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ON THE GROBAL DIMENSION OF ORE-EXTENSIONS

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Introduction. Let S be a ring and d be a derivation of S. The Ore-extension $S(X, d)$ is the ring generated by S and an indeterminate X satisfying the ralation $Xa - aX = da$ for all a in S.

It can be deduced from [3, Theorem 2] that if S is a commutative noetherian ring and d is a derivation of S, such that there exists a maximal ideal $\mathfrak{m}$ of S with (i) $d(\mathfrak{m}) \subset \mathfrak{m}$ (ii) gl. dim $S =$ gl. dim S_m, then l.gl. dim $S(X, d) = 1 +$ gl. dim S. In § 1, we prove the converse of the above proposition (see theorem 1.1) if S is a Dedekind ring containing field Q of rationals. This is a generalization of theorem of Rinehart [5, Propsition 2].

In § 2 we compute the l.gl. dim of $S(X, d)$ when S is a commutative noetherian ring containing Q and d is a derivation of S, such that $1 \in d(S)$ and for every $a \in S$ there exists an integer $n \geqslant 1$ such that $d^n(a) = 0$.

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§ 1. In this section we prove the following.

THEOREM 1.1. *Let S be a Dedekind ring which contains Q. Let d be a derivation of S, such that for every maximal ideal $\mathfrak{m}$ of S, $d\mathfrak{m} \not\subset \mathfrak{m}$. Then*

$$\text{l.gl. dim}\, S(X, d) = 1 .$$

For the proof of the theorem, we need two lemmas. We start with

LEMMA 1.2. *Under the hypothesis of Theorem 1.1, for every maximal ideal $\mathfrak{m}$ of S, $R\mathfrak{m}$ (resp. $\mathfrak{m}R$) is a maximal left (resp. right) ideal of R, where R denotes $S(X, d)$.*

Proof. Let I be a left ideal of R such that $R\mathfrak{m} \subset I \subset R$, where $\mathfrak{m}$

is a maximal ideal of S. Suppose $I \neq R$. Then we will show that $R\mathfrak{m} = I$.

For, if not, then there exists $f \in I$ such that $f \notin R\mathfrak{m}$. Consider an element g of I of smallest degree and not belonging to $R\mathfrak{m}$. Without loss of generality we can take g to be of the form $g = X^k + \sum_{0 \leqslant i \leqslant k-1} X^i a_i$, $k \geqslant 1$.

Since $d\mathfrak{m} \not\subset \mathfrak{m}$, there exists $b \in \mathfrak{m}$ such that $db \notin \mathfrak{m}$. Consider $g' = X^k b - bg$. It is easy to see that $g' \in I$ and $g' = X^{k-1}(kdb - ba_{k-1}) + \sum_{0 \leqslant i \leqslant k-2} X^i a'_i$. This shows that $g' \in R\mathfrak{m}$. Therefore $kdb - ba_{k-1} \in \mathfrak{m}$, i.e. $kdb \in \mathfrak{m}$. But $db \notin \mathfrak{m}$ and k is a unit in S. Hence we get a contradiction. Therefore $R\mathfrak{m} = I$.

This completes the proof of lemma 1.2.

LEMMA 1.3. *Let S and R be as given in Theorem* 1.1. *If J is a nonzero projective left ideal of R and $J_1 = J + R\phi$ for some $\phi \in R$ such that $\mathfrak{m}\phi \subset J$ for some maximal ideal $\mathfrak{m}$ of S, then J_1 is also a projective ideal of R.*

Proof. $\mathfrak{m}\phi \subset J$ implies that if $J \neq J_1$ then $J_1/J \simeq R/R\mathfrak{m}$. Also $J_1 \neq J$ implies that $\operatorname{Hom}_R(J_1, R) \xrightarrow{\operatorname{Hom}(i,R)} \operatorname{Hom}_R(J, R)$ is not a surjective map, where $i: J \to J_1$ inclusion map.

For, if $\operatorname{Hom}(i, R)$ is a surjective map, then $\operatorname{Hom}_R(J_1, F) \xrightarrow{\operatorname{Hom}(i,F)} \operatorname{Hom}_R(J, F)$ is surjective for every finitely generated free left module F of R.

