

ON THE DIVISOR CLASS GROUPS OF A TWO-DIMENSIONAL LOCAL RING AND ITS FORM RING

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Introduction

Let A be a noetherian ring and let I be an ideal of A contained in the Jacobson radical of A : $\text{Rad}(A)$. We assume that the form ring of A with respect to the ideal I : $G = \text{Gr}(A, I)$, is a normal integral domain. Hence A is a normal integral domain and one can ask for the links between $\text{Cl}(A)$ and $\text{Cl}(G)$.

Let $R = \bigoplus_{n \in \mathbb{Z}} I^n$ be the Rees algebra of A with respect to the ideal I (see § 2). In a previous paper [20], the authors have proved that $\text{Cl}(A) \simeq \text{Cl}(R)$; moreover there exists a “canonical” map $j: \text{Cl}(R) \rightarrow \text{Cl}(G)$ deduced from the hypersurface section $R \rightarrow G = R/uR$ (§ 1). Following the ideas of Lipman’s paper [18], in [20] an attempt was made to find out sufficient conditions for $\ker(j) = 0$, (resp.: for $\ker(j)$ to be a torsion group). But this sufficient conditions become almost tautological when $\dim(A) = 2$ and $\text{ht}(I) = 2$ (i.e. when A is a local ring and $I = \text{Rad}(A)$; see §1). This paper deals with this last case.

The main result of the paper is Theorem 4; this theorem can be proved also by using the geometrical machinery of Grothendieck, Danilov, Boutot and Bădescu-Fiorentini [15, 8, 9, 6, 3] (see also the Remark 3 after the proof of Theorem 4).

Our proof mainly uses simple tools of Commutative Algebra and standard facts of Local Cohomology theory. A key point is the finiteness of a suitable local cohomology module which we derive from [15].

It is also interesting that the short exact sequences which appear in Theorem 1 of [18] are the same which appear in our proof. In a certain sense, this circumstance unifies the two techniques.

However the problem of the injectivity of $j: \text{Cl}(R) \rightarrow \text{Cl}(G)$ for a general hypersurface section $R \rightarrow G = R/uR$, $\dim(G) = 2$, is rather different

from that considered in this paper as the example given in Section 3 shows.

§ 1.

In [10] Danilov has studied the links between the groups $\text{Cl}(A[[T]])$ and $\text{Cl}(A)$, where A is a normal integral domain. To do this, he has defined a canonical map $j: \text{Cl}(A[[T]]) \rightarrow \text{Cl}(A)$. But in fact Danilov's definition works more generally to give a map $j: \text{Cl}(R) \rightarrow \text{Cl}(R/uR)$ for any normal integral domain R and nonunit $u \in R$ such that $R/uR = G$ is also a normal integral domain. Let us recall the construction of j from the viewpoint of this paper.

First of all, $\text{Cl}(R)$ can be thought as the group of isomorphism classes of finitely generated, reflexive, rank one R -modules [18, 31]; a similar interpretation holds for $\text{Cl}(G)$. Let F be a finitely generated, reflexive, rank one R -module; we set: $[F]_R = (\text{isomorphism class of } F) \in \text{Cl}(R)$. Let $E = F \otimes_R G$ with F as above; then $E^{**} = \text{Hom}_G(\text{Hom}_G(E, G), G)$ is a finitely generated, reflexive, rank one G -module. By this interpretation of the class group, we have, following [18]: $j([F]_R) = [E^{**}]_G$.

From now on we assume that R is a \mathbb{Z} -graded ring, i.e. $R = \bigoplus_{n \in \mathbb{Z}} R_n$, and $u \in R$ is a homogeneous element. Let $\xi \in \text{Cl}(R)$; then $\xi = [\mathfrak{b}]_R$ for some homogeneous integral divisorial ideal \mathfrak{b} of R . If $\mathfrak{b} = \mathfrak{b}' \cap u^n R$, ($n \geq 0$), where \mathfrak{b}' is a homogeneous divisorial ideal with $\mathfrak{b}' \not\subseteq uR$; then $[\mathfrak{b}]_R = [\mathfrak{b}']_R$ since u is a prime element of R . Therefore there is no loss of generality in assuming that $\mathfrak{b} \not\subseteq uR$, or equivalently that u is regular for R/\mathfrak{b} . Then we get: $\alpha = \mathfrak{b} \otimes_R G \simeq \mathfrak{b}/u\mathfrak{b} \simeq \mathfrak{b} + uR/uR$ and $j([\mathfrak{b}]_R) = [((\alpha)^{-1})^{-1}]_G$ where $((\alpha)^{-1})^{-1}$ denotes, as usual, the divisorial ideal associated to α . In the sequel we will always refer to this simpler setup whenever the map j is concerned.

The homomorphism j ties together the groups $\text{Cl}(R)$ and $\text{Cl}(G)$. In particular one can ask the following questions for j : when is j surjective? and: when is j injective?

