

THE GROWTH OF THE POSITIVE SOLUTIONS OF $Lu=0$ NEAR THE BOUNDARY OF AN INNER NTA DOMAIN

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§ 1. Introduction

Let D be a bounded domain in the Euclidean space R^n ($n \geq 2$) and L a uniformly elliptic partial differential operator of second order with α -Hölder continuous coefficients ($0 < \alpha \leq 1$) on D .

According to N. Suzuki [3], D is said to be associated with the cone of angle $\theta < \pi/2$ if there exist positive constants h, d_0 and $K_0 \geq 1$ such that:

(i) For any $z \in \partial D$, there exists $e_z \in R^n$ with $|e_z| = 1$ such that $\Gamma_\theta(z, e_z) \subset D$, where $\Gamma_\theta(z, e_z)$ is the half cone obtained from $\{x \in R^n; \sqrt{x_2^2 + \cdots + x_n^2} < x_1 \tan \theta, 0 < x_1 < h\}$ by the translation z and the rotation e_z .

(ii) Put $A_D = \{y = z + te_z \in R^n; z \in \partial D, 0 < t < h/2\}$. Then for any $x \in D$ with $d(x) \leq d_0$, there exist $y_x \in A_D$ and a polygonal line L_x from x to y_x such that $d(x) \leq d(y_x)$ and the length of L_x is $\leq K_0 d(L_x, \partial D)$.

In [4] he proved the following result:

If D is associated with a cone, there exist constants $m, m' \geq 1$ such that for any positive solution of $Lu = 0$ in D ,

$$(1) \quad C_u^{-1}(d(x))^m \leq u(x) \leq C_u(d(x))^{-m'}$$

with some constant $C_u \geq 1$ depending on u , where $d(x)$ denotes the distance between x and ∂D , the boundary of D . In this paper, we shall define inner NTA (non-tangentially accessible) domains and show that for an inner NTA domain, we can choose two positive constants $m, m' \geq 1$ satisfying (1) for all positive solutions of $Lu = 0$ in D . This is a direct extension of N. Suzuki's result. As applications of our main result, we shall establish the uniqueness theorem for L -superharmonic functions on an inner NTA domain and the Harnack inequality for inner NTA domains.

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§ 2. Preliminaries

Let D be a domain in \mathbf{R}^n . For three numbers $0 < \alpha \leq 1$, $\lambda \geq 1$ and $\eta \geq 0$, we denote by $\mathcal{L}(\alpha, \lambda, \eta; D)$ the set of all uniformly elliptic differential operators L of the form

$$L = \sum_{i,j=1}^n a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^n b_i(x) \frac{\partial}{\partial x_i} + c(x)$$

with

$$\begin{aligned} \lambda^{-1} |\xi|^2 &\leq \sum_{i,j=1}^n a_{ij}(x) \xi_i \xi_j \leq \lambda |\xi|^2, \\ \sum_{i,j=1}^n |a_{ij}(x) - a_{ij}(y)| + \sum_{i=1}^n |b_i(x) - b_i(y)| + |c(x) - c(y)| &\leq \eta |x - y|^\alpha, \\ \sum_{i=1}^n |b_i(x)| &\leq \eta \quad \text{and} \quad -\eta \leq c(x) \leq 0 \end{aligned}$$

for all $x, y \in D$ and $\xi \in \mathbf{R}^n$, where $|x - y|$ is the distance between x and y . For $L \in \mathcal{L}(\alpha, \lambda, \eta; D)$, a function u of class C^2 on D is said to be L -harmonic in D if $Lu = 0$ on D . We denote by $H_L(D)$ the set of all L -harmonic functions on D and put $H_L^+(D) = \{u \in H_L(D); u > 0 \text{ on } D\}$.

A lower semi-continuous function u on D is said to be L -superharmonic if u satisfies the following conditions:

(i) $-\infty < u \leq +\infty$, $u \not\equiv +\infty$.

(ii) For any open ball B with $\bar{B} \subset D$ and any $v \in H_L(B)$ which is continuous on \bar{B} , we have

$$u \geq v \text{ on } \partial B \implies u \geq v \text{ in } B.$$

For $x \in \mathbf{R}^n$ and $r > 0$, $B(x, r)$ (resp. $\dot{B}(x, r)$) denotes the closed (resp. open) ball with center x and radius r . For an open or closed ball B , $r(B)$ denotes the radius of B .