Let $p: F_0 \to J$ be a surjection from a finitely generated free module F_0 on to J. Consider the commutative diagram

$$\begin{array}{ccc} \operatorname{Hom}_R(J_1, F_0) & \xrightarrow{\operatorname{Hom}(i,F_0)} & \operatorname{Hom}_R(J, F_0) \\ {\scriptstyle \operatorname{Hom}(J_1,p)}\downarrow & & \downarrow{\scriptstyle \operatorname{Hom}(J,p)} \\ \operatorname{Hom}_R(J_1, J) & \xrightarrow{\operatorname{Hom}(i,J)} & \operatorname{Hom}_R(J, J) \end{array}$$

Since J is a projective module, we get $\operatorname{Hom}(J, p)$ to be a surjection. Hence $\operatorname{Hom}(i, J)$ is a surjective map. This implies that J is a direct summand of J_1. Since $J \neq 0$ and $J \neq J_1$, this gives a contradiction. Thus $\operatorname{Hom}(i, R)$ is not a surjective map.

Assume $J \neq J_1$. Consider the exact sequence

$$0 \longrightarrow J \xrightarrow{i} J_1 \longrightarrow J_1/J \longrightarrow 0 .$$

This gives rise to an exact sequence of right R-modules

$$\operatorname{Hom}_R(J_1, R) \xrightarrow{\operatorname{Hom}(J_1, R)} \operatorname{Hom}_R(J, R) \longrightarrow \operatorname{Ext}^1_R(J_1/J, R)$$
$$\longrightarrow \operatorname{Ext}^1_R(J_1, R) \longrightarrow 0 .$$

$J_1/J \simeq R/R\mathfrak{m}$ implies that $\operatorname{Ext}^1_R(J_1/J, R) \simeq \operatorname{Ext}^1_S(S/\mathfrak{m}, S) \otimes_S R$ as right R-modules. But $\operatorname{Ext}^1_S(S/\mathfrak{m}, S) \simeq S/\mathfrak{m}$. Therefore $\operatorname{Ext}^1_R(J_1/J, R) \simeq S/\mathfrak{m} \otimes_S R \simeq R/\mathfrak{m}R$. By Lemma 1.2, $R/\mathfrak{m}R$ is a simple right R-module. Also, $\operatorname{Hom}(i, R)$ is not a surjective map. Hence we get an exact sequence

$$\operatorname{Hom}_R(J_1, R) \to \operatorname{Hom}_R(J, R) \to \operatorname{Ext}^1_R(J_1/J, R) \to 0 .$$

This shows that $\operatorname{Ext}^1_R(J_1, R) = 0$. By a 'direct sum' argument, we can show that $\operatorname{Ext}^1_R(J_1, F) = 0$ for every finitely generated free left module F of R.

Let M be a finitely generated left module of R. Let $0 \to C \to F \to M \to 0$ be an exact sequence of left R-modules where F is free module of finite rank.

Then we get an exact sequence

$$0 = \operatorname{Ext}^1_R(J_1, F) \to \operatorname{Ext}^1_R(J_1, M) \to \operatorname{Ext}^2_R(J_1, C) .$$

But we know that $\text{l.gl.}\dim R \leqslant 2$. Also, since R is not semisimple, from [1, Theorem 1] it follows that

$$\text{l.gl.}\dim R = 1 + \sup_I \text{hd.}\, I .$$

where I ranges over all left ideals of R.

Therefore $\text{hd.}\, I \leqslant 1$ for every left ideal I of R. This gives $\operatorname{Ext}^2_R(J_1, C) = 0$. Therefore $\operatorname{Ext}^1_R(J_1, M) = 0$. Thus for every finitely generated R-module M we get $\operatorname{Ext}^1_R(J_1, M) = 0$. This proves that J_1 is a projective left ideal of R.

If $J = J_1$ then there is nothing to prove.

Thus the proof of Lemma 1.3 is complete.

Proof of Theorem 1.1. Let R denote $S(X, d)$. From [1, Theorem 1] it follows that it is enough to prove that every left ideal of R is projective.

