

A NOTE ON GORENSTEIN RINGS OF EMBEDDING CODIMENSION THREE

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1. Let $A = R/\alpha$, where R is a regular local ring of arbitrary dimension and α is an ideal of R . If A is a Gorenstein ring and if height $\alpha = 2$, it is easily proved that A is a complete intersection, i.e., α is generated by two elements (Serre [5], Proposition 3). Hence Gorenstein rings which are not complete intersections are of embedding codimension at least three. An example of these rings is found in Bass' paper [1] (p. 29). This is obtained as a quotient of a three dimensional regular local ring by an ideal which is generated by five elements, i.e., generated by a regular sequence plus two more elements. In this paper, suggested by this example, we prove that if A is a Gorenstein ring and if height $\alpha = 3$, then α is minimally generated by an odd number of elements. If A has a greater codimension, presumably there is no such restriction on the minimal number of generators for α , as will be conceived from the proof.

In the following the basic results of the two famous papers Bass [2] and Matlis [4] are taken for granted.

2. In this paper we shall consider only Noetherian local rings. If R is a local ring with the maximal ideal \mathfrak{m} , we sometimes say that the pair (R, \mathfrak{m}) is a local ring. Let R be a ring. If x, y, \dots, z are elements of R , (x, y, \dots, z) denotes the ideal they generate. For an R -module M , $\text{hd } M$ denotes the homological dimension of M over R . If R is a regular local ring, $\text{hd } M < \infty$ for any finite R -module M and it holds that $\text{hd } M + \text{depth } M = \dim R$.

LEMMA 1. *Let R be a regular local ring and let \mathfrak{q} be a primary ideal belonging to the maximal ideal of R . Suppose that $\mathfrak{q} = \bigcap_{i=1}^n \mathfrak{q}_i$ is an irredundant decomposition of \mathfrak{q} by n irreducible ideals \mathfrak{q}_i . Let*

$0 \rightarrow F_d \rightarrow F_{d-1} \rightarrow \cdots \rightarrow F_1 \rightarrow F_0 \rightarrow 0$ be a minimal free resolution of R/\mathfrak{q} . Then the rank of F_d is equal to n .

Proof. Since $\text{depth } R/\mathfrak{q} = 0$, $\dim R = d$. Therefore we have an isomorphism $\text{Ext}_R^d(R/\mathfrak{q}, R) \cong \text{Hom}_R(R/\mathfrak{q}, E)$, where E denotes the injective envelope of the residue class field. (See [2] Theorem 4.1) Thus the rank of F_d is equal to the minimal number of generators for $\text{Hom}_R(R/\mathfrak{q}, E)$. On the other hand, the injective envelope of the module $\text{Hom}_R(\text{Hom}_R(R/\mathfrak{q}, E), E) \cong R/\mathfrak{q}$ is an n copies of E , and in general these two numbers are identical, because a minimal surjection $F \rightarrow \text{Hom}_R(R/\mathfrak{q}, E) \rightarrow 0$ with F free gives an essential injection $0 \rightarrow \text{Hom}_R(\text{Hom}_R(R/\mathfrak{q}, E), E) \rightarrow \text{Hom}_R(F, E)$. (cf. [4] Theorem 2.3 and Theorem 4.2)

COROLLARY. Let R be a Gorenstein ring and \mathfrak{q} a perfect ideal of grade d . Let $0 \rightarrow F_d \rightarrow F_{d-1} \rightarrow \cdots \rightarrow F_0 \rightarrow 0$ be as in the Lemma. Then the rank of F_d is the “type” of the Cohen-Macaulay ring R/\mathfrak{q} .

Proof. Let x_1, x_2, \dots, x_r be a maximal regular sequence for both R and R/\mathfrak{q} . Then it is well known that the complex:

$$0 \longrightarrow F_d \otimes R/\mathfrak{x} \longrightarrow F_{d-1} \otimes R/\mathfrak{x} \longrightarrow \cdots \longrightarrow F_0 \otimes R/\mathfrak{x} \longrightarrow 0$$

is a minimal free resolution of $R/\mathfrak{q} + \mathfrak{x}$, over R/\mathfrak{x} , where $\mathfrak{x} = (x_1, \dots, x_r)$. (To prove this we only have to show the acyclicity, and this can be done by induction on r .) Since the isomorphisms used in the proof of Lemma 1 hold for a Gorenstein ring R/\mathfrak{x} , the assertion follows.