The following Harnack inequality for L -harmonic functions plays an essential role in this paper.

PROPOSITION 1 ([1], p. 109). *For given $\lambda \geq 1$, $0 < \alpha \leq 1$ and $\eta \geq 0$, there exists a constant $K \geq 1$ depending only on λ , α and η such that for any $x \in \mathbf{R}^n$, $0 < r < 1$, $L \in \mathcal{L}(\alpha, \lambda, \eta; \dot{B}(x, r))$, $u \in H_L^+(\dot{B}(x, r))$ and any $0 < s < 1$, we have*

$$(2) \quad K^{-1}(1-s)(1+s)^{1-n}u(x) \leq u(y) \leq K(1-s)^{1-n}(1+s)u(x)$$

for all $y \in B(x, sr)$.

For a bounded domain D and $x \in D$, we denote by $d(x) = d_D(x)$ the distance between x and ∂D .

DEFINITION 1. Let D be a bounded domain in \mathbf{R}^n , M a constant > 1 and N a positive integer. An M -Harnack chain of the length N in D is a finite sequence of closed balls $(B_j)_{j=1}^N$ contained in D such that $B_j \cap B_{j+1} \neq \emptyset$ ($j = 1, \dots, N - 1$) and

$$M^{-1} \leq r(B_j)/d(B_j, \partial D) \leq M,$$

where $d(B_j, \partial D)$ denotes the distance between B_j and ∂D .

Let $x, y \in D$. We say that y can be connected with x by an M -Harnack chain $(B_j)_{j=1}^N$ of the length N in D if x is the center of B_1 and $y \in B_N$. For $x \in D$, we denote by $H_{M,N}(x)$ the set of the points which can be connected with x by an M -Harnack chain of the length N in D .

DEFINITION 2. Let $M > 1$ be a constant, N a positive integer and $0 < \nu < 1$ a constant. A bounded domain D in \mathbf{R}^n is called an (M, N, ν) -inner NTA domain if there exist a constant $r_0 > 0$ and a mapping $\Phi(z) = (z_j(z))_{j=1}^\infty$ from ∂D to sequences in D with $d(z_1(z)) \geq r_0$ and $\lim_{j \rightarrow \infty} z_j(z) = z$ satisfying the following two conditions:

(I) For any $z \in \partial D$,

$$z_{j+1}(z) \in H_{M,N}(z_j(z)) \quad (j = 1, 2, \dots)$$

and

$$(3) \quad \sup_{z \in \partial D} \sup_{1 \leq j < \infty} d(z_j(z))/\nu^j < +\infty.$$

(II) For each $x \in D$, we put $R_x = \bigcup_{z \in \partial D} \{z_j(z); d(x) \leq d(z_j(z))\}$. Then

$$\sup_{\substack{x \in D \\ d(x) \leq r_0}} \inf_{H_{P,Q}(x) \cap R_x \neq \emptyset} P + Q < +\infty.$$

A bounded domain D in \mathbf{R}^n is simply called an inner NTA domain if there exist $M > 1$, $0 < \nu < 1$ and a positive integer N such that D is an (M, N, ν) -inner NTA domain.

Remark 1. NTA domains (cf. [2], p. 93) are inner NTA domains. Here an NTA domain is a bounded domain in \mathbf{R}^n such that there exist $M > 1$ and $r_0 > 0$ satisfying the following conditions:

(i) For any $z \in \partial D$ and any $r \leq r_0$, there exists $a = a_r(z) \in D$ such that $d(a) \geq M^{-1}r$ and $M^{-1}r \leq |a - z| \leq r$.

(ii) The complement of \bar{D} also satisfies the condition (i).

(iii) For any $\varepsilon > 0$ and any $x, y \in D$ such that $d(x) \geq \varepsilon$, $d(y) \geq \varepsilon$ and $|x - y| \leq \delta$, there exists an M -Harnack chain from x to y whose length depends only on δ/ε .

We remark that there are inner NTA domains which are not NTA domains. For example, $D = \{(r, \theta) \in \mathbf{R}^2 \setminus \{0\}; r \neq e^\theta, \theta < 0, r < 1\}$ is such a domain.

Remark 2. Put $M = \sin \theta / (1 - \sin \theta)$, $N = 1$ and $\nu = 1 - \sin^2 \theta$. Then the domain being associated with the cone of angle θ is an (M, N, ν) -inner NTA domain.

