E. Ballico and A. J. Sommese
Nagoya Math. J.
Vol. 136 (1994), 57–79

PROJECTIVE SURFACES WITH K–VERY AMPLE LINE BUNDLES OF DEGREE $\leq 4K + 4$

EDOARDO BALLICO AND ANDREW J. SOMMESE

Introduction

A line bundle, L, on a smooth, connected projective surface, S, is defined [7] to be k-very ample for a non-negative integer, k, if given any 0-dimensional subscheme $(Z, \mathcal{O}_Z) \subset S$ with length $(Z, \mathcal{O}_Z) \leq k + 1$, it follows that the restriction map $\Gamma(L) \to \Gamma(L \otimes \mathcal{O}_Z)$ is onto. L is 1-very ample (respectively 0-very ample) if and only if L is very ample (respectively spanned at all points by global sections). For a smooth surface, S, embedded in projective space by $|L|$ where L is very ample, L being k-very ample is equivalent to there being no k-secant $\mathbf{P}^{k-1}$ to S containing $\geq k + 1$ points of S.

In this article we study pairs (S, L), where S is a smooth, projective surface and L is a k-very ample line bundle satisfying $L \cdot L \leq 4k + 4$.

In [8] M. Beltrametti and the second author studied the question of when L being k-very ample implies that $K_S \otimes L$ is k-very ample. This question generalizes classical questions for very ample bundles, and has a nice interpretation as a question about adjunction on $S^{[k]}$, the space of 0-dimensional subschemes of length k on S (see the introduction to [8] for details).

That question breaks up naturally into the cases when $d := L \cdot L \geq 4k + 5$ and the cases when $d \leq 4k + 4$. In [8], Beltametti and the second author gave a complete answer to the question for $d \geq 4k + 5$ using their generalization, [8], of the Reider criterion for spannedness and very ampleness. This division into two parts exists in the classical case for very ample line bundles (see [18]).

In §2 and §3 we prove a number of general results for k-very ample line bundles on curves and surfaces respectively.

With these results we turn in §4 to the study of special pairs (S, L) with $d \leq 4k + 4$, mainly $\mathbf{P}^1$-bundles and k-conic bundles. The study of such special classes is required by our approach based on [8, Theorem (3.1)]. That theorem says that either (S, L) is on a list of very special pairs or $kK_S + L$ is spanned

Received March 22, 1993.

and big.

In §5 we classify all pairs (S, L) where L is a k-very ample line bundle on S with $k \geq 2$ and $d \leq \max\{11, 4k+2\}$.

In §6 we show that for $k \geq 9$, if L is k-very ample and $\kappa(S) = -\infty$, then $L \cdot L \geq 4k+5$. We also show that for $k \geq 5$, if L is k-very ample and $\kappa(S) \geq 0$, then $L \cdot L \geq 4k+5$ except for S a $K3$-surface with $d = 4k$, $4k+2$, $4k+4$, or an Enriques surface with $d = 4k+4$. In Remark (6.2), we discuss the k-very ampleness of $K_S + L$ in view of our results.

We especially thank the referee for the proof of (2.3), which we conjectured in our original paper, for the useful result (3.6), and for a number of simplifications of our original arguments. We thank S. Di Rocco for helpful suggestions including a simplification of our original proof of Lemma (3.9), and a proof of a version of (2.3) between our original result of and the complete statement proved by the referee. We would both like to thank the University of Notre Dame for making this collaboration possible. The first author was partially supported by MURST and GNSAGA of CNR (Italy). The second author would also like to thank the NSF (DMS 89-21702 and DMS 93-02021), and especially the Sonderforschungsbereich 170 at the University of Göttingen.

1. Background material

Throughout this paper we will follow the notation of [8]. All surfaces will be smooth, connected, and projective.

We need the following result, [17, Proposition (0.9)], which is due to Weil in the non-ruled case and the second author in the ruled case.

LEMMA 1.1. *Let L be a very ample line bundle on a smooth projecive surface, S. If E is a line bundle on S with $E_C \cong \mathcal{O}_C$ for an open set of curves $C \in |L|$, then $E \cong \mathcal{O}_S$ unless S is a $\mathbf{P}^1$-bundle over some curve with $L_f \cong \mathcal{O}_{\mathbf{P}^1}(1)$.*

Note that if L is k-very ample for some $k \geq 2$, then there are no curves f on S with $L \cdot f = 1$.