The following proposition, concerning the latter question, has been proved in [20] following the general ideas of Lipman's paper [18]:

Let $R = \bigoplus_{n \in \mathbb{Z}} R_n$ be a \mathbb{Z} -graded normal integral domain and let $u \in h\text{-Rad}(R)$ be a non-zero homogeneous element such that $G = R/uR$ is also a normal domain. Suppose that the canonical map $j_Q: \text{Cl}(R_Q) \rightarrow \text{Cl}(G_P)$ is injective (resp.: $\ker(j_Q)$ is a torsion group) for every homogeneous prime ideal Q of R such that: $u \in Q$ and $\text{depth}(R_Q) \leq 3$ (of course $P = Q/uR$). Then also $j: \text{Cl}(R) \rightarrow \text{Cl}(G)$ is an injective map (resp.: $\ker(j)$ is a torsion

group).

Let us observe that the hypotheses of this proposition forces $\dim(G) \geq 2$. But if G is a normal integral domain and $\dim(G) = 2$, then G is a C.M. ring. Hence also R is a C.M. ring ($u \in h\text{-Rad}(R)$; see [7], Proposition 2.2) and the above proposition becomes almost tautological. If $\dim(G) \leq 1$, $\text{Cl}(R)$ is simple to compute.

Therefore only the case $\dim(G) = 2$ remains still open. After all, this is not so surprising; in fact the case $\dim(A) = 2$ was the hardest to solve also for the problem of Danilov-Samuel, i.e. for the hypersurface section $A[[T]] \rightarrow A$ (see [26, 25, 28, 9]). Essentially, there are two (non tautological) ways to handle the case of a general hypersurface section $R \rightarrow G$ when $\dim(G) = 2$. The most recent one is due to Flenner (see Lemma 3.4 of [12]) and is inspired to Theorem 1 of [18]. The other one is used in this paper and comes from Hilfsatz 3 of [28], or Remarque p. 164 of [27]; it is summarized in the following proposition:

PROPOSITION 1. *Let $R = \bigoplus_{n \in \mathbb{Z}} R_n$ be a \mathbb{Z} -graded, normal integral domain and let $u \in h\text{-Rad}(R)$ be a non-zero, homogeneous element such that $G = R/uR$ is also a normal integral domain. Assume that G is a C.M. ring. Let $\xi \in \text{Cl}(R)$ and let $\mathfrak{b} \subset R$ be a homogeneous, proper, divisorial ideal such that $\xi = [\mathfrak{b}]_R$, $\mathfrak{b} \not\subset uR$ and α^{-1} is a h -free G -module, where $\alpha = \mathfrak{b} \otimes_R G$. Then $\xi = 0$, i.e. \mathfrak{b} is h -free, if and only if R/\mathfrak{b} is a C.M. ring.*

Proof. Let R/\mathfrak{b} be a C.M. ring; since u is regular for R/\mathfrak{b} , the ideal $\mathfrak{b} + uR/\mathfrak{b}$ is an unmixed ideal of height one of R/\mathfrak{b} . Therefore $\mathfrak{b} + uR$ is an unmixed ideal of height two of R , hence $\alpha = \mathfrak{b} + uR/uR$ is an unmixed ideal of height one of G . It follows that $\alpha = ((\alpha)^{-1})^{-1}$, so α is h -free and then \mathfrak{b} is h -free (see [5], Ch. II, 3.2, Proposition 5; with suitable modifications to the homogeneous case). The converse is trivial because R is a C.M. ring.

§ 2.

Let A be a ring and $I \subseteq \text{Rad}(A)$ an ideal of A . We fix the following notation: $R = R(A, I) = \bigoplus_{n \in \mathbb{Z}} I^n$ ($I^n = A$ for $n \leq 0$) is the Rees algebra of A with respect to I . If T is an indeterminate over A , let $u = T^{-1}$. We have $R = A[a_1T, \dots, a_rT, u] \subseteq A[T, u]$ where $I = (a_1, \dots, a_r)$. $G = \text{Gr}(A, I) = \bigoplus_{n \geq 0} I^n/I^{n+1}$ is the form ring of A with respect to I ; the irrelevant ideal of G is $G_+ = \bigoplus_{n > 0} I^n/I^{n+1}$; let $\alpha^* = \alpha \cdot A[T, u] \cap R$ where α is an ideal of

A ; α^* is a graded ideal of R ; $\text{In}_I(\alpha)$ is the graded ideal of G generated by the initial forms $\text{In}(x)$ for all $x \in \alpha$.