According to N. Suzuki [4], a bounded domain in \mathbf{R}^n is said to be associated with the ball of radius $r > 0$ if there exist positive constants r , d_0 and $K_0 \geq 1$ such that:

(i) For any $z \in \partial D$, there exists $e_z \in D$ with $d(e_z, z) = r$ such that $\dot{B}(e_z, r) \subset D$.

(ii) Put $A_D = \{y = z + t(e_z - z) \in \mathbf{R}^n; z \in \partial D, 0 < t \leq 2\}$. Then for any $x \in D$ with $d(x) \leq d_0$, there exist $y_x \in A_D$ and a polygonal line L_x from x to y_x such that $d(x) \leq d(y_x)$ and the length of L_x is $\leq K_0 d(L_x, \partial D)$.

Remark 3. The above domain being associated with a ball is an $(M, 1, 1/(M + 1))$ -inner NTA domain for all $M > 1$.

§ 3. Main result

THEOREM 1. *Let $M > 1$ be a constant, N a positive integer, $0 < \nu < 1$ a constant and D an (M, N, ν) -inner NTA domain in \mathbf{R}^n . For a fixed $x_0 \in D$, we set $H_L^0(D) = \{u \in H_L^+(D); u(x_0) = 1\}$. Put*

$$(4) \quad m = m(M, N, \nu) = \frac{(2N - 1) \log K^{-1}(M + 1)^{n-2}(2M + 1)^{1-n}}{\log \nu}$$

and

$$(5) \quad m' = m'(M, N, \nu) = \frac{(2N - 1) \log K(M + 1)^{n-2}(2M + 1)}{-\log \nu},$$

where K is the constant in Proposition 1. Then there exist positive constants C and C' such that for any $u \in H_L^0(D)$,

$$(6) \quad C(d(x))^m \leq u(x) \leq C'(d(x))^{-m'}$$

on D .

Remark 4. (1) If a domain is associated with the cone of angle $\theta < \pi/2$, then $m = \log \{K^{-1}(1 - \sin \theta)(1 + \sin \theta)^{1-n}\} / (2 \cdot \log \cos \theta)$ and $m' = -\log \{K(1 - \sin \theta)^{1-n}(1 + \sin \theta)\} / (2 \cdot \log \cos \theta)$, which are also obtained by N. Suzuki [4].

(2) If the domain D is associated with a ball, we can choose $m = 1$ and $m' = n - 1$.

(3) In the case $n = 2$ and $L = \Delta$, Kuran-Schiff [3] obtained a more precise estimate for rather specific domains.

Proof of Theorem 1. Put $F = \{x \in D; d(x) \geq r_0\}$, then F is compact in D . From Proposition 1, it follows that there exist two positive constants A_1 and A_2 depending only on D and x_0 such that for any $u \in H_L^0(D)$ and any $x \in F$,

$$A_1 \leq u(x) \leq A_2.$$

For any $z \in \partial D$, we have $z_1(z) \in F$, so

$$(7) \quad A_1 \leq u(z_1(z)) \leq A_2.$$

Let $z \in \partial D$. Then for any k , there exists an M -Harnack chain $(B_j)_{j=1}^N$ from $z_k(z)$ to $z_{k+1}(z)$. We choose $b_j \in B_j \cap B_{j+1}$ ($1 \leq j \leq N - 1$) and γ_k the polygonal line $\cup_{j=0}^{N-1} \overline{b_j b_{j+1}}$, where $b_0 = z_k(z)$, $b_N = z_{k+1}(z)$ and $\overline{b_j b_{j+1}}$ is the closed segment between b_j and b_{j+1} . Put $\gamma = \{z\} \cup (\cup_{k=1}^\infty \gamma_k)$. Then γ is a rectifiable curve from z to $z_1(z)$. Put $C_0 = K^{-1}(M + 1)^{n-2}(2M + 1)^{1-n}$ and $\tilde{C}_0 = K(M + 1)^{n-2}(2M + 1)$. Proposition 1 shows that for any $x \in \gamma_k$,

$$\begin{aligned} C_0^{2N-1} u(z_k(z)) &\leq u(x) \leq \tilde{C}_0^{2N-1} u(z_k(z)) \\ C_0^{(2N-1)k} u(z_1(z)) &\leq u(x) \leq \tilde{C}_0^{(2N-1)k} u(z_1(z)). \end{aligned}$$