The next result, [8, Proposition (2.6)], will let us assume without loss of generality that $d := L \cdot L \geq 2k+4$.

THEOREM 1.2. *Assume that L is k-very ample for $k \geq 2$ and $d \leq 2k+3$. Then $2 \leq k \leq 3$ and $(S, L) \cong (\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(k))$.*

2. k-very ample line bundles on curves

In [6] the following result is shown.

THEOREM 2.1. *Let L be a k-very ample line bundle on a smooth curve, C, of genus $g \geq 1$. Then $\deg L \geq k+2$, and if $h^1(L) \geq 1$, then K_C is k-very ample with $g \geq 2k+1$.*

THEOREM 2.2. *Let L be a k-very ample line bundle on a smooth curve, C. If $h^1(L) \geq 3$ and $2g-2-d < k+2h^1(L)-2$ then K_C is $(k+1)$-very ample.*

Proof. Consider the following procedure.

- Given an effective cycle, $Z \subset C$, such that $h^0(K_C - L - Z) \geq 3$ and $\deg Z < 2h^0(K_C - L) - 4$, choose if possible two not necessarily distinct points $\{a, b\} \subset C$ such that $h^0(K_C - L - Z - a - b) \geq h^0(K_C - L - Z) - 1$. Let Z be redefined as the old cycle Z plus the points a, b.

Starting with the empty cycle Z, repeat the above procedure until it stops. We end up with a cycle Z such that either

1. $\deg Z \geq 2h^0(K_C - L) - 4$ and $h^0(K_C - L - Z) \geq 2$; or
2. $\deg Z < 2h^0(K_C - L) - 4$ and $h^0(K_C - L - Z) \geq 3$.

In the first case, the k-very ampleness of K_C (see (2.1)) and the fact that $h^0(K_C - L - Z) \geq 2$ imply that $\deg(K_C - L - Z) \geq k+2$, i.e., $\deg(K_C - L) \geq 2h^0(K_C - L) + k - 2$. Therefore assuming that $2g-2-d < k+2h^1(L)-2$, the second possibility must hold.

Here by construction given any 2, possibly equal points, a, b, we have $h^0(K_C - L - Z - a - b) = h^0(K_C - L - Z) - 2$. Therefore $K_C - L - Z$ is 1-very ample. Thus by [6, Lemma (0.3.5)], we conclude that $K_C - Z = L + (K_C - L - Z)$ is $(k+1)$-very ample. Since $h^1(K_C - Z) = h^0(Z) \geq 1$ we have by (2.1) that K_C is $(k+1)$-very ample. Q.E.D.

In our original article we conjectured the following result, and proved a partial version of it. We are very grateful to the referee for the following proof of the full conjecture.

THEOREM 2.3. *Let L be a k-very ample line bundle of degree d on an irreducible, non-singular curve of genus g. If $h^1(L) \geq 2$, then $2g-2-d \geq k+2h^1(L)-2$.*

Proof. If $h^1(L) = 2$, then $2g - 2 - d \geq k + 2$ since K_C is k-very ample by (2.1). Hence we need only consider the case when $h^1(L) \geq 3$.

We will assume that $2g - 2 - d < k + 2h^1(L) - 2$ and derive a contradiction. If K_C is l-very ample but not $(l+1)$-very ample, then C is $(l+2)$-gonal, and we have $l \geq k + 1$ by (2.2). Since $h^0(L)$ and $h^1(L)$ are both greater than 1, we have $d \geq l + 2$, $2g - 2 - d \geq l + 2$. By a result of Coppens and Martens [9, Theorem B] applied to $K_C - L$, we get $2g - 2 - d \geq (l - 1) + 2h^1(L) - 2$. Since $l > k$, this contradicts the assumption. Q.E.D.

Theorem (2.5) below gives added information when $h^1(L) \geq 1$. The following lemma is a simple corollary of the Brill-Noether existence theorem and an ampleness result of Fulton-Lazarsfeld, [10, Lemma (2.7)]. In what follows, $\rho(g, x, y)$, denotes the Brill-Noether number, $g - (x + 1)(g - y + x)$.