LEMMA 2. Let A be an Artin Gorenstein local ring and \mathfrak{a} and \mathfrak{b} be two ideals of A . If $0:\mathfrak{a} = 0:\mathfrak{b}$, then $\mathfrak{a} = \mathfrak{b}$.

Proof. Since for any ideal \mathfrak{a} of A , we have $0:[0:\mathfrak{a}] = \mathfrak{a}$, the assertion is clear. (cf. [3] Satz 1.44)

LEMMA 3. Let (R, \mathfrak{m}) be a local ring and \mathfrak{q} an \mathfrak{m} -primary irreducible ideal, and let y be an element of R which is not in \mathfrak{q} . Assume that $\mathfrak{q}:y = \mathfrak{q} + (f_1, f_2, \dots, f_n)$. Then we have $\bigcap_{i=1}^n [\mathfrak{q}:f_i] = \mathfrak{q}:(f_1, \dots, f_n) = \mathfrak{q} + (y)$. Moreover the following two conditions are equivalent to each other:

- i) the intersection of ideals $\bigcap_{i=1}^n [\mathfrak{q}:f_i]$ is irredundant.
- ii) $\{f_1, f_2, \dots, f_n\}$ is (a set of representatives of) a minimal generators for the ideal $\mathfrak{q} + (f_1, f_2, \dots, f_n)$ modulo \mathfrak{q} .

Proof. The equality $\bigcap_{i=1}^n [q : f_i] = q : (f_1, \dots, f_n)$ is easily verified (without the assumption that q is irreducible and \mathfrak{m} -primary). We prove the second equality. It is obvious that $q + (y) \subset q : (f_1, \dots, f_n)$. Assume $z \in q : (f_1, \dots, f_n)$. Then $zf_i \in q$ for each i , which implies that $q : z \supset q : y$. Therefore $q : (y) = q : (y, z)$, and considering everything modulo q , we conclude by Lemma 2 that $q + (y) = q + (y, z)$, which proves $q : (f_1, \dots, f_n) \subset q + (y)$. (Recall that R/q is an Artin Gorenstein ring if and only if q is \mathfrak{m} -primary and irreducible.)

To prove the second assertion assume $\bigcap_{i=1}^n [q : f_i] = \bigcap_{i=2}^n [q : f_i]$ for instance. Then $q : (f_1, f_2, \dots, f_n) = q : (f_2, \dots, f_n)$, and again by Lemma 2, $q + (f_1, \dots, f_n) = q + (f_2, \dots, f_n)$. This shows that ii) implies i). The other implication is immediate.

LEMMA 4. *Let α be an irreducible \mathfrak{m} -primary ideal of a local ring (R, \mathfrak{m}) . If \mathfrak{b} is another irreducible ideal which contains α , then there is an element y such that $\mathfrak{b} = \alpha : y$. Conversely for any element y of R which is not in α , $\alpha : y$ is irreducible.*

Proof. Let E be the injective envelope of R/\mathfrak{m} . From the canonical epimorphism $R/\alpha \rightarrow R/\mathfrak{b} \rightarrow 0$ we obtain a monomorphism $0 \rightarrow \text{Hom}_R(R/\mathfrak{b}, E) \rightarrow \text{Hom}_R(R/\alpha, E)$. Since R/α and R/\mathfrak{b} are both self-injective, $\text{Hom}_R(R/\alpha, E) \cong R/\alpha$ and $\text{Hom}_R(R/\mathfrak{b}, E) \cong R/\mathfrak{b}$. Therefore the above monomorphism shows the existence of y satisfying $\mathfrak{b} = \alpha : y$. An \mathfrak{m} -primary ideal q is irreducible if and only if $\dim_k \text{Hom}_R(k, R/q) = 1$, where $k = R/\mathfrak{m}$. Consequently the irreducibility of $\alpha : y$ follows immediately from the fact that we can define a monomorphism $R/[\alpha : y] \rightarrow R/\alpha$ by $1 \bmod [\alpha : y] \mapsto y \bmod \alpha$.