Let I be a left ideal of R. For any integer $k \geqslant 0$ let

$$I_k = \left\{ a \,\middle|\, \begin{array}{l} a \in S, \text{ such that } a \text{ is leading coefficient} \\ \text{of some element of } I \text{ of degree } k \end{array} \right\} .$$

Then it is easy to see that we get an increasing sequence $I_0 \subset I_1 \subset I_2 \cdots$

of ideals of S. Let m be the least integer such that $I_m = I_n$ for $n \geqslant m$. Let k_0 be the least integer such that $I_{k_0} \neq 0$. Let $(b_k^1, \cdots, b_k^{n_k})$ be a set of generators of I_k for $k_0 \leqslant k \leqslant m$. By definition of I_k, there exist elements $(f_k^1, \cdots, f_k^{n_k})$ of I such that f_k^i is of degree k and with leading coefficient b_k^i for every i, $1 \leqslant i \leqslant n_k$.

Let $J_k = \sum R f_l^i$, $1 \leqslant i \leqslant n_l$, $k_0 \leqslant l \leqslant k$. Then we get an increasing sequence $0 \neq J_{k_0} \subset \cdots \subset J_m$ of left ideals of R such that $J_m = I$. It is easy to prove that $J_0 \simeq R \otimes_S I_{k_0}$ as left ideals of R.

Let $r = m - k_0$. We will prove the result by induction on r.

If $r = 0$, then $I = J_m = J_{k_0} \simeq R \otimes_S I_{k_0}$. Since S is a Dedekind ring, I_{k_0} is a projective ideal of S. This shows that I is a projective left ideal of R.

Assume the result for $r - 1 \geqslant 0$. Then by induction hypothesis J_{m-1} is a projective left ideal of R. Since $I_{m-1} \neq 0$, there exists an increasing sequence

$I_{m-1} = \mathscr{B}_0 \subset \mathscr{B}_1 \subset \mathscr{B}_2 \cdots \mathscr{B}_p = S$ of ideals of S such that $\mathscr{B}_i / \mathscr{B}_{i-1} \simeq S/\mathfrak{m}_i$ for some maximal ideal $\mathfrak{m}_i$ of S, $1 \leqslant i \leqslant p$.

Therefore $\mathscr{B}_i = \mathscr{B}_{i-1} + S\theta_i$ for some $\theta_i \in S$. We can take $\theta_p = 1$. Then there exists a maximal ideal $\mathfrak{m}_i$ such that $\mathfrak{m}_i \theta_i \subset \mathscr{B}_{i-1}$. Let $\mathscr{A}_i^j = J_{m-1} + R(f_m^1, \cdots, f_m^{i-1}) + R\theta_1 f_m^i + R\theta_2 f_m^i + \cdots + R\theta_j f_m^i$, $1 \leqslant i \leqslant n_m$, $1 \leqslant j \leqslant p$. Then $\mathscr{A}_i^j \subset \mathscr{A}_l^k$ if either $i \leqslant l$, or $i = l$ and $j \leqslant k$. Also $\mathscr{A}_{n_m}^p = J_m$. From the definition of $\mathscr{A}_i^j$ it follows that either $\mathscr{A}_i^j = \mathscr{A}_i^{j+1}$ or $\mathscr{A}_i^{j+1}/\mathscr{A}_i^j \simeq R/R\mathfrak{m}$ for some maximal ideal $\mathfrak{m}$ of R. Also, either $\mathscr{A}_1^1 = J_{m-1}$ or $\mathscr{A}_1^1/J_{m-1} \simeq R/R\mathfrak{m}$. Since J_{m-1} is R-projective by our assumption, by using Lemma 1.3 step by step, we get J_m $(=I)$ is a projective left ideal of R.

This proves theorem 1.1.

Remark. Theorem 1.1 shows that if S is a Dedekind ring containing $\boldsymbol{Q}$ and d is a derivation of S then

$$\text{l.gl. dim. } S(X, d) = 2 = 1 + \text{gl. dim. } S \text{ iff}$$

there exists a maximal ideal $\mathfrak{m}$ of S such that $d\mathfrak{m} \subset \mathfrak{m}$.

§ **2.** In this section we prove the following theorem.

THEOREM 2.1. *Let S be a commutative noetherian ring of global dimension $n < \infty$, such that $\boldsymbol{Q} \subset S$. Let d be a derivation of S such that $1 \in d(S)$ and for every $a \in S$ there exists an integer $k \geqslant 1$ such that $d^k(a) = 0$ then*

$$\text{l.gl. dim}\, S(X, d) = n \ .$$

First we state a lemma. [4, p. 78].