LEMMA 2.4. *Let A be an effective divisor of degree $t > 0$ on a curve C. Assume that $\rho(g, x, y) \geq t - x$. Assume that $g + x \geq y \geq t \geq x$. Then there is an effective divisor $D \subset C$ such that,* $\deg D = y$, $h^0([D]) \geq x + 1$, *and $A \subset D$.*

Proof. By the Brill-Noether existence theorem [2], if $x \geq y - g$ then there is an algebraic set, $V \subset C^{(y)}$, of dimension at least t where each fiber under the map to $\mathrm{Pic}(C)$ is a linear series of dimension at least x. By [10, Lemma (2.7)], it follows that $A + C^{(y-t)}$ meets V non-trivially. D can be taken to be any point in the intersection. Q.E.D.

THEOREM 2.5. *Let L be a k-very ample line bundle on a smooth curve, C. Assume that $k \geq 1$. If $h^1(L) > 0$ and $K_C \not\cong L$, $d := \deg L \geq k + g - h^1(L) + \dfrac{k+1}{h^1(L)}$.*

Proof. Assume that the inequality is false, i.e., that $d \leq k + g - 1 - h^1(L) + \dfrac{k+1}{h^1(L)}$. Since $h^1(L) \neq 0$, and $K_C \not\cong L$, we can choose an effective divisor $A \in |K_C - L|$. Set $w := 2g - 2 - d$. Note that $\rho(g, h^1(L), w + k + 1) \geq w - h^1(L)$, is equivalent to $d \leq k + g - 1 - h^1(L) + \dfrac{k+1}{h^1(L)}$.

Note that $g + h^1(L) \geq w + k + 1 \geq w \geq h^1(L)$. The first inequality is equivalent to $h^0(L) \geq k$ which is immediate since L is k-very ample. The second is obvious. Since $g \geq 1$, $h^1(L) \leq \deg(K_C - L)$, and the third inequality follows.

We conclude from Lemma (2.4) that there exists an effective divisor $D \subset C$ such that $\deg D = k + 1 + w$, $h^0([D]) \geq h^1(L) + 1$, and $A \subset D$.

Set $Z := D - A$. Note that $\deg Z = k + 1$. We will be done if we show that $h^1(L - Z) \geq h^1(L) + 1$. Indeed otherwise it would follow from the exact sequence, $0 \to L - Z \to L \to L_Z \to 0$, that $\Gamma(L) \to \Gamma(L_Z)$ is not onto. Note that $h^1(L - Z) = h^1(L - D + A) = h^1(K_C - D) = h^0([D])$, which is $> h^1(L) + 1$ by the second property of D stated above. Q.E.D.

COROLLARY 2.6. *Let L be a k-very ample line bundle on a smooth curve, C. Assume that $h^1(L) \neq 0$. Then either $K_C \cong L$ or $g \geq 2k + 3$. If $g = 2k + 3$, then $h^1(L) = 1$.*

Proof. If $h^1(L) = 1$, then Theorem (2.5) gives $d \geq 2k + g$. Since $K_C \not\cong L$, $d \leq 2g - 3$. This gives $g \geq 2k + 3$.

If $h^1(L) \geq 2$, then $2g - 2 \geq d + k + 2h^1(L) - 2$ by (2.3). Using the inequality from Theorem (2.5), we obtain

$$(1) \qquad 2g - 2 \geq d + k + 2h^1(L) - 2 \geq 2k + g + h^1(L) - 2 + \frac{k+1}{h^1(L)}.$$

If $h^1(L) = 2$ then inequality (1) gives $2g - 2 \geq 2k + g + \frac{3}{2}$, and if $h^1(L) \geq 3$, then inequality (1) gives $2g - 2 \geq 2k + g + 2$. In either case we get $g \geq 2k + 4$.

Q.E.D.

LEMMA 2.7. *Let L be a k-very ample line bundle on an irreducible curve, C with $k \geq 2$. If the arithmetic genus, γ, of C is* 2, 3, 4, then $L \geq k + \gamma + 2$.