THEOREM. *Let (R, \mathfrak{m}) be a regular local ring and α be an ideal of height three, such that R/α is a Gorenstein ring. Then α is minimally generated by an odd number of elements.*

Proof. We denote by $\mu(I)$ the number of minimal generators of an ideal I of a local ring. With this notation it is easy to see that if $x \in \mathfrak{m}$ is a regular element on R/α , then $\mu(\alpha) = \mu(\alpha + (x)/(x))$, where $\alpha + (x)/(x)$ is an ideal of $R/(x)$. Note also that, in this case, $\text{height } \alpha = \text{height } \alpha + (x)/(x)$ and $R/\alpha + (x) = R/(x)/\alpha + (x)/(x)$ is a Gorenstein ring. Thus we may assume $\text{depth } R/\alpha = 0$, because whenever $\text{depth } R/\alpha > 0$, there is a regular element on R/α in $\mathfrak{m} - \mathfrak{m}^2$. This amounts to assuming that

α is \mathfrak{m} -primary and dimension $R = 3$, since R/α is a Cohen-Macaulay ring.

Let $\mu(\alpha) = N = n + 3$. If $n = 0$, there is nothing to prove. Let $n > 0$, and let $\alpha = (x_1, x_2, x_3, f_1, f_2, \dots, f_n)$, where we may assume that $(x_1, x_2, x_3) = \mathfrak{x}$ is already \mathfrak{m} -primary. Since both α and \mathfrak{x} are irreducible, by Lemma 4, there is y such that $\alpha = \mathfrak{x} : y$.

This y can be chosen in such a way that x_2, x_3, y is a regular sequence. For suppose that $\mathfrak{p}_1, \mathfrak{p}_2, \dots, \mathfrak{p}_t$ are the associated primes of (x_2, x_3) and that

$$\begin{aligned} y \notin \mathfrak{p}_i & \quad i = 1, 2, \dots, s \\ y \in \mathfrak{p}_i & \quad i = s + 1, \dots, t. \end{aligned}$$

Since x_2, x_3 is a regular sequence, height $\mathfrak{p}_i = 2$ for every i . If $s = t$, the sequence x_2, x_3, y is a regular sequence. Let $0 \leq s < t$ and $D = \mathfrak{x} \cap \mathfrak{p}_1 \cap \dots \cap \mathfrak{p}_s$. Then $\mathfrak{p}_{s+1} \cup \dots \cup \mathfrak{p}_t \not\supseteq D$. For any element $z \in D$ such that $z \notin \mathfrak{p}_{s+1} \cup \dots \cup \mathfrak{p}_t$, $x_2, x_3, y + z$ is a regular sequence and obviously $\mathfrak{x} : y + z = \alpha$. From now on y is assumed to be chosen in this way.

We are interested in the ideal \mathfrak{q} generated by x_1, x_2, x_3 and y . We assert first that these four elements are a minimal generating set for \mathfrak{q} . For if $x_1 \in (x_2, x_3, y)$, $x_1 = a_2x_2 + a_3x_3 + by$ with suitable elements a_2, a_3, b . This b is an element of $\mathfrak{x} : y$, so that b is a linear combination of x_1 and f_i . But this contradicts the fact that x_i and f_i form a minimal basis for α . The same is true with x_2 and x_3 . Since it is clear that y cannot be omitted, the above assertion is proved. On the other hand, by Lemma 3, $\mathfrak{q} = \bigcap_{i=1}^n [\mathfrak{x} : f_i]$, and therefore, by Lemma 1 and Lemma 4, $\dim_k \operatorname{Tor}_3^R(k, R/\mathfrak{q}) = n$, where $k = R/\mathfrak{m}$.