LEMMA 2.2. *Under the hypothesis of Theorem* 2.1, *if* $d(b) = 1$ *for* $b \in S$, *then the mapping*

$$\chi : S \to (S/Sb)[Y]$$
$$\chi(a) = \bar{a} + \overline{da}Y + \overline{d^2a} \quad Y^2/2\,! + \overline{d^3a} \quad Y^3/3\,! + \cdots$$

is an isomorphism of rings, where $\overline{d^i a}$ *denotes the image of* $d^i a$ *in* S/Sb *under the canonical mapping* $\eta : S \to S/Sb$.

Moreover, if D *is the* S/Sb*-derivation of* $S/Sb[Y]$ *given by* $DY = 1$, *then* χ *is an isomorphism of differential rings.*

This shows that it is sufficient to prove the theorem if $S = A[Y]$ where A is a commutative noetherian ring of finite global dimension which contains $\boldsymbol{Q}$ and d is the A-derivation of S given by $dY = 1$. Also it is easy to see that it is enough to prove the result in case A is a local ring.

So we prove the following theorem.

THEOREM 2.3. *Let* A *be a commutative noetherian local ring of global dimension* $n < \infty$ *such that* $\boldsymbol{Q} \subset A$. *Let* $S = A[Y]$ *and* d *be the* A*-derivation of* S *given by* $dY = 1$. *Then*

$$\text{l.gl. dim}\, S(X, d) = n + 1 \ .$$

Before proceeding further we will give some definitions and results which can be found in [7, § 15].

Let B be a ring, not necessarily commutative. Let T be a multiplicatively closed subset of B such that $1 \in T$.

DEFINITION. T is called *right* (*resp. left*) *permutable* if given $a \in B$ and $t \in T$, there exist $b \in A$ and $s \in T$ such that $tb = as$ (resp. $bt = sa$).

DEFINITION. T is called *right* (*resp. left*) *reversible* if $ta = 0$ (resp. $at = 0$) with $t \in T$, $a \in B$ implies $as = 0$ (resp. $sa = 0$) for some $s \in T$.

DEFINITION. A right (resp. left) ring of fractions of B with respect to T is a ring $B[T^{-1}]$ (resp. $[T^{-1}]B$) and a ring homomorphism $\phi : B \to B[T^{-1}]$ (resp. $\psi : B \to [T^{-1}]B$) satisfying

i) $\phi(s)$ (resp. $\psi(s)$) is invertible for every $s \in T$.

ii) every element in $B[T^{-1}]$ (resp. $[T^{-1}]B$) has the form

$$\phi(a)\phi(s)^{-1} \text{ (resp. } \psi(s)^{-1}\psi(a)) \quad \text{with } s \in T .$$

iii) $\phi(a) = 0$ (resp. $\psi(a) = 0$) iff $as = 0$ (resp. $sa = 0$) for some $s \in T$. Some results concerning $B[T^{-1}]$.

(a) If $B[T^{-1}]$ exists, it is unique up to isomorphism.

(b) $B[T^{-1}]$ exists iff T is right permutable and right reversible set.

(c) $B[T^{-1}]$ is B-flat as a left B-module.

(d) If both $B[T^{-1}]$ and $[T^{-1}]B$ exist, they are isomorphic.

We have similar results for $[T^{-1}]B$.

DEFINITION. *A ring B is said to be left coherent if every finitely generated left ideal of B is finitely presented.*

Let w.gl. dim B denote the weak global dimension of B. If B is left noetherian then l.gl. dim $B =$ w.gl. dim B. [2, Chapt. VI].

The proof of Theorem 2.3 depends upon the following proposition.

PROPOSITION 2.4. *Let $(T_i)_{i \in I}$ be a finite family of multiplicatively closed subsets of a ring R such that*

(i) *Each T_i is right permutable and right reversible.*

(ii) *For every family $(t_i)_{i \in I}$ of elements of R with $t_i \in T_i$ we have $\sum_{i \in I} t_i R = R$.*

(iii) *Every R_i is R-flat as a left R-module and as a right R-module, where $R_i = R[T_i^{-1}]$*

(iv) w.gl. dim $R < \infty$

(v) *R is left coherent.*

Then w.gl. dim $R \leqslant \sup_{i \in I}$ w.gl. dim R_i.

For a proof, see [6, Proposition 1].

Proof of Theorem 2.3. Let R denote the ring $S(X, d)$. Then under the hypothesis of Theorem 2.1, R is nothing but the A-algebra $A\{X, Y\}$ in two variables X and Y and with the relation $XY - YX = 1$.

Let $\mathfrak{m}$ be the maximal ideal of A. Let $T_1 = A[X] - \mathfrak{m}[X]$ and $T_2 = A[Y] - \mathfrak{m}[Y]$ be two multiplicatively closed subsets of R.

Since $S(X, d)$ is without proper divisors of zero T_1 and T_2 are right as well as left reversible.

To prove that T_1 is right permutable it is enough to show that given f in T_1 and Y^n there exist g in T_1 and h in $S(X, d)$ such that $Y^n g = fh$. Taking $g = f^{n+1}$ we see that $Y^n f^{n+1} = \sum_{0 \leqslant i \leqslant n} {}^nC_i d^i(f^{n+1}) Y^{n-i}$. But

$d^i(f^{n+1}) = fh_i$ for some h_i in $A[X]$. Therefore $Y^n f^{n+1} = fh$ where $h = \sum_{0\leqslant i\leqslant n} {}^nC_i h_i Y^{n-i}$.

Similarly we prove that T_1 is left permutable and T_2 is right and left permutable. This shows that $R[T_i^{-1}]$ is R-flat as a right R-module as well as left R-module for every $i = 1, 2$.

Since R is left noetherian and $\text{l.gl. dim}\, R \leqslant n + 2$ we see that all the conditions of the previous proposition except the second condition are satisfied.

Assume for the time being that the second condition is also satisfied. Then

$$\text{w.gl. dim}\, R \leqslant \max_i \text{w.gl. dim}\, R_i$$

when $R_i = R[T_i^{-1}]$.

Let d be the A-derivation of $A[Y]$ given by $dY = 1$. If S' is the localization of $A[Y]$ with respect to the prime ideal $\mathfrak{m}[Y]$ and d' is the derivation of S' induced by d then $R[T_2^{-1}]$ is nothing but the Ore-extension of S' with respect to d'. Hence $\text{w.gl. dim}\, R[T_2^{-1}] = \text{l.gl. dim}\, R[T_2^{-1}] \leqslant 1 + \text{gl. dim}\, S'$. But $\text{gl. dim}\, S' = n$. Therefore $\text{w.gl. dim}\, R[T_2^{-1}] \leqslant n + 1$.

Similarly we can show that $\text{w.gl. dim}\, R[T_1^{-1}] \leqslant n + 1$.

Hence $\text{w. gl. dim}\, R \leqslant n + 1$. But we already know that $n + 1 \leqslant \text{l.gl. dim}\, R = \text{w.gl. dim}\, R$.

Hence the equality.

The lemma given below shows that T_1 and T_2 satisfy the second condition of the proposition.

LEMMA 2.5. *Under the hypothesis of Theorem* 2.3, *if* $f \in T_1$ *and* $g \in T_2$ *then* $fR + gR = R$.

Proof. We will prove the result by using induction on the global dimension of A.

If $\text{gl. dim}\, A = 0$, then A is a field of char $= 0$. The result in this case is proved in [6, p. 25–26].

Assume the result for $n - 1$. Let $\text{gl. dim}\, A = n$. If $\mathscr{B} = fR + gR$ then by our induction hypothesis there exists an integer $r \geqslant 1$ such that $\mathfrak{m}^r \subset \mathscr{B} \cap A$, where $\mathfrak{m}$ is the maximal ideal of A. (Since for every prime ideal $\mathfrak{p}$ of A other than $\mathfrak{m}$, $\mathscr{B}_{\mathfrak{p}} = R_{\mathfrak{p}}$.) We will prove that $A \subset \mathscr{B} \cap A$ by proving that $\mathfrak{m}^{r-1} \subset \mathscr{B} \cap A$.

Let $a \in \mathfrak{m}^{r-1}$. We can write $f = f_0 + f_1$ and $g = g_0 + g_1$ where all the coefficients of f_1 and g_1 are in $\mathfrak{m}$ and all nonzero coefficients of f_0 and g_0 are units in A. Because of the choice of f and g we get $f_0 \neq 0$, $g_0 \neq 0$.

Since $\mathfrak{m}^r \subset \mathscr{B} \cap A$, $f_1 a \in \mathscr{B}$ and $g_1 a \in \mathscr{B}$. This shows that $f_0 \cdot a \in \mathscr{B}$ and $g_0 \cdot a \in \mathscr{B}$. We will prove $a \in \mathscr{B}$ by showing that $f_0 R + g_0 R = R$.