Proof. Assume that $L \leq k + \gamma + 1$. Choose $k - 1$ smooth points $\{x_1, \ldots, x_{k-1}\}$ of C. Let $\mathscr{L} := L - \sum_{i=1}^{k-1} x_i$. By [8, Lemma (1.1)] it follows that $\mathscr{L}$ is very ample with $\mathscr{L} = \deg L - k + 1 \leq \gamma + 2$ and $h^1(L) = h^1(\mathscr{L})$.

If $\gamma = 2$, then $|\mathscr{L}|$ cannot embed C as a plane curve. Using Castelmuovo's bound for the genus, [8, (0.2)] with $h^0(\mathscr{L}) \geq 4$ we have that $\deg \mathscr{L} \geq 5$. This gives that $L = \deg \mathscr{L} + k - 1 \geq 5 + k - 1$ proving the Lemma in the case $\gamma = 2$.

If $\gamma = 3$ then either $|\mathscr{L}|$ embeds C as a plane curve, necessarily of degree 4, or $h^0(\mathscr{L}) \geq 4$. Note in the former case $\mathscr{L} \cong K_C$ and $h^1(\mathscr{L})$ is thus 1. This contradicts the fact that $h^1(L) = h^1(\mathscr{L})$ with $\deg L = \deg \mathscr{L} + k - 1 \geq \deg \mathscr{L} + 1$. If $h^0(\mathscr{L}) \geq 4$, then by Castelnouvo's bound for the genus, we have that $\deg \mathscr{L} \geq 6$. This gives that $\deg L = \deg \mathscr{L} + k - 1 \geq 6 + k - 1$ proving the Lemma in the case $\gamma = 3$.

If $\gamma = 4$, then $|\mathscr{L}|$ cannot embed C as a plane curve. Castelnuovo's bound shows that $\mathrm{dg}\,\mathscr{L} \geq 6$. Note that if $\deg \mathscr{L} = 6$, then since $h^0(\mathscr{L}) \geq 4$, it follows that $\mathscr{L} \cong K_C$ which gives the same cohomology contradiction as for $\gamma = 3$. Therefore we have that $\deg \mathscr{L} \geq 7$. This proves the Lemma. Q.E.D.

In [5] the first author showed that given a k-very ample line bundle on a smooth surface (respectively a ruled surface), then $h^0(L) \geq 2k$ (respectively $h^0(L) \geq 2k+2$). The argument as written there actually proves more. First there is the useful Lemma [5, Lemma (1.3)].

LEMMA 2.8. *Let $\mathscr{L}$ be a line bundle on be a smooth curve, C. Assume that there is a proper linear subspace, $V \subset \Gamma(\mathscr{L})$ such that given any effective divisor Z on C with $\deg Z = k+1$, the evaluation map $C \times V \to \Gamma(\mathscr{L} \otimes \mathcal{O}_Z)$ is onto. Then $\dim V \geq 2k+2$ and $\dim \Gamma(\mathscr{L}) \geq 2k+3$. In particular if L is a k-very ample line bundle on a smooth surface S, and $\Gamma(L) \to \Gamma(L_C)$ is not onto for some smooth $C \in |L|$, then $h^0(L) \geq 2k+3$ and $h^0(L_C) \geq 2k+3$.*

The following is proved by step (c) of the proof of the main theorem of [5] with no change.

THEOREM 2.9. *Let L be a k-very ample line bundle on a smooth curve. Assume that $h^1(L) \neq 0$. Then $h^0(L) \geq 2k+1$.*

The following is proved by step (b) of the proof of main theorem of [5].

