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HOMEOMORPHISMS WITHOUT THE PSEUDO-ORBIT TRACING PROPERTY

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§0. Introduction

Recently, A. Morimoto [5] proved that every isometry of a compact Riemannian manifold of positive dimension has not the pseudo-orbit tracing property, and that if a homeomorphism of a compact metric space has the pseudo-orbit tracing property then $E_{\varphi} = O_{\varphi}$ (see §1 for definition). The purpose of this paper is to show that every distal homeomorphism of a compact connected metric space has not the pseudo-orbit tracing property.

The author benefited from reading the papers by A. Morimoto [5, 6].

§1. Definitions

Let $\varphi: X \to X$ be a (self-) homeomorphism of a compact metric space $b \leqslant \infty$) is called a δ -pseudo-orbit of φ if $d(\varphi(x_i), x_{i+1}) < \delta$ for $i \in (a, b-1)$. A sequence $\{x_i\}$ is called to be ε -traced by $x \in X$ if $d(\varphi^i(x), x_i) < \varepsilon$ holds for $i \in (a, b)$. We say (X, φ) to have the pseudo-orbit tracing property (abbrev. P.O.T.P.) if for every $\varepsilon > 0$ there is $\delta > 0$ such that every δ pseudo-orbit of φ can be ε -traced by some point $x \in X$. The system (X, φ) is said to be *minimal* if a φ -invariant closed set K is necessarily $K = \emptyset$ or K = X. Let A be a subset of the integer group Z. Then A is syndetic if there is a finite subset K of Z with Z = K + A. Let $x \in X$. Then x is an almost periodic point if $\{n \in \mathbb{Z} : \varphi^n(x) \in U\}$ is a syndetic set for all neighborhoods U of x. Let (X, φ) be distal, that is, if $\inf_{n \in \mathbb{Z}} d(\varphi^n(x), \varphi^n(y))$ = 0 then x = y. Then every $x \in X$ is an almost periodic point and the converse is true (p. 36 of [2]). It is clear that every equi-continuous homeomorphism has this property and is hence distal. But the converse does not hold. To check this for example, let T^2 be a 2-dimensional torus

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and define a homeomorphism $\varphi: T^2 \to T^2$ by $\varphi(x_1, x_2) = (\alpha + x_1, nx_1 + x_2)$ $((x_1, x_2) \in T^2)$ where $\alpha \in T^1$ and $0 \neq n \in \mathbb{Z}$. Then it will be easily checked that φ is distal but not equi-continuous. A point $x \in X$ is said to be nonwandering (with respect to φ) if for every neighborhood U of x, there is an n > 0 with $U \cap \varphi^n(U) \neq \emptyset$. The set of all nonwandering points is called the nonwandering set and denoted by $\Omega(\varphi)$. Since X is compact, we get $\Omega(\varphi) \neq \emptyset$. If in particular (X, φ) is distal, then it is easily proved that $\Omega(\varphi) = X$ since every $x \in X$ is almost periodic. We know (cf. p. 132 of [7]) that there is always a Borel probability measure μ on X which is preserved by φ and φ^{-1} , and (cf. p. 135 of [7]) that if (X, σ) is minimal then $\mu(U) > 0$ for all non-empty open set U.

The set 2^x of all closed non-empty subsets of X will be a compact metric space by the distance function \overline{d} defined by

$$\overline{d}(A, B) = \operatorname{Max} \left\{ \operatorname{Max}_{b \in B} d(A, b), \operatorname{Max}_{a \in A} d(a, B) \right\} \quad (A, B \in 2^{X})$$

where $d(A, b) = \inf_{a \in A} d(a, b)$ (cf. p. 45 of [4]). We denote by $\operatorname{Orb}^{\delta}(\varphi)$ the set of all δ -pseudo-orbits of φ and by $\overline{\operatorname{Orb}^{\delta}(\varphi)}$ the set of all $A \in 2^{x}$, for which there is $\{x_{i}\} \in \operatorname{Orb}^{\delta}(\varphi)$ such that $A = \operatorname{Cl}\{x_{i}: i \in \mathbb{Z}\}$, Cl denoting the closure. Let E_{φ} denote the set of all $A \in 2^{x}$ such that for every $\varepsilon > 0$ there is $A_{\epsilon} \in \overline{\operatorname{Orb}^{\epsilon}(\varphi)}$ with $\overline{d}(A, A_{\epsilon}) < \varepsilon$. An element A of E_{φ} is called an *extended orbit* of φ . On the other hand, we define $O_{\varphi} = \operatorname{Cl}\{O_{\varphi}(x): i \in \mathbb{Z}\}$ where $O_{\varphi}(x) = \operatorname{Cl}\{\varphi^{i}(x): i \in \mathbb{Z}\}$. We can easily see that E_{φ} is closed in 2^{x} and that $O_{\varphi} \subset E_{\varphi}$ holds.