We refer to [7, 20, 22, 24] for the general properties of these rings and ideals. However, for the sake of completeness, let us recall the following ones: first $G \simeq R/uR$; u is a homogeneous element and $\deg(u) = -1$. Moreover $u \in h - \text{Rad}(R)$ and, finally, $\text{In}_I(\alpha) = \alpha^* + uR/uR$. If G is a normal integral domain, then also R and A are normal integral domains. If G is a normal integral domain we can consider the map $j: \text{Cl}(R) \rightarrow \text{Cl}(G)$ defined in Section 1. Moreover, since u is a prime element of R it is easy to see that $\text{Cl}(A) \xrightarrow[\psi]{} \text{Cl}(R)$ (see [20], Proposition 1); to be precise, the isomorphism ψ between $\text{Cl}(A)$ and $\text{Cl}(R)$ is given by $\psi([\alpha]_A) = [\alpha^*]_R$, where α is an integral, divisorial ideal of A . Therefore, by composition, we get a homomorphism $i: \text{Cl}(A) \rightarrow \text{Cl}(G)$ such that $i([\alpha]_A) = [(\text{In}_I(\alpha))^{-1}]_G$ where α is as above.

We begin the study of the map $j: \text{Cl}(R(A, I)) \rightarrow \text{Cl}(G(A, I))$ with a statement concerning the surjectivity of j .

PROPOSITION 2. *Let (A, m) be a local, henselian ring with $\dim(A) = 2$. Suppose that $G = \text{Gr}(A, m)$ is a normal integral domain. Then the map $j: \text{Cl}(R) \rightarrow \text{Cl}(G)$ is surjective.*

Proof. Let P be a homogeneous, height one, prime ideal of G . Pick $\bar{x} \in P - P^{(2)}$ with \bar{x} homogeneous. Let $x \in A$ be an element such that $\text{In}(x) = \bar{x}$. Expand xA to a prime ideal Q , maximal among those disjointed from the multiplicatively closed set $\{y \in A \mid \text{In}(y) \notin P\}$; clearly $\text{ht}(Q) = 1$. From the isomorphism $G(A/Q, m/Q) \simeq G(A, m)/\text{In}(Q)$ and the choice of \bar{x} it follows that $(\text{In}(Q))^{-1} = P$ (see Lemma 6 of [1]). Therefore $j([Q^*]_R) = [P]_G$, where $Q^* = Q \cdot A[T, u] \cap R$. Since $\text{Cl}(G)$ is generated by the classes of homogeneous, height one prime ideals of G , the thesis follows.

Remark. The following example shows that we cannot delete the requirement “ G is normal” in Proposition 2. Let $A = \mathbf{R}[[X, Y, Z]]/(x^2 + Y^2 + Z^3)$ (\mathbf{R} is the field of real numbers); A is a local complete factorial ring, and $\dim(A) = 2$ (see ex. (25, 4) of [17]). But $G(A, m) \simeq \mathbf{R}[X, Y, Z]/(X^2 + Y^2)$ is not even normal, hence it cannot be factorial.

The next proposition deals with the case $\text{ht}(I) \leq 1$.

PROPOSITION 3. *Let A be a ring with $\dim(A) = 2$, and let $I \subset \text{Rad}(A)$ be an ideal of A such that $\text{ht}(I) \leq 1$ and $G = \text{Gr}(A, I)$ is a normal integral*

domain.

- a) If I is invertible, then $\text{Cl}(A)$ is embedded in $\text{Cl}(G)$;
- b) If G is an almost factorial ring (in particular if G is a factorial ring) then I is invertible.

Proof. If $\text{ht}(I) = 0$ there is nothing to prove, so we assume that $\text{ht}(I) = 1$. We have that $G_0 = A/I$ is a Krull domain. Since $\dim(A/I) = 1$, A/I is a Dedekind domain; in particular it satisfies the property (R_1) of Serre and moreover $\text{Cl}(A/I) \simeq \text{Pic}(A/I)$. But I is invertible and $I \subseteq \text{Rad}(A)$; by localization at the maximal ideals, we have that A is an (R_2) ring, hence A is locally factorial and $\text{Cl}(A) \simeq \text{Pic}(A)$. Since $I \subseteq \text{Rad}(A)$, the canonical map $\text{Pic}(A) \rightarrow \text{Pic}(A/I)$ is injective (see [2], Proposition 1.4). From the hypothesis “ I is invertible” it follows that G is a flat G_0 -module (Lemma 2.1 of [23]). So the extension $G_0 \rightarrow G$ satisfies condition (PDE) and the induced homomorphism $\text{Cl}(G_0) \rightarrow \text{Cl}(G)$ is injective ([13], Proposition 10.7). This completes the proof of a).

The irrelevant ideal G_+ of G is a prime ideal. We easily get $G_+^{(p)} = \bigoplus_{n \geq p} G_n = G_+^p$ for all $p > 0$. Therefore, since $G_+ = \text{In}(I)$ and $\text{ht}(\text{In}(I)) = \text{ht}(I) = 1$, we have that G_+ is a projective G -module, since G is an almost factorial ring. But $I/I^2 \simeq G_+/G_+^2 \simeq G_+ \otimes_G G_0$, so I/I^2 is a projective A/I -module. Then I is locally principal since A_I is a DVR (see Lemma 2.1 of [23]).