LEMMA 2.10. *Let L be a k-very ample line bundle on a smooth curve C. If $h^1(L) = 0$, then $\deg L \geq 2k+g+1$ or $\deg L \geq 2g+k$.*

Proof. Let $d := \deg L$ and assume to the contrary that $d \leq 2k+g$ and $d \leq 2g+k-1$. We will be done if we show that there is a length $k+1$, 0-cycle, $Z \subset C$, with $\Gamma(L) \to \Gamma(L_Z)$ not onto. This is equivalent to producing a length $k+1$, 0-cycle, $Z \subset C$ with $h^1(L-Z) \geq 1$. This will be done if we produce an effective (possibly empty) 0-cycle $M \subset C$ of length $2g-2-d+k+1$ and an effective, length $k+1$, 0-cycle, $Z \subset C$ such that $K_C - M \cong L - Z$. Note that $\deg Z + \deg M = k+1+2g-2-d+k+1$ and this is $\geq g$ by hypothesis. Note also that $\deg M \geq 0$ by hypothesis. Thus the difference map $C^{\deg Z} \times C^{\deg M} \to \mathrm{Jac}(C)$ is onto. By the identification of $\mathrm{Jac}(C)$ with the component of $\mathrm{Pic}(C)$ parametrizing $\deg(L-K_C)$ line bundles we have produced the desired Z and M giving the contradiction. Q.E.D.

THEOREM 2.11. *Let L be a k-very ample line bundle on a smooth, connected projective surface, S If $k \geq 2$ and $h^1(L_C) = 0$ for some smooth $C \in |L|$, then* $\deg L > 2k + g + 1$.

Proof. Let $d := \deg L$ and assume to the contrary that $d \leq g + 2k$. By (2.10) we can assume without loss of generality that

$$2g + k \leq d \leq g + 2k. \tag{2}$$

Since $L \cdot \mathscr{E} \geq k$ for all irreducible curves, $\mathscr{E} \subset S$, we see that for $S \cong \mathbf{P}^2$, a line bundle L is k-very ample only if it is of the form $L \cong \mathscr{O}_{\mathbf{P}^2}(a)$ for some $a \geq d$. For such an L with $k \geq 2$, (2) is impossible.

Similarly for $S \cong \mathbf{P}^1 \times \mathbf{P}^1$, a line bundle L is k-very ample only if it is of the form $L \cong \mathscr{O}_{\mathbf{P}^1 \times \mathbf{P}^1}(a, b)$ for some $a \geq k$ and $b \geq k$. For such an L with $k \geq 2$ (2) is impossible.

By (1.2) we can assume without loss of generality that $d \geq 2k + 4$. Combined with (2) we conclude that $k \geq g \geq 4$. Since $L \cdot \mathscr{E} \geq k \geq 4$ for all irreducible curves $\mathscr{E} \subset S$, we conclude from the main theorem of [18] that $K_S + L$ is very ample. Furthermore using the spannedness criterion for the adjoint of a very ample bundle, e.g., [18], we see that $K_S + (K_S + L)$ is spanned by global sections unless $S \cong \mathbf{P}^2$, $S \cong \mathbf{P}^1 \times \mathbf{P}^1$, or S is a $\mathbf{P}^1$ bundle over a curve with $(K_S + L)_f \cong \mathscr{O}_{\mathbf{P}^1}(1)$ for a fiber f of the bundle. The first two surfaces have already been dealt with. The fact that $L \cdot f \geq k \geq 4$ implies that $(K_S + L) \cdot f \geq 2$, which rules out the last case.

Moreover $L \not\cong -2K_S$. Indeed if this happened then we would have that either $S \cong \mathbf{P}^2$ or $S \cong \mathbf{P}^1 \times \mathbf{P}^1$ or S is not minimal. The first two cases have been dealt with. If S is not minimal then there is a smooth rational curve $\mathscr{E}$ with $K_S \cdot \mathscr{E} = \mathscr{E} \cdot \mathscr{E} = -1$. Thus $L \cdot \mathscr{E} = 2$ which contradicts the fact that $L \cdot \mathscr{E} \geq k \geq 4$.

Thus there exists a nontrivial $\mathscr{E} \in |K_S + (K_S + L)|$, which implies that $L \cdot (2K_S + L) = L \cdot \mathscr{E} \geq k$. This gives $4g - 4 - d \geq k$. Using $d \geq 2g + k$ from (2) we get $k \leq 2g - 4 - k$. Thus $k \leq g - 2$ in contradiction to (2). Q.E.D.

