,\xi)P(t)(\xi^2+h+A(t))^{-1}A(t)^{1+\delta}w\|_0 \le C(\xi^2+1)^{-\delta}\|w\|_0\,. \tag{6.20}
$$

If we have proved (6.20), then we have

$$\left\|\left(\frac{d}{d\xi}\right)\{F(t,\xi)\}P(t)(\xi^2+h+A(t))^{-1}A(t)^{1+\delta}w\right\|_0 \le C(\xi^2+1)^{-1/2-\delta}\|w\|_0 \tag{6.21}$$

since $|||F(t,\xi)T(t,\xi)|||_0 \le C(\xi^2+1)^{-1/2}$.
Since the other terms in (6.19) obey the estimate of the same type as (6.21), we see that for $w\in\mathscr{D}(A(t)^{1+\delta})$.

$$\|\mathrm{II}\,(t,\xi)A(t)^{1+\delta}w\|_0 \le C(\xi^2+1)^{-1/2-\delta}\|w\|_0\,.$$

This implies that

$$\|K_{t,2}^{(0)}(t-s\,;h)A(t)^{1+\delta}w\|_0 \le C\|w\|_0\,. \tag{6.22}$$

Similarly we have

$$\|K_{t,3}^{(0)}(t-s\,;h)A(t)^{1+\delta}w\|_0 \le C\|w\|_0\,. \tag{6.23}$$

Hence, by combining (6.18), (6.22) and (6.23), we have (6.14).

Now we shall prove (6.20). To this end, we rewrite $F(t,\xi)P(t)(\xi^2+h+A(t))^{-1}A(t)^{1+\delta}$ as follows:

$$\begin{aligned}
&F(t,\xi)P(t)(\xi^2+h+A(t))^{-1}A(t)^{1+\delta}\\
&\qquad=[F(t,\xi)A(t)][A(t)^{-1}P(t)A(t)^{2\delta}][(\xi^2+h+A(t))^{-1}A(t)^{1-\delta}]\\
&\qquad=\mathrm{II}_1(t,\xi)\,\mathrm{II}_2(t)\,\mathrm{II}_3(t,\xi)\,.
\end{aligned}$$

The operators $\mathrm{II}_1(t,\xi)$ and $\mathrm{II}_3(t,\xi)$ can be extended to bounded operators in X and satisfy the estimates $|||\mathrm{II}_1(t,\xi)|||_0\le C$ and $|||\mathrm{II}_3(t,\xi)|||_0\le C(\xi^2+1)^{-\delta}$ respectively. Hence, in order to prove (6.20), it is sufficient to show that $\mathrm{II}_2(t)$ can be extended to a bounded operator in X and that $|||\mathrm{II}_2(t)|||_0\le C$. But this fact readily follows from Lemma 3.1. In fact, we have for $w\in\mathscr{D}(A(t)^{2\delta})$,

$$\begin{aligned}
&\|A(t)^{-1}P(t)A(t)^{2\delta}w\|_0\\
&\qquad\le C\|A(t)^{-2\delta}A(t)^{-1/2}P(t)A(t)^{2\delta}w\|_0\\
&\qquad\le Cq(t)^{2\delta}\|A(t)^{-1/2}P(t)A(t)^{2\delta}w\|_{-2\delta}\\
&\qquad\le Cq(t)^{2\delta}\|A(t)^{2\delta}w\|_{-2\delta}\le C\|w\|_0\,,
\end{aligned}$$

which implies that $|||\mathrm{II}_2(t)|||_0\le C$ since $\mathscr{D}(A(t)^{2\delta})$ is dense in X. Q.E.D.

LEMMA 6.2. *Let* $g_{j,p}(t,\varepsilon)$ $(j=1,2,\cdots,p=0,1,\cdots,n)$ *be the functions defined in* (6.2). *Then, for any sufficiently small* $\delta>0$, *there exists* $\varepsilon(\delta)$ *such that for* $\varepsilon<\varepsilon(\delta)$,

$$\sum_{j=1}^{\infty}\int_{-\infty}^{\infty}\|g_{j,p}(t,\varepsilon)\|_0^2\,dt \le \delta h^{1/2+1/2m-2(n+1)} \qquad (0<m<1)\,.$$
$$(\le \delta h^{1-2(n+1)}\log h \qquad (m=1).)$$

Here we should note that $\varepsilon(\delta)$ *is taken independently of* $h>2$.