Remark. In general the embedding $f: \text{Cl}(A) \rightarrow \text{Cl}(G)$ constructed in the proof is different from the map obtained by the composition of the isomorphism $\psi: \text{Cl}(A) \xrightarrow{\sim} \text{Cl}(R)$ and $j: \text{Cl}(R) \rightarrow \text{Cl}(G)$, i.e. from the map i .

If $\text{ht}(I) = 2$, from $I \subseteq \text{Rad}(A)$ and $\dim(A) = 2$ it follows that (A, I) is a local ring, i.e. $I = \text{Rad}(A)$. In the next theorem a sufficient condition is given for the map $j: \text{Cl}(R) \rightarrow \text{Cl}(G)$ to be injective when $\text{ht}(I) = 2$.

Local cohomology is the key tool in the proof of Theorem 4, so let us make some general remarks on it. Let S be a graded ring, $J \subset S$ a graded ideal and M a graded S -module. Since $H_J^i(M) = \varinjlim_n \text{Ext}_S^i(S/J^n, M)$ for all $i \geq 0$, the local cohomology modules are graded modules in this case. Moreover, let

$$0 \longrightarrow M' \xrightarrow[(d)]{\rho} M \xrightarrow[(t)]{\pi} M'' \longrightarrow 0$$

be a short exact sequence of graded S -modules. Suppose ρ and π graded,

respectively of degree d and t . Then the corresponding long exact cohomology sequence is graded as follows:

$$\cdots \longrightarrow H_J^i(M') \xrightarrow[(d)]{H_J^i(\rho)} H_J^i(M) \xrightarrow[(t)]{H_J^i(\pi)} H_J^i(M'') \xrightarrow[(-d-t)]{\theta_i} H_J^{i+1}(M') \longrightarrow \cdots$$

THEOREM 4. *Let (A, Q) be a local ring with $\dim(A) = 2$. Assume that:*

$$(1) \quad (H_{G+}^2(G))_n = 0 \text{ for all } n > 0$$

Then the map $j: \text{Cl}(R) \longrightarrow \text{Cl}(G)$ is injective.

Proof. We shall give the proof in several steps.

Step 1. Let \mathfrak{b} be a homogeneous integral (proper) divisorial ideal of R such that $\mathfrak{b} \not\subseteq uR$; suppose that α^{-1} is a h -free G -module (where α denotes, as usual, $\mathfrak{b} \otimes_R G \simeq \mathfrak{b} + uR/uR$). By Proposition 1 we have only to show that R/\mathfrak{b} is a C.M. ring. R is a h -local ring; indeed $\mathfrak{m} = (Q^*, u)$ is a maximal ideal of R and $\mathfrak{m} = h - \text{Rad}(R)$. Then also R/\mathfrak{b} is a h -local ring and $\mathfrak{n} = \mathfrak{m}/\mathfrak{b} = h - \text{Rad}(R/\mathfrak{b})$. R/\mathfrak{b} is a C.M. ring if and only if $(R/\mathfrak{b})_{\mathfrak{n}}$ is a C.M. ring (see [19], Theorem 1.1). But $(R/\mathfrak{b})_{\mathfrak{n}}$ is a C.M. ring if and only if $H_{\bar{\mathfrak{n}}}^0((R/\mathfrak{b})_{\mathfrak{n}}) = H_{\bar{\mathfrak{n}}}^1((R/\mathfrak{b})_{\mathfrak{n}}) = 0$ (where $\bar{\mathfrak{n}} = \mathfrak{n} \cdot (R/\mathfrak{b})_{\mathfrak{n}}$). Now $H_{\bar{\mathfrak{n}}}^0((R/\mathfrak{b})_{\mathfrak{n}}) = 0$ since u is a regular element for R/\mathfrak{b} . R is a C.M. ring and $\text{depth}(R_{\mathfrak{m}}) = 3$; then from the long exact sequence for the local cohomology and from Theorem 4.3 of [29] we get: $H_{\mathfrak{n}}^1(R/\mathfrak{b}) \simeq H_{\mathfrak{m}}^1(R/\mathfrak{b}) \simeq H_{\mathfrak{m}}^2(\mathfrak{b})$. As $H_{\mathfrak{n}}^1(R/\mathfrak{b}) \otimes_{R/\mathfrak{b}} (R/\mathfrak{b})_{\mathfrak{n}} \simeq H_{\bar{\mathfrak{n}}}^1((R/\mathfrak{b})_{\mathfrak{n}})$ (see [29], Theorem 5.1), it will be sufficient to show that $H_{\mathfrak{m}}^2(\mathfrak{b}) = 0$.