COROLLARY 2.12. *Let L be a k-very ample line bundle on a smooth, connected, projective surface, S. Assume that $k \geq 2$ and that $h^1(L_C) = 0$ for some smooth $C \in |L|$. Then $h^0(L_C) \geq 2k + 2$ and $d \geq 4k + 4 + K_S \cdot L$. In particular if $d \leq 4k + 4$ then $K_S \cdot L \leq 0$ with equality implying $d = 4k + 4$.*

Proof. Note that $h^0(L_C) = d - g + 1 \geq 2k + 2$. Simply rewrite the inequality in Theorem (2.11) using $2g - 2 = K_S \cdot L + d$. Q.E.D.

As a consequence we have the result that the first author's proof in [5] actually yields.

THEOREM 2.13. *Let L be a k-very ample line bundle on a smooth surface, S, with $k \geq 2$. Then $h^0(L) \geq 2k + 2$ and $h^0(L_C) \geq 2k + 1$ for a smooth $C \in |L|$. If equality holds in either inequality, then $K_S \cong \mathcal{O}_S$ or $h^1(L_C) \geq 2$ and $\Gamma(L) \to \Gamma(L_C)$ is onto.*

Proof. By (2.8), we can assume without loss of generality that the map $\Gamma(L) \to \Gamma(L_C)$ is onto. If $h^1(L_C) = 0$ then $h^0(L_C) = d - g + 1 \geq 2k + 2$ by Theorem (2.11). If $h^1(L_C) = 1$, and $K_C \not\cong L_C$, then $h^0(L_C) = d - g + 1 + 1 \geq 2k + 2$ by Theorem (2.5). If $h^1(L_C) = 1$, and $K_C \cong L_C$ then $K_S \cong \mathcal{O}_S$ by Theorem (1.1). If $h^1(L_C) \geq 2$, then use Theorem (2.9). Q.E.D.

3. k-very ampleness for line bundles on surfaces

LEMMA 3.1. *Let L be a k-very ample line bundle on a smooth, connected, projective surface, S, with $k \geq 2$, and $d \leq 4k + 4$. Assume that $h^1(L_C) \neq 0$ for some smooth curve, $C \in |L|$. Then $d \geq 2h^0(L) - 4$ with equality only if $K_S \cong \mathcal{O}_S$. In this case $d \geq 4k$.*

Proof. This is just Clifford's inequality. Indeed given a smooth, $C \in |L|$, $h^0(L_C) \leq \frac{d}{2} + 1$ with equality only if $K_C \cong L_C$, or L_C is a multiple of the hyperelliptic line bundle on a hyperelliptic curve. If there was a hyperelliptic $C \in |L|$ with $k \geq 2$, then $h^1(L_C) = 0$. Since $k \geq 2$, (S, L) can't be scroll, and therefore by (1.1), we have equality only if $K_S \cong \mathcal{O}_S$. Note that in this case, $K_C \cong L_C$, $d = 2g - 2$, and $h^1(L_C) = 1$. Thus by (2.13), $\frac{d}{2} + 1 = g = h^0(L_C)$ is $\geq 2k + 1$, i.e., $d \geq 4k$. Q.E.D.

COROLLARY 3.2. *Let L be a k-very ample line bundle on a smooth, connected, surface, S, with $d \leq 4k + 4$. Assume that $K_S \sim 0$, but $K_S \not\cong \mathcal{O}_S$. Then $d = 4k + 4$, and S is an Enriques surface, i.e., $2K_S \cong \mathcal{O}_S$ with the double cover of S simply connected. If $K_S \cong \mathcal{O}_S$, then d equals $4k$, $4k + 2$, or $4k + 4$, with S a $K3$-surface.*

Proof. If $K_S \not\cong \mathcal{O}_S$, but $K_S \sim 0$, it follows that $h^2(\mathcal{O}_S) = 0$. Moreover by Kodaira's Vanishing Theorem, $h^1(L) = h^2(L) = 0$. Thus $h^1(L_C) = 0$ for any smooth $C \in |L|$. By Corollary (2.12) we know that $d = 4k + 4$. Thus $2k + 2 = g(L) - 1 = h^0(L_C)$. Note that if $q \neq 0$, then the restriction $\Gamma(L) \to \Gamma(L_C)$ is not onto and (2.8) gives the absurdity, $h^0(L_C) \geq 2k + 3$. Since $q = 0$, the result is a standard result of surface theory.

Assume