§2. Results

Throughout this section, X will be a compact metric space with distance function d and φ will be a self-homeomorphism of X.

THEOREM. Assume that X is connected. If (X, φ) is distal, then (X, φ) has not P.O.T.P.

LEMMA 1. If (X, φ) has P.O.T.P., for every $\varepsilon > 0$ and every $x_0 \in \Omega(\varphi)$ there is a point $y \in X$ and an integer $k = k(x_0, \varepsilon) > 0$ such that $O_{\varphi^k}(y) \subset U_{\epsilon}(x_0)$.

Proof. Since $x_0 \in \Omega(\varphi)$, for $\delta > 0$ with $\delta < \varepsilon$ there are a point $x \in X$ and an integer k > 0 such that x and $\varphi^k(x)$ belong to $U_{\delta/2}(x_0)$. Now, set $x_{nk+i} = \varphi^i(x)$ for $n \in \mathbb{Z}$ and $0 \le i < k$. Obviously, $\{x_i\}_{i \in \mathbb{Z}} = \{\cdots, x, \varphi(x), \cdots, \varphi^{k-1}(x), \cdots\} \in \operatorname{Orb}^{\delta}(\varphi)$. Hence we can find a point $y \in X$ such that $d(\varphi^i(y), \varphi^{k-1}(x), \cdots) \in \mathbb{Q}$ $x_i) < \varepsilon$ for $i \in \mathbb{Z}$. In particular, $d(\varphi^{nk}(y), x_{nk}) < \varepsilon$ and hence $d(\varphi^{nk}(y), x) < \varepsilon$ for $n \in \mathbb{Z}$. Therefore we have $O_{\varphi^k}(y) \subset U_{\epsilon}(x_0)$.

COROLLARY 1. Assume that X is connected and not one point. If (X, φ) is minimal, then (X, φ) has not P.O.T.P.

Proof. Let $\varepsilon = \text{diameter } (X)/3$ and assume that (X, φ) has P.O.T.P. By Lemma 1 we have that for some $x_0 \in X$ there are $y \in X$ and k > 0 with $O_{\varphi^k}(y) \subset U_{\epsilon}(x_0)$. Since X is connected, $O_{\varphi^k}(y) = O_{\varphi}(y) = X$ and so diameter $(X) \leq 2\varepsilon$. This is a contradiction.

COROLLARY 2. If (X, φ) is minimal, then $E_{\varphi} = O_{\varphi}$.

Proof. It is proved by A. Morimoto that every $A \in E_{\varphi}$ is φ -invariant $(\varphi(A) = A)$. In fact, for every $\varepsilon > 0$ there is $\varepsilon > \varepsilon_1 > 0$ such that $d(\varphi(x), \varphi(y)) < \varepsilon$ when $d(x, y) < \varepsilon_1$. By definition we can find $\{x_i\} \in \operatorname{Orb}^{\epsilon_1}(\varphi)$ with $\overline{d}(A, \operatorname{Cl}\{x_i\}) < \varepsilon_1$. Set $y_i = \varphi(x_i)$ for $i \in \mathbb{Z}$, then $d(y_i, x_{i+1}) < \varepsilon_1$ and so $\overline{d}(\operatorname{Cl}\{x_i\}, \operatorname{Cl}\{y_i\}) < \varepsilon_1$. It is clear that $d(\varphi(y_i), y_{i+1}) < \varepsilon$ for $i \in \mathbb{Z}$. Hence, $\{y_i\} \in \operatorname{Orb}^{\epsilon}(\varphi)$. Let $A' = \operatorname{Cl}\{x_i\}$. Then $\overline{d}(A', \varphi(A')) < \varepsilon_1$ and since $\overline{d}(A, A') < \varepsilon_1$ we get $\overline{d}(\varphi(A), \varphi(A')) < \varepsilon$. Therefore

$$ar{d}(arphi(A),A) < ar{d}(arphi(A),arphi(A')) + ar{d}(arphi(A'),A') + ar{d}(A',A) < 3arepsilon$$

and so $\overline{d}(\varphi(A), A) = 0$; i.e. $\varphi(A) = A$. Therefore we get $E_{\varphi} = \{X\} = O_{\varphi}$.

LEMMA 2. If (X, φ) has P.O.T.P., for every integer k > 0, (X, φ^k) has also P.O.T.P.