Step 2. $H_{\mathfrak{m}}^2(\mathfrak{b})$ is a finitely generated R -module. To see this it is sufficient to prove that $H_{\mathfrak{m}}^2(\hat{\mathfrak{b}})$ is a finitely generated \hat{R} -module, where $\hat{R} = (\widehat{R, \mathfrak{m}}) \simeq (\widehat{R_{\mathfrak{m}}, \mathfrak{m}R_{\mathfrak{m}}})$. In fact we have (see [30], Theorem 4.5): $H_{\mathfrak{m}}^2(\hat{\mathfrak{b}}) \simeq H_{\mathfrak{m}R_{\mathfrak{m}}}^2(\mathfrak{b}R_{\mathfrak{m}}) \otimes_{R_{\mathfrak{m}}} \hat{R}$; therefore $H_{\mathfrak{m}}^2(\hat{\mathfrak{b}})$ is finitely generated over \hat{R} if and only if $H_{\mathfrak{m}R_{\mathfrak{m}}}^2(\mathfrak{b}R_{\mathfrak{m}})$ is a finitely generated $R_{\mathfrak{m}}$ -module (see [5], Proposition 11, Ch. I. 3.6.) and this last condition is equivalent to " $H_{\mathfrak{m}}^2(\mathfrak{b})$ is finitely generated over R " since $H_{\mathfrak{m}R_{\mathfrak{m}}}^2(\mathfrak{b}R_{\mathfrak{m}}) \simeq H_{\mathfrak{m}}^2(\mathfrak{b}) \otimes_R R_{\mathfrak{m}}$ (see [21], Proposition 11). Since \hat{R} is a C.M. ring and since $\hat{\mathfrak{b}}$ is an unmixed ideal of height one of \hat{R} (see [14], 9.3 and 13.8), for every prime ideal P of \hat{R} such that $\text{ht}(P) = 2$, $\text{depth}(\hat{\mathfrak{b}}_P) = 2$. The finite generation of $H_{\mathfrak{m}}^2(\hat{\mathfrak{b}})$ over \hat{R} then follows from [15], Expose VIII, Corollaire 2.3.

Step 3. The hypothesis " α^{-1} is h -free" implies $((\alpha)^{-1})^{-1} = xG$, where x is a homogeneous element of G of degree $d > 0$. Then $\alpha = \mathfrak{b} + uR/uR = xG \cap I$, where I is an eventual embedded primary component; since α is homogeneous and $\dim(G) = 2$, I is irrelevant, i.e. $\sqrt{I} = G_+$. Now we have $H_{G_+}^i(G) = H_{G_+}^0(G) = 0$ since G is a C.M. ring. But $((\alpha)^{-1})^{-1} = xG \simeq G(-d)$, hence $H_{G_+}^i(((\alpha)^{-1})^{-1}) \simeq H_{G_+}^i(G)(-d)$. From the short exact sequence:

$$(2) \quad 0 \longrightarrow \alpha \longrightarrow ((\alpha)^{-1})^{-1} \longrightarrow C \longrightarrow 0$$

it follows that $H_{G_+}^1(\alpha) \simeq H_{G_+}^0(C) = C$ where the isomorphism is of degree zero. Then from $((\alpha)^{-1})^{-1}_n = 0$ for all $n < d$ we get: $(H_{G_+}^1(\alpha))_n = 0$ for all $n < d$. Since $\text{Supp}(C) \subseteq \{G_+\}$ we have $H_{G_+}^i(C) = 0$ for all $i > 0$. Therefore from the long exact cohomology sequence associated to (2) we get: $H_{G_+}^2(\alpha) \simeq H_{G_+}^2(((\alpha)^{-1})^{-1}) \simeq H_{G_+}^2(G)(-d)$ where both isomorphisms are of degree zero. Now the hypothesis (1) comes into play to get: $(H_{G_+}^2(\alpha))_n = 0$ for all $n > d$.

Finally, from the canonical isomorphisms (of degree zero) $H_{G_+}^1(\alpha) \simeq H_{\mathfrak{m}}^1(\alpha)$ and $H_{G_+}^2(\alpha) \simeq H_{\mathfrak{m}}^2(\alpha)$ we get:

$$(3) \quad (H_{\mathfrak{m}}^1(\alpha))_n = 0 \quad \text{for all } n < d,$$

$$(4) \quad (H_{\mathfrak{m}}^2(\alpha))_n = 0 \quad \text{for all } n > d.$$

Step 4. Let

$$(5) \quad \dots \longrightarrow (H_{\mathfrak{m}}^1(\alpha))_n \longrightarrow (H_{\mathfrak{m}}^2(\mathfrak{b}))_{n+1} \xrightarrow{\cdot u} (H_{\mathfrak{m}}^2(\mathfrak{b}))_n \longrightarrow (H_{\mathfrak{m}}^2(\alpha))_n \longrightarrow \dots$$

be the long exact cohomology sequence corresponding to the short exact sequence:

$$0 \longrightarrow \mathfrak{b} \xrightarrow[(-1)]{\cdot u} \mathfrak{b} \xrightarrow{(0)} \alpha \longrightarrow 0$$