Let $\hat{A}$ be the completion of A with respect to the $\mathfrak{m}$-adic topology. $\hat{A}$ contains a subfield k isomorphic to $A/\mathfrak{m}$.

We can regard f_0 and g_0 as elements of $k[X]$ and $k[Y]$ respectively. Since $\operatorname{char} k \neq 0$, there exist h_1 and h_2 in $k\{X, Y\}$ such that $f_0 h_1 + g_0 h_2 = 1$. This shows that $f_0 \hat{R} + g_0 \hat{R} = \hat{R}$ where $\hat{R} = \hat{A} \otimes_A R$.

Let $\mathscr{A} = f_0 R + g_0 R$. Since $\hat{A}$ is faithfully flat over A, and since we have $R/\mathscr{A} \otimes_A \hat{A} = \widehat{R/\mathscr{A}} = 0$, we get $R/\mathscr{A} = 0$, i.e. $f_0 R + g_0 R = R$. Therefore $a \in \mathscr{B} = fR + gR$. This shows that $A \subset \mathscr{B} \cap A$ i.e. $\mathscr{B} = R$. This completes the proof of Lemma 2.5.

The proof of Theorem 2.4 is complete.

COROLLARY 2.6. *Let $A_n(S) = S\{X_1, \cdots, X_n, \partial/\partial X_1, \cdots, \partial/\partial X_n\}$ be the Weyl algebra of index n with coefficients in S, where S is a commutative noetherian ring which contains $\boldsymbol{Q}$. Then*

$$\operatorname{gl.\,dim} A_n(S) = n + \operatorname{gl.\,dim} S .$$

Proof of Corollary 2.6. We will prove the result by induction on n. Theorem 2.3 proves the result when $n = 1$. Assume the result for $n - 1$.

Let $A_n(S) = S\{X_1, \cdots, X_n, \partial/\partial X_1, \cdots, \partial/\partial X_n\}$. We can assume without loss of generality that S is a regular local ring with maximal ideal $\mathfrak{m}$. Let $T_1 = S[X_n] - \mathfrak{m}[X_n]$ and $T_2 = S[\partial/\partial X_n] - \mathfrak{m}[\partial/\partial X_n]$ be the multiplicatively closed sets satisfying the conditions of the Proposition 2.4.

If $B = S\{X_1, \cdots, X_n, \partial/\partial X_1, \cdots, \partial/\partial X_{n-1}\}$, then T_1 consists of central elements of B. Therefore the S-derivation of B given by $\partial/\partial X_n$ can be extended to a derivation d' of $B[T_1^{-1}]$. Since $A_n(S)$ is the Ore-extension of B with respect to derivation $\partial/\partial X_n$, $A_n(S)[T_1^{-1}]$ is the Ore-extension of $B[T_1^{-1}]$ with respect to the derivation d'. Therefore $\operatorname{l.gl.\,dim} A_n(S)[T^{-1}] \leqslant 1 + \operatorname{l.gl.\,dim} B[T_1^{-1}]$.

But $B[T_1^{-1}] \simeq S'\{X_1, \cdots, X_{n-1}, \partial/\partial X_1, \cdots, \partial/\partial X_{n-1}\}$ where S' is the localization of $S[X_n]$ with respect to T_1. Therefore by induction hypothesis $\operatorname{l.gl.\,dim} B[T_1^{-1}] = n - 1 + \operatorname{gl.\,dim} S' = n - 1 + \operatorname{gl.\,dim} S$. This shows that $\operatorname{l.gl.\,dim} A_n(S)[T_1^{-1}] \leqslant n + \operatorname{gl.\,dim} S$. Similarly we prove that l.gl. $\dim A_n(S)[T_2^{-1}] \leqslant n + \text{gl.} \dim S$. Therefore by Proposition 2.4 we get that l.gl. $\dim A_n(S) \leqslant n + \text{gl.} \dim S$. But we already know that l.gl. $\dim A_n(S) \geqslant n + \text{l.gl.} \dim S$. Hence the equality.

Remark. Theorem 2.3 is a generalization of a Theorem of Rinehart [5, Proposition 2].

Remark. Corollary 2.6 is a generalization of a Theorem of Roos [6, Theorem 1].

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