Proof. For every $\varepsilon > 0$ there is $\delta > 0$ such that $\{x_i\} \in \operatorname{Orb}^{\delta}(\varphi)$ is ε traced by a point in X. Take $\{y_i\} \in \operatorname{Orb}^{\delta}(\varphi)$ and put $x_{nk+i} = \varphi^i(y_n)$ for n $\in \mathbb{Z}$ and $0 \leq i \leq k-1$. Obviously, $\{x_i\} \in \operatorname{Orb}^{\delta}(\varphi)$. Hence there is $y \in X$ with $d(\varphi^i(y), x_i) < \varepsilon$ for $i \in \mathbb{Z}$. In particular, $d(\varphi^k)^n(y), y_n) = d(\varphi^{nk}(y), x_{nk})$ $< \varepsilon$ for $n \in \mathbb{Z}$. This completes the proof of Lemma 2.

LEMMA 3. Let (X, φ) be distal. Then for every $x \in X$, $(O_{\varphi}(x), \varphi)$ is minimal.

Proof. Since every $x \in X$ is almost periodic under φ , for a neighborhood U of x there is a finite set $K = \{n_1, \dots, n_k\}$ of Z such that Z = A + K where $A = \{n \in Z : \varphi^n(x) \in U\}$. Hence $O_{\varphi}(x) = \operatorname{Cl} \{\varphi^n(x) : n \in A\} \cup$ $\operatorname{Cl} \{\varphi^{n+n_1}(x) : n \in A\} \cup \cdots \cup \operatorname{Cl} \{\varphi^{n+n_k}(x) : n \in A\}$. Let $y \in O_{\varphi}(x)$. Then $O_{\varphi}(y) \cap U \neq \emptyset$. This implies that $x \in O_{\varphi}(y)$. Hence $O_{\varphi}(x) = O_{\varphi}(y)$. Remark 1. If (X, φ) is distal and topologically transitive, then it is clearly minimal (by Lemma 3).

We shall now give a proof of the theorem.

Assuming that (X, φ) has P.O.T.P., we shall draw a contradiction. To do this, let $\varepsilon =$ diameter (X)/9. Then there is $\delta > 0$ with $\delta < \varepsilon$ such that every $\{z_i\} \in \operatorname{Orb}^{\delta}(\varphi)$ is ε -traced by a point of X. Lemma 1 insures us that for $y_0 \in \Omega(\varphi)$ there are $y \in X$ and k > 0 with $O_{\varphi^k}(y) \subset U_{\epsilon}(y_0)$. Put $\psi = \varphi^k$. Then (X, ψ) has P.O.T.P. (by Lemma 2) and is distal. Since X is connected and compact, we can take a sequence of points $\{p_i\}_{i=1}^N$ in X such that p_1 $= y, \ d(p_i, p_{i+1}) < \delta/2$ for $1 \leq i \leq N-1$ and such that $\bigcup_{i=1}^N U_{\delta}(p_i) = X$. Since (X, ψ) is distal, every point of X is almost periodic. Hence for $1 \leq i \leq N$ there is an integer c(i) > 0 such that $d(p_i, \psi^{c(i)}(p_i)) < \delta/2$. Let us put

$x_{\scriptscriptstyle i}=\psi^{\scriptscriptstyle -i}(p_{\scriptscriptstyle 1})$	(i < 0)
$x_{i}=\psi^{i}(p_{i})$	$(0\leqslant i\leqslant c(1)-1)$
$x_{\scriptscriptstyle c(1)+i}=\psi^i(p_2)$	$(0 \leqslant i \leqslant c(2) - 1)$
•••	
$x_{\scriptscriptstyle c(1)+\ldots+c(N-1)+i}=\psi^i(p_{\scriptscriptstyle N})$	$(0\leqslant i\leqslant c(N)-1)$
$x_{_{c(1)}+\ldots+_{c(N)}+i}=\psi^{i}(p_{_{N-1}})$	$(0\leqslant i\leqslant c(N-1)-1)$
• • •	
$x_{\mathfrak{c}(1)+2\mathfrak{c}(2)+\dots+2\mathfrak{c}(N-1)+\mathfrak{c}(N)+i}=\psi^i(p_1)$	$(i \geqslant 0)$.