Let $0 \rightarrow F_3 \rightarrow F_2 \rightarrow F_1 \rightarrow F_0 \rightarrow R/\mathfrak{q} \rightarrow 0$ be a minimal free resolution of R/\mathfrak{q} . So far we have proved that $\operatorname{rank} F_1 = 4$ and $\operatorname{rank} F_3 = n$. Since $\operatorname{rank} F_0 = 1$, $\operatorname{rank} F_2 = N$. We may assume that the homomorphism $F_1 \rightarrow F_0$ is defined by the column vector φ such that ${}^t\varphi = [x_1 \ x_2 \ x_3 \ y]$ (where ${}^t\varphi$ denotes the transposed matrix of φ); if elements of F_1 are represented by row vectors, their images by φ are obtained by the usual matrix product. Let M be a matrix that defines $F_2 \rightarrow F_1$, and let I_i be the ideal generated by those elements that appear in the i -th column of M ($i = 1, 2, 3, 4$). It is easy to see that these I_i depend only on the vector φ and not on the choice of M . In fact I_1 is nothing but $(x_2, x_3, y) : x_1$, for example. Note $I_4 = \alpha$. By Lemma 4 and by the choice of y , I_1 is irreducible. We are going to prove that $\mu(I_1) = N - 2$, which completes

the proof of the theorem by induction on μ of irreducible \mathfrak{m} -primary ideals, because the least μ is three.

Consider the following $N \times 4$ matrix M_1 :

$$M_1 = \begin{bmatrix} -y & 0 & 0 & x_1 \\ 0 & -y & 0 & x_2 \\ 0 & 0 & -y & x_3 \\ a_{11} & a_{12} & a_{13} & f_1 \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & a_{n3} & f_n \end{bmatrix}$$

where a_{ij} are elements satisfying $a_{i1}x_1 + a_{i2}x_2 + a_{i3}x_3 + f_iy = 0$, their existence being a consequence of the assumption that $f_i \in \mathfrak{x} : y$. Since each row of M_1 is in $\text{Ker } \varphi$, there is an $N \times N$ matrix T such that $TM = M_1$. This T can be regarded as an R -endomorphism of each I_i . Then since T must be an R -automorphism of I_i , T is invertible and it follows that $I_1 = (y, a_{11}, a_{21}, \dots, a_{n1})$. We want to show that these elements are precisely a minimal basis for I_1 . Let v denote the first and v_j the $(3+j)$ -th row of M_1 , where $j = 1, 2, \dots, n$. Assume for instance $a_{11} \in (y, a_{21}, \dots, a_{n1})$. Then there are elements b and c_j such that the first component of $u = bv + \sum_{j=2}^n c_j v_j$ is a_{11} . Set $u' = v_1 - u$. Then the first component of u' is 0, and the fourth component of u' has the form $f_1 - \alpha$, where α is an element of (x_1, f_2, \dots, f_n) . Let d_2 and d_3 be the 2nd and the 3rd component of u' respectively. Since $u' \in \text{Ker } \varphi$, d_2, d_3 and $f_1 - \alpha$ give a relation of x_2, x_3, y , i.e., $d_2x_2 + d_3x_3 + (f_1 - \alpha)y = 0$. Since x_2, x_3, y is a regular sequence, it follows that $f_1 - \alpha \in (x_2, x_3)$, whence $f_1 \in (x_1, x_2, x_3, f_2, \dots, f_n)$, which is impossible. That y is not superfluous is similarly proved. Q.E.D.

COROLLARY. *Let R be a Gorenstein ring and α be an ideal of homological dimension two. If R/α is a Gorenstein ring, then $\mu(\alpha)$ is odd.*

Proof. By the first part of the proof of Corollary to Lemma 1, we may assume that R/α is Artinian.

Let $\mathfrak{x}, y, \mathfrak{q}$ etc. be as in the proof of the theorem. In order to repeat the same argument as before we only have to show that $\text{hd } R/\mathfrak{q}$ is finite. But we have an exact sequence: $0 \rightarrow R/\alpha \xrightarrow{\varphi} R/\mathfrak{x} \rightarrow R/\mathfrak{q} \rightarrow 0$, where φ is defined by $\varphi(1 \bmod \alpha) = y \bmod \mathfrak{x}$. Since $\text{hd } R/\mathfrak{x} = \text{hd } R/\alpha = 3$, it follows that $\text{hd } R/\mathfrak{q} \leq 3$. (In fact $\text{hd } R/\mathfrak{q} = 3$, since R/\mathfrak{q} is Artinian.) Q.E.D.

Remark. It can be proved that over a local ring R the existence of an ideal α of finite homological dimension such that R/α is a Gorenstein ring implies that R itself is a Gorenstein ring. Therefore in the above corollary the condition that R is a Gorenstein ring is unnecessary.

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