From (5) and (3) it follows that u is a regular element for all homogeneous elements of $H_{\mathfrak{m}}^2(\mathfrak{b})$ of degree $\leq d$. Let $x \in (H_{\mathfrak{m}}^2(\mathfrak{b}))_n$ with $n \leq d$; by definition of local cohomology there exists a positive integer t such that $\mathfrak{m}^t \cdot x = 0$; but $u \in \mathfrak{m}$, hence $u^t \cdot x = 0$; therefore $x = 0$ and $(H_{\mathfrak{m}}^2(\mathfrak{b}))_n = 0$ for all $n \leq d$. (this is essentially the proof of Lemma 1.2 of [30]). Therefore, from (5) and (4) it follows $H_{\mathfrak{m}}^2(\mathfrak{b}) = uH_{\mathfrak{m}}^2(\mathfrak{b})$. But $u \in h - \text{Rad}(R)$, hence $H_{\mathfrak{m}}^2(\mathfrak{b}) = 0$ by Step 2 and the homogeneous Nakayama's lemma.

Remarks.

1) Since \hat{R} is a local ring we can derive the finite generation of

$H_m^2(\hat{b})$ over \hat{R} , also from [15], Exposé V, Corollaire 3.6. Since \hat{R} is flat over R , u is a regular element for \hat{R} . Therefore: $\text{Gr}(\hat{R}, u\hat{R}) \simeq \hat{R}/u\hat{R}[T] \simeq \hat{G}[T]$, where T is an indeterminate over $\hat{R}/u\hat{R}$ and $\hat{G} = (\widehat{G}, \widehat{G}_+)$. It follows that \hat{R} is a normal integral domain (see § 3, Proposition 6, a)).

2) With the same notations of Theorem 4, but without the hypothesis (1), we can prove the following result:

For every ideal J of A , let $c = JA[T, u] \cap R$. Then $(H_m^2(c))_n = 0$ for all $n \leq 0$. Let $\{u, x, y\}$ be a homogeneous system of parameters in R with $\deg(x) = \deg(y) = 1$; thus the Čech complex of c is given by:

$$C(u, x, y; c); 0 \xrightarrow{d_0} c_u \oplus c_x \oplus c_y \xrightarrow{d_1} c_{ux} \oplus c_{uy} \oplus c_{xy} \xrightarrow{d_2} c_{uxy} \longrightarrow 0$$

All the modules have a natural grading and the maps d_i are as usual. Assume $\sigma = (d/(ux)^p; e/(uy)^p; f/(xy)^p) \in \ker(d_2)$, (i.e. $-dy^p + ex^p - fu^p = 0$) with d, e, f homogeneous elements of c , and $\deg(d) = \deg(e) = \deg(f) - 2p = n \leq 0$.

We prove that there exists $\rho = (a/u^p; b/x^p; c/y^p)$, with a, b, c homogeneous elements of c , such that $d_1(\rho) = \sigma$ i.e. $ax^p - bu^p = d$ and $ay^p - cu^p = e$ (the third equation $by^p - cx^p = f$ is dependent upon the others). If $p \leq 0$, the proof is trivial. Let $p > 0$. Since (u^p, x^p, y^p) is an R -regular sequence, we have $d \in (x^p, u^p)$, $e \in (y^p, u^p)$. On the other hand, one can easily prove that $\bigoplus_{n \leq 0} (x^p, u^p)_n$ and $\bigoplus_{n \leq 0} (y^p, u^p)_n$ are included in $\bigoplus_{n \leq 0} (u^p)_n$; hence $d, e \in (u^p)$.

We now recall that u^p is regular for R/c ; therefore the system

$$\begin{cases} ax^p - bu^p = d \\ ay^p - cu^p = e \end{cases}$$

has solutions if we take $a = 0$.

3) Theorem 4 has many sources; in particular see [8, 12, 3, 4]. Instead of condition (1) of Theorem 4 in these papers is used the equivalent condition:

$$(6) \quad H^1(Y, \Theta_Y(n)) = 0 \text{ for all } n > 0$$

where $Y = \text{Proj}(G)$ (see [16], Ch. III, Proposition 2.1.5.)

For the sake of completeness we briefly show how the geometrical techniques work to get results as in Theorem 4.

Let $\mathcal{R} = \bigoplus_{n \geq 0} Q^n$ be the blow-up algebra with respect to the ideal Q , and set: $X = \text{Proj}(\mathcal{R})$, $Y = \text{Proj}(G)$; let $\chi: Y \rightarrow X$ be the closed immersion

deduced from the canonical map $\mathcal{R} \rightarrow \mathcal{R}/Q\mathcal{R} \simeq G$.