By (7), we have

$$A_1 C_0^{(2N-1)k} \leq u(x) \leq A_2 \tilde{C}_0^{(2N-1)k}.$$

By (3), there exists a positive constant β such that for all $k \geq 1$,

$$d(z_k(z)) \leq \beta \nu^k.$$

Then for any $x \in \gamma_k$, we have

$$d(x) \leq C_2^{N-1} d(z_k(z)) \leq C_2^{N-1} \beta \nu^k,$$

where $C_2 = (2M + 1)^2$. Putting $C_3 = A_1(\beta^{-1} C_2^{1-N})^m$ and $\tilde{C}_3 = A_2(\beta^{-1} C_2^{1-N})^{-m'}$, we have

$$(8) \quad C_3(d(x))^m \leq u(x) \leq \tilde{C}_3(d(x))^{-m'}$$

for all $x \in \gamma \cap D$.

Let $x \in D \setminus F$. By the condition (II) in Definition 2, there exist a constant $P > 1$, a positive integer Q and $z_k(z) \in R_x$ such that x can be connected to $z_k(z)$ by a P -Harnack chain of the length $\leq Q$. From Proposition 1, it also follows that

$$(9) \quad C_4^{2Q}u(z_k(z)) \leq u(x) \leq \tilde{C}_4^{2Q}u(z_k(z)),$$

where $C_4 = K^{-1}(P+1)^{n-2}(2P+1)^{1-n}$ and $\tilde{C}_4 = K(P+1)^{n-2}(2P+1)$.

Combining (8) and (9), we have

$$(10) \quad C_3C_4^{2Q}(d(x))^m \leq u(x) \leq \tilde{C}_3\tilde{C}_4^{2Q}(d(x))^{-m'}$$

for all $x \in D \setminus F$. Put $C = C_3C_4^{2Q}$ and $C' = \tilde{C}_3\tilde{C}_4^{2Q}$. Then we have

$$C(d(x))^m \leq u(x) \leq C'(d(x))^{-m'}$$

for all $x \in D$, which completes the proof of our theorem.

§ 4. Applications

We apply our main result to the following uniqueness theorem for L -superharmonic functions.

THEOREM 2. *Let D be an (M, N, ν) -inner NTA domain, $L \in \mathcal{L}(\alpha, \lambda, \eta; D)$ and let m be the constant obtained in (4). If a non-negative L -superharmonic function u in D satisfies*

$$\liminf_{x \rightarrow z} u(x)/(d(x))^m = 0$$

for some $z \in \partial D$, then u is identically equal to 0.

Proof. Let G be the Green function on D with respect to L . Assume that there exists $x_0 \in D$ such that $u(x_0) > 0$. We can choose $r > 0$ such that $B(x_0, r) \subset D$ and $u(x) > 0$ on $B(x_0, r)$. There exists a positive measure $\mu \neq 0$ supported by $B(x_0, r/2)$ such that $G\mu(x)$ is finite continuous on D and $G\mu(x) \leq u(x)$ on $B(x_0, r)$, where $G\mu(x) = \int G(x, y)d\mu(y)$. By the maximum principle, we have $G\mu(x) \leq u(x)$ on D . Put $D' = D \setminus B(x_0, r)$; D' is an (M, N, ν) -inner NTA domain and the restriction of $G\mu$ to D' is L -harmonic in D' . Since $G\mu > 0$ on D' , Theorem 1 shows that for any $x \in D'$, $G\mu(x) \geq C(d'(x))^m$ with some $C > 0$, where $d'(x) = d(x, \partial D')$. Hence $u(x) \geq C(d'(x))^m$ for all $x \in D'$, which contradicts our assumption. Thus Theorem

2 is proved.

The following theorem is a generalization of the Harnack inequality on a ball.

THEOREM 3. *Let D and L be the same as in Theorem 1 and let m and m' be the constants obtained in (4) and (5). Then there exist positive constants C and C' such that for any $u \in H_L^+(D)$ and any relatively compact open subset Ω of D ,*

$$C(d_\Omega)^m \leq u(y)/u(x) \leq C'(d_\Omega)^{-m'}$$

for all $x, y \in \Omega$, where $d_\Omega = d(\Omega, \partial D)$.

The above theorem immediately follows from Theorem 1.

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