Obviously, $\{x_i\}_{i\in\mathbb{Z}} \in \operatorname{Orb}^{\delta}(\psi)$ and $\overline{d}(\operatorname{Cl}\{x_i\}, X) < \delta$. By assumption, there is $z \in X$ with $d(\psi^i(z), x_i) < \varepsilon$ $(i \in \mathbb{Z})$ so that $\overline{d}(O_{\psi}(z), X) < \delta + \varepsilon < 2\varepsilon$, and in particular

where $c = c(1) + c(N) + 2 \sum_{i=2}^{N-1} c(i)$. This implies that

where $U_{\epsilon}(O_{\psi}(y)) = \bigcup_{h \in O_{\psi}(y)} U_{\epsilon}(h)$. Put $O_{\psi}^{-}(z) = \operatorname{Cl} \{\psi^{i}(z) : i < 0\}$ and $O_{\psi}^{+}(z)$ = $\operatorname{Cl} \{\psi^{i}(z) : i \ge 0\}$. Then we have that $O_{\psi}^{-}(z) \subset U_{\epsilon}(O_{\psi}(y))$ and $\psi^{c}O_{\psi}^{+}(z) \subset U_{\epsilon}(O_{\psi}(y))$. Since $O_{\psi}^{-}(z) \cup O_{\psi}^{+}(z) = O_{\psi}(z)$, by Baire's theorem either $O_{\psi}^{-}(z)$ or $O_{\psi}^{+}(z)$ has non-empty interior in the set $O_{\psi}(z)$.

Let μ be a ψ -invariant Borel probability measure of $O_{\psi}(z)$. Since

 $(O_{\psi}(z), \psi)$ is minimal by Lemma 3, every non-empty open set in $O_{\psi}(z)$ has μ -positive measure. When the interior of $O_{\psi}^{-}(z)$ in $O_{\psi}(z)$ is non-empty, it is easy to see that $O_{\psi}^{-}(z) = \psi O_{\psi}^{-}(z)$ and so $O_{\psi}^{-}(z) = O_{\psi}(z)$. Indeed, assume $\psi^{-1}O_{\psi}^{-}(z) \subseteq O_{\psi}^{-}(z)$. Then $V = \bigcap_{k \geq 0} \psi^{-k}O_{\psi}^{-}(z)$ does not contain the interior of $O_{\psi}^{-}(z)$ in $O_{\psi}(z)$. Hence $\mu(O_{\psi}^{-}(z) \setminus V) > 0$. Since $O_{\psi}^{-}(z) = \bigcup_{k \geq 0} \psi^{-k} \{O_{\psi}^{-}(z) \setminus \psi^{-1}O_{\psi}^{-}(z)\} \cup V$, we get $\mu(O_{\psi}^{-}(z) \setminus \psi^{-1}O_{\psi}^{-}(z)) > 0$, thus contradicting $\mu(O_{\psi}^{-}(z)) \leqslant$ 1. If the interior of $O_{\psi}^{+}(z)$ in $O_{\psi}(z)$ is non-empty; i.e. $\mu(O_{\psi}^{+}(z)) > 0$, then it follows that $O_{\psi}^{+}(z) = O_{\psi}(z)$. Obviously $O_{\psi}(z) = \psi^{c}O_{\psi}^{+}(z)$. In any case we get $O_{\psi}(z) \subset U_{i}(O_{\psi}(y))$ so that $O_{\psi}(z) \subset U_{i}(O_{\psi}(y)) \subset U_{2i}(y_{0})$ (because $O_{\psi}(y)$ $\subset U_{\varepsilon}(y_{0})$). Since $2\varepsilon > \overline{d}(O_{\psi}(z), X) = \max_{x \in X} d(O_{\psi}(z), x)$, we have $X = U_{2i}(O_{\psi}(z))$ from which $X = U_{4i}(y_{0})$; i.e. diameter $(X) \leq \varepsilon$. This is a contradiction.

Remark 2. We know (Application 2 of [1]) that every (group) automorphism σ of a zero-dimensional compact metric group X has P.O.T.P. If (X, σ) has zero topological entropy (the existence of such automorphisms is known), then we can prove (cf. Lemma 14 of [1]) that X contains a sequence $X = X_0 \supset X_1 \supset \cdots$ of completely σ -invariant normal subgroups such that $\cap X_n$ is trivial and for every $n \ge 0$, X_n/X_{n+1} is a finite group. Hence for $x, y \in X$ ($x \neq y$) there is n > 0 such that $xy^{-1} \notin X_n$. Since $\sigma^j(X_n)$ $= X_n$ for all $j \in \mathbb{Z}$, we get easily $\sigma^j(xy^{-1}) \notin X_n$ ($j \in \mathbb{Z}$), which implies that $d(\sigma^j(x), \sigma^j(y)) \ge d(\sigma^j(x)X_n, \sigma^j(y)X_n) > 0$ (the distance function d is a translation invariant metric of X). Since X_n/X_{n+1} is a finite group, we get $\inf_j d(\sigma^j(x), \sigma^j(y)) > 0$; i.e. (X, σ) is distal. Therefore every zero-dimensional automorphism with zero topological entropy is distal and has P.O.T.P. This shows that the assumption of connectedness in the theorem can not drop out.

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