Since A and G are normal integral domains of dimension two, it follows easily that the canonical morphism $X \rightarrow \text{Spec}(A)$ is a desingularization of $\text{Spec}(A)$. In particular we get that the canonical morphism $\varphi: \text{Pic}(X) \rightarrow \text{Pic}(X - Y)$ is surjective (see [16] IV, 21. 6. 11). Moreover: $\text{Ker } \varphi = [\theta_X(1)] \cdot Z$, and this is an infinite cyclic group. But $X - Y \simeq \text{Spec}(A) - \{\mathfrak{m}\}$; therefore $\text{Pic}(X - Y) \simeq \text{Pic}(\text{Spec}(A) - \{\mathfrak{m}\}) \simeq \text{Cl}(A)$ (see [13], 18. 10) and we get the short exact sequence:

$$0 \longrightarrow Z \longrightarrow \text{Pic}(X) \longrightarrow \text{Cl}(A) \longrightarrow 0$$

Another well known short exact sequence is the following:

$$0 \longrightarrow Z \longrightarrow \text{Pic}(Y) \longrightarrow \text{Cl}(G) \longrightarrow 0$$

where the first morphism maps 1 to $[\theta_Y(1)]$. Finally, we consider the morphism $\chi^*: \text{Pic}(X) \rightarrow \text{Pic}(Y)$ deduced from the closed immersion $\chi: Y \hookrightarrow X$. Putting all together we get the following diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & Z & \longrightarrow & \text{Pic}(X) & \longrightarrow & \text{Cl}(A) \longrightarrow 0 \\ & & \parallel & & \downarrow \chi^* & & \downarrow \bar{\chi} \\ 0 & \longrightarrow & Z & \longrightarrow & \text{Pic}(Y) & \longrightarrow & \text{Cl}(G) \longrightarrow 0 \end{array}$$

where $\bar{\chi}$ is deduced from χ^* . It is easily checked that $\bar{\chi} = i$. From the "snake-lemma" it follows that the maps χ^* and $\bar{\chi} = i$ have isomorphic kernels and cokernels. The geometrical techniques developed in [15, 9, 6, 3] allow a direct study of χ^* . Now we sketch their use.

First of all, we can define for all $n \geq 0$ a graded ring $G_n = \bigoplus_{i \geq 0} Q^i / Q^{i+n+1}$; in particular we have $G_0 = G$. If $m \geq n$ we get an epimorphic map $G_m \rightarrow G_n$. Let $Y_n = \text{Proj}(G_n)$; we have a closed immersion $Y_n \hookrightarrow Y_m$ whenever $m \geq n$.

If (A, Q) is henselian, then the sequence $\{\text{Pic}(Y_n)\}_n$ is essentially constant and $\text{Pic}(X) = \varprojlim_n \text{Pic}(Y_n)$ (see [6], Ch. IV, Proposition 6.2). By virtue of well-known Theorem of Mori ([13], Corollary 6.12), we can reduce to the case " (A, Q) henselian" by replacing A with $\hat{A} = (\hat{A}, \hat{Q})$. Moreover, for all $n \geq 0$ we have a short exact sequence of abelian sheaves on the topological space of Y :

$$(7) \quad 0 \longrightarrow (i_{n+1})_* \theta_Y(n+1) \longrightarrow (\theta_{Y_{n+1}})^* \longrightarrow (\theta_{Y_n})^* \longrightarrow 1$$

where $i_{n+1}: Y \rightarrow Y_{n+1}$ is the canonical closed immersion. Since $\dim(Y) = 1$

the long exact sequence deduced from (7) is:

$$(8) \quad H^1(Y, \mathcal{O}_Y(n+1)) \longrightarrow \text{Pic}(Y_{n+1}) \longrightarrow \text{Pic}(Y_n) \longrightarrow 0$$

If condition (6) holds, we get $\chi^*: \text{Pic}(X) \simeq \text{Pic}(Y)$ and Theorem 4 follows. On the other hand, without any hypothesis on G , we get easily the following:

PROPOSITION 5. *If $\text{char}(k) = p > 0$, then $\ker(\chi^*) = \ker(i)$ is a p -torsion group.*

With the same notations of Theorem 4, let \mathfrak{b} be a homogeneous, proper, divisorial ideal of R , $\mathfrak{b} \not\subseteq uR$ such that $[\mathfrak{b}]_R \in \ker(j)$. If $\mathfrak{b} = P_1^{(n_1)} \cap \dots \cap P_r^{(n_r)}$ is the primary decomposition of \mathfrak{b} , put $\mathfrak{b}^{(p^m)} = P_1^{(n_1 p^m)} \cap \dots \cap P_r^{(n_r p^m)}$ for $m > 0$. Then Proposition 5 means that $H_m^2(\mathfrak{b}^{(p^m)}) = 0$ for some $m \geq 0$. The authors were unable to prove directly this fact.