Proof. We shall consider only the case of $0<m<1$. Two different methods of estimates will be employed in proving this lemma.

Case 1, $0\le p<n+1-1/4-1/4m$: By the definition of $g_{j,p}(t,\varepsilon)$, we have

$$\sum_{j=1}^{\infty}\int_{-\infty}^{\infty}\|g_{j,p}(t,\varepsilon)\|_0^2\,dt \le \sum_{j=1}^{\infty}(\mu_j+h)^{-2(n-p+1)}\int_{-\infty}^{\infty}dt\Big(\int_{-\infty}^{\infty}\|K_t^{(p)}(t-s;h)H_2(t,s,\varepsilon)u_j(s)\|_0\,ds\Big)^2\,.$$

Set $\mathrm{I}(t,s,j)=\|K_t^{(p)}(t-s;h)H_2(t,s,\varepsilon)u_j(s)\|_0$. Then, by virtue of Lemma 6.1 and (2.14) in Lemma 2.2, $\mathrm{I}(t,s,j)$ is estimated as follows:

$$\begin{aligned}\mathrm{I}(t,s,j)&=\|[K_t^{(p)}(t-s;h)A(t)][A(t)^{-1}(A(t)-A(s))\varphi_{t,\varepsilon}(s)]u_j(s)\|_0\\&\le Ch^{-p}|t-s|^{1-\alpha}\|u_j(s)\|_0\,, &&\text{for } |t-s|\le 2\varepsilon\,.\\&(=0 &&\text{for } |t-s|>2\varepsilon)\end{aligned}$$

Hence, it follows that

$$\begin{aligned}\int_{-\infty}^{\infty}dt\Big(\int_{-\infty}^{\infty}\mathrm{I}(t,s,j)ds\Big)^2 &\le Ch^{-2p}\int_{-\infty}^{\infty}dt\Big(\int_{|t-s|\le 2\varepsilon}|t-s|^{1-\alpha}\|u_j(s)\|_0\,ds\Big)^2\\&\le Ch^{-2p}\int_{|s|\le 2\varepsilon}|s|^{1-\alpha}ds\int_{|r|\le 2\varepsilon}|r|^{1-\alpha}dr\\&\quad\cdot\int_{-\infty}^{\infty}\|u_j(t+s)\|_0\|u_j(t+r)\|_0\,dt\\&\le Ch^{-2p}\varepsilon^{4-2\alpha}\,.\end{aligned}$$

(We have used that $0<\alpha<2$ and $\int_{-\infty}^{\infty}\|u_j(t+s)\|_0\|u_j(t+r)\|_0\,dt\le 1$ (Schwarz's inequality).) On the other hand, we have proved in Proposition 2.1 that if $2(n-p+1)>1/2+1/2m$,

$$\sum_{j=1}^{\infty}(\mu_j+h)^{-2(n-p+1)}\le Ch^{1/2+1/2m-2(n-p+1)}\,. \tag{6.25}$$

Hence, by combining (6.24) and (6.25), we have the desired estimate.

Case 2, $n \geq p \geq n + 1 - 1/4 - 1/4m > (1/4 + 1/4m)$: $(m \leq 1/3$[1]

It is easily seen that the operator $K_t^{(p)}(t-s\,; h)H_2(t, s, \varepsilon)$ belongs to $B_2(X)$ since $(\xi^2 + h + E(t, \xi))^{-1}A(t)$ can be extended to a bounded operator in X. Therefore, by virtue of Proposition 3.1, we have

$$\begin{aligned}\sum_{j=1}^{\infty} \|g_{j,p}(t,\xi)\|_0^2 &\leq h^{-2(n-p+1)} \sum_{j=1}^{\infty} \left\| \int_{-\infty}^{\infty} K_t^{(p)}(t-s\,; h)H_2(t,s,\varepsilon)u_j(s)ds \right\|_0^2 \\ &= h^{-2(n-p+1)} \int_{-\infty}^{\infty} |||K_t^{(p)}(t-s\,; h)H_2(t,s,\varepsilon)|||_2^2\, ds\,.\end{aligned} \tag{6.26}$$

Furthermore, it follows from (2.14) in Lemma 2.2 that

$$\begin{aligned}&|||K_t^{(p)}(t-s\,; h)H_2(t,s,\varepsilon)|||_2 \\ &\quad = |||[K_t^{(p)}(t-s\,; h)A(t)][A(t)^{-1}(A(t)-A(s))\varphi_{t,\varepsilon}(s)]|||_2 \\ &\quad \leq C\varepsilon\, |||K_t^{(p)}(t-s\,; h)A(t)|||_2\,.\end{aligned} \tag{6.27}$$

On the other hand, with the aid of the Parseval equality we have

$$\begin{aligned}&\int_{-\infty}^{\infty} |||K_t^{(p)}(t-s\,; h)A(t)|||_2^2\, dt \\ &\quad = C\int_{-\infty}^{\infty} |||(\xi^2 + h + E(t,\xi))^{-(p+1)}A(t)|||_2^2\, d\xi \\ &\quad \leq C\int_{-\infty}^{\infty} |||(\xi^2 + h + E(t,\xi))^{-p}|||_2^2\, d\xi \\ &\quad = C\sum_{j=1}^{\infty}\int_{-\infty}^{\infty} (\xi^2 + h + \alpha_j(t,\xi))^{-2p} d\xi\,.\end{aligned} \tag{6.28}$$

Hence, by combinig (6.27) and (6.28) with (6.26), we have

$$\sum_{j=1}^{\infty} \|g_{j,p}(t,\varepsilon)\|_0^2 \leq C\varepsilon h^{-2(n-p+1)} \sum_{j=1}^{\infty}\int_{-\infty}^{\infty} (\xi^2 + h + \alpha_j(t,\xi))^{-2p} d\xi\,,$$

which together with Lemma 4.1 implies that

$$\sum_{j=1}^{\infty}\int_{-\infty}^{\infty} \|g_{j,p}(t,\varepsilon)\|_0^2\, dt \leq C\varepsilon h^{1/2+1/2m-2(n+1))}\,.$$

This completes the proof. Q.E.D.

LEMMA 6.3. *Let $e_{j,p}(t,\varepsilon)$ be the function defined in* (6.2). *Then, for any sufficiently small $\delta > 0$, there exist $\varepsilon(\delta)$ and $h(\delta)$ such that for $h > h(\delta)$,*

$$\sum_{j=1}^{\infty}\int_{-\infty}^{\infty} \|e_{j,p}(t,\varepsilon(\delta))\|_0^2\, dt \leq \delta h^{1/2+1/2m-2(n+1)} \qquad (0 < m < 1)\,,$$

$$(\leq \delta h^{1-2(n+1)} \log h \qquad (m = 1.)$$

1) If $m > 1/3$, it is enough to consider only the case 1.

Proof.

Case 1, $n \geq p \geq 1$:

By using (6.7) instead of (6.5), the proof is obtained exactly in the same way as in the proof of Lemma 6.2.

Case 2, $p = 0$:

Recalling the definition of $H_0(t, s, \varepsilon)$ given by (6.1), we rewrite $H_0(t, s, \varepsilon)$ as follows:

$$\begin{aligned} H_0(t, s, \varepsilon) &= -(P(t) - P(s))\varphi_{t,\varepsilon}(s)\frac{d}{ds} - (P(t) - P(s))\varphi'_{t,\varepsilon}(s) + P'(s)\varphi_{t,\varepsilon}(s) \\ &= \mathrm{I}_1(t, s, \varepsilon) + \mathrm{I}_2(t, s, \varepsilon) + \mathrm{I}_3(t, s, \varepsilon) , \end{aligned}$$

where we put $P'(s) = \dfrac{d}{ds}P(s)$ and $\varphi'_{t,\varepsilon}(s) = \dfrac{d}{ds}\varphi_{t,\varepsilon}(s)$. Then, $e_{j,0}(t, \varepsilon)$ is rewritten as follows:

$$\begin{aligned} e_{j,0}(t, \varepsilon) &= (\mu_j + h)^{-(n+1)} \int_{-\infty}^{\infty} K_t^{(0)}(t - s; h) \sum_{k=1}^{3} \mathrm{I}_k(t, s, \varepsilon)u_j(s)ds \\ &= e_{j,0,1}(t, \varepsilon) + e_{j,0,2}(t, \varepsilon) + e_{j,0,3}(t, \varepsilon) . \end{aligned}$$

(1) Case 2–1, estimate of $\sum_{j=1}^{\infty} \int_{-\infty}^{\infty} \| e_{j,0,1}(t, \varepsilon) \|_0^2 \, dt$:

We note that there exists a constant C independent of j such that

$$\left\| \frac{d}{ds}u_j(s) \right\|_0^2 ds \leq C\mu_j . \tag{6.29}$$

Put $\mathrm{II}_1(t, s, j) = \|K_t^{(0)}(t - s; h)\, \mathrm{I}_1(t, s, \varepsilon)u_j(s)\|_0$. Then, an argument similar to the proof of the case 1 in Lemma 6.2 shows that

$$\begin{aligned} \mathrm{II}_1(t, s, j) &\leq Ch^{-1/2}|t - s|^{1-\alpha} \left\| \frac{d}{ds}u_j(s) \right\|_0 , &\quad \text{for } |t - s| \leq 2\varepsilon . \\ (&= 0 , &\quad \text{for } |t - s| > 2\varepsilon,) \end{aligned}$$

Furthermore, by using (6.29) and this estimate, we obtain

$$\int_{-\infty}^{\infty} dt\left(\int_{-\infty}^{\infty} \mathrm{II}_1(t, s, j)ds\right)^2 \leq Ch^{-1}\varepsilon^{4-2\alpha}\mu_j .$$

Hence, by virtue of Proposition 2.1, it readily follows that

$$\begin{aligned} \sum_{j=1}^{\infty} \int_{-\infty}^{\infty} \| e_{j,0,1}(t, \varepsilon) \|_0^2 \, dt &\leq Ch^{-1}\varepsilon^{4-2\alpha} \sum_{j=1}^{\infty} (\mu_j + h)^{-2(n+1)}\mu_j \\ &\leq Ch^{-1}\varepsilon^{4-2\alpha}h^{1/2+1/2m-2n-1} . \end{aligned}$$

Choosing $\varepsilon(\delta)$ in the above estimate so that $C\varepsilon^{4-2\alpha} < \delta$, we can get the desired estimate for the term $\sum_{j=1}^{\infty} \int_{-\infty}^{\infty} \| e_{j,0,1}(t, \varepsilon) \|_0^2 \, dt$.

(2) Case 2–2, estimate of $\sum_{j=1}^{\infty} \int_{-\infty}^{\infty} \| e_{j,0,2}(t, \varepsilon) \|_0^2 \, dt$:

Set $\mathrm{II}_2 (t, s, j) = \| K_t^{(0)}(t - s; h) \, \mathrm{I}_2 (t, s, \varepsilon) u_j(s) \|_0$. Then, by virtue of (6.6) in Lemma 6.1, we have

$$\begin{aligned} (6.30) \qquad \mathrm{II}_2 (t, s, j) &\le C(\varepsilon) h^{-1/2} |t - s|^{1-\alpha} \| u_j(s) \|_0 , \qquad && \text{for } |t - s| \le 2\varepsilon . \\ (&= 0 && \text{for } |t - s| \ge 2\varepsilon .) \end{aligned}$$

Hence, as in the proof of the above case 2–1, we have

$$\sum_{j=1}^{\infty} \int_{-\infty}^{\infty} \| e_{j,0,2}(t, \varepsilon) \|_0^2 \, dt \le C(\varepsilon) \cdot h^{-1} h^{1/2 + 1/2m - 2(n+1)} .$$

Choosing h such that $C(\varepsilon) h^{-1} < \delta$, we have have the desired estimate for the term $\sum_{j=1}^{\infty} \int_{-\infty}^{\infty} \| e_{j,0,2}(t, \varepsilon) \|_0^2 \, dt$.

Similarly we can see that $\sum_{j=1}^{\infty} \int_{-\infty}^{\infty} \| e_{j,0,3}(t, \varepsilon) \|_0^2 \, dt$ is estimated as in $\sum_{j=1}^{\infty} \int_{-\infty}^{\infty} \| e_{j,0,2}(t, \varepsilon) \|_0^2 \, dt$.

Combining the results of the cases 2–1 and 2–2, the proof is completed. Q.E.D.

A method similar to those given in the proofs of Lemmas 6.2 and 6.3 can be applied also to $d_{j,p}(t, \varepsilon)$ and $f_{j,p}(t, \varepsilon)$. Thus the proof of Lemma 4.4 is completed.

7. Generalizations.

The method developed in the preceding sections can be applied to more general problems.

7.1. Multi-dimensional case.

Let us consider the following problem:

$$(7.1) \qquad -\sum_{j=1}^{k} \frac{\partial^2}{\partial x_j^2} u - \Delta_y u = \lambda u , \qquad u \in H_0^1(\Omega) ,$$

where $\Delta_y = \sum_{j=1}^{n} \frac{\partial^2}{\partial y_j^2}$.

Here we impose the following assumptions on a domain Ω in $\boldsymbol{R}^{n+k}$.

(A–1) Ω is a domain of the form $\Omega = \{(x, y) \mid x \in \boldsymbol{R}^k, y \in \Omega(x) \subset \boldsymbol{R}^n\}$, where $\Omega(x)$ is a bounded domain for each fixed $x \in \boldsymbol{R}^k$.

(A–2) There exists a family of differentiable mappings of class C^∞, $\{g(x, y) = (g_i(x, y))_{i=1}^n\}$, from $\Omega(x)$ onto $\Omega(0)$ satisfying the following assumptions:

Let m be a positive constant such that $0 < m \leq k/n$ and let C be a positive constant independent of $x, y \in \Omega(x)$ and $\xi \in \boldsymbol{R}^n$.

(1) For the Jacobian $J(x, y) = \left|\det\left(\frac{\partial}{\partial y_j} g_i(x, y)\right)\right|$,

$$C(1 + |x|)^{mn} \leq J(x, y) \leq C(1 + |x|)^{mn} .$$

(2) For $g_{ij}(x, y) = \frac{\partial}{\partial y_i} g_i(x, y)$,

$$C(1 + |x|)^m |\xi|^2 \leq \left(\sum_{i,j=1}^n g_{ij}(x, y)\xi_i\right)^2 \leq C(1 + |x|)^m |\xi|^2 .$$

(3) For any multi-index α with $|\alpha| \leq 2$, $\left|\left(\frac{\partial}{\partial x}\right)^\alpha g_i(x, y)\right| \leq C(1 + |x|)^{m-|\alpha|}$.

If a domain Ω satisfies the above assumptions (A–1) and (A–2), we say that Ω belongs to $D(m)$.

THEOREM 7.1. *Let Ω be a domain belonging to $D(m)$ with $0 < m \leq k/n$. Let $\{\alpha_j(x)\}_{j=1}^\infty$ be eigenvalues of the operator $-\Delta_y$ with the domain of definition $\mathscr{D}(\Delta_y) = H_0^1(\Omega(x)) \cap H^2(\Omega(x))$. $N(h)$ denotes the number of eigenvalues less than h of the problem* (7.1). *Then,*

$$N(h) \sim C \sum_{j=1}^\infty \int_{\Omega_j(h)} (h - \alpha_j(x))^{k/2} dx \qquad \text{as } h \to \infty ,$$

where $C = ((2\sqrt{\pi})^k \Gamma(1 + k/2))^{-1}$ and $\Omega_j(h) = \{x \in \boldsymbol{R}^K \mid \alpha_j(x) > h\}$.

7.2. Case of domains with a finite number of holes:

Consider the following eigenvalue problem:

$$(7.2) \qquad -\frac{\partial^2}{\partial x^2} u - \frac{\partial^2}{\partial y^2} u = \lambda u , \qquad u \in H_0^1(\Omega) .$$

Here we assume that an open domain $\Omega = \{(x, y) \mid -\infty < x < \infty, y \in \Omega(x)\}$ with the smooth boundary is decomposed into

$$(-\infty, -R) \times (0, q(x)) \cup \Omega_0 \cup (R, \infty) \times (0, q(x)) ,$$

where R is some constant and $q(x)$ is a smooth function belonging to $K(m)$, while Ω_0 is not necessarily a simply connected domain but may have a finite number of holes.

THEOREM 7.2. *Let Ω be an open domain satisfying the above assumptions. Let $\{\alpha_j(x)\}_{j=1}^{\infty}$ be eigenvalues of the operator* $-\dfrac{\partial^2}{\partial y^2}$ with the *domain of definition $H_0^1(\Omega(x)) \cap H^2(\Omega(x))$. Then,*

$$N(h) \sim (\pi)^{-1} \int_{\Omega_j(h)} (h - \alpha_j(x))^{1/2} dx \qquad \text{as } h \to \infty .$$

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