§ 3. Concluding remarks

1) As the following counterexample shows, the hypothesis (1) of Theorem 4 does not suffice to deduce the injectivity of $j: \text{Cl}(R) \rightarrow \text{Cl}(G)$ when $R \rightarrow R/uR = G$ is a general hypersurface section. Let k be an algebraically closed field. Let $R = k[X, Y, Z, W]/(XY - ZW) = k[x, y, z, w]$ and let $G = R/(x - y)$. G is the homogeneous coordinate ring of a smooth conic in \mathbf{P}_k^2 , hence G satisfies the hypothesis (1) of Theorem 4. But $\text{Cl}(R) \simeq \mathbf{Z}$ and $\text{Cl}(G) \simeq \mathbf{Z}/2\mathbf{Z}$.

2) If a form ring G is given, we can consider two rings of special relevance for our problem: G_{σ_+} and $\hat{G} = (\widehat{G}, \widehat{G}_+)$. This relevance is partially explained by the properties collected in the following:

PROPOSITION 6. *Let $G = G(A, I)$ be a normal integral domain, where $I \subseteq \text{Rad}(A)$ as always. Then we have:*

a) $\text{Gr}(\hat{G}, \hat{G}_+) \simeq \text{Gr}(G, G_+) \simeq G$, hence \hat{G} is a normal integral domain.

b) Let \mathfrak{c} be a homogeneous ideal of G ; then $\text{In}_{\hat{G}_+}(\mathfrak{c}\hat{G}) \simeq \mathfrak{c}$ where the isomorphism is that of a). In particular, let $m: \text{Cl}(G) \rightarrow \text{Cl}(\hat{G})$ the homomorphism deduced from the flat extension $G \rightarrow \hat{G}$, i.e. $m([\mathfrak{c}]_G) = [\mathfrak{c}\hat{G}]_{\hat{G}}$ for every integral divisorial ideal \mathfrak{c} of G . Then $i \cdot m = 1_{\text{Cl}(G)}$.

c) $\text{Gr}(G_{\sigma_+}, G_+G_{\sigma_+}) \simeq G \otimes_{G/G_+} K$ (graded isomorphism) where K is the quotient field of A/I . In particular, if I is maximal, then G is a h -local ring and $\text{Gr}(G_{G_+}, G_+G_{G_+}) \simeq G$.

d) Let \mathfrak{c} be a homogeneous ideal of G ; then $\text{In}_{G+G_{G_+}}(\mathfrak{c}_{G_+})$ is graded iso-

morphic to $c \otimes_{G/G_+} K$, where K is the residue field of G_{G_+} . In particular, if I is maximal, we have $\text{In}_{G_+G_{G_+}}(c_{G_+}) \simeq c$; if we consider $\text{Cl}(G) \xrightarrow{\sigma} \text{Cl}(G_{G_+}) \xrightarrow{\bar{i}} \text{Cl}(G)$ where σ is the canonical isomorphism ([13], Corollary 10.3), we have $\bar{i} \cdot \sigma = 1_{\text{Cl}(G)}$ and consequently $\bar{i} = \sigma^{-1}$.

Proof. Easy calculations.

Now let $G = \bigoplus_{n \geq 0} G_n$ be a graded two dimensional normal domain such that G_0 is a field, $G = G_0[G_1]$ and G is a finitely generated algebra over G_0 . Since Danilov's condition DCG (i.e. $\text{Cl}(G) \simeq \text{Cl}(G[[T]])$) is equivalent to $(H_{G_+}^1(G))_n = 0$ for all $n \geq 0$ (see [12], Satz 4.4), from DCG condition it trivially follows that $\text{Cl}(G) \simeq \text{Cl}(\hat{G})$, since this is equivalent to $(H_{G_+}^1(G))_n = 0$ for all $n > 0$. (see [12], Theorem 4.1). Moreover there exist factorial graded rings as G such that $\text{Cl}(G) \simeq \text{Cl}(\hat{G})$ but not satisfying the DCG condition (see [9], page 128).

However, condition (1) is not necessary for j to be injective as the following example shows.

Let $G = \mathbb{Q}[X, Y, Z]/(X^4 + Y^4 - Z^4)$ where \mathbb{Q} is the field of rational numbers; $\text{Cl}(G)$ is finite (see [11]); but $\text{Cl}(G) \neq \text{Cl}(\hat{G})$, since $\text{Cl}(\hat{G}) \simeq \text{Cl}(G) \oplus \mathbb{Q}$ (see [12]); then take $A = G_{G_+}$. Since \bar{i} is an isomorphism (see Proposition 6.d)), j is surjective by definition of \bar{i} . Moreover $\text{Cl}(R)$, $\text{Cl}(A)$ and $\text{Cl}(G)$ are finite sets with the same number of elements; hence j is injective.

3) The authors do not know the existence of factorial graded ring G satisfying the general above-mentioned hypotheses and such that $\text{Cl}(G) \neq \text{Cl}(\hat{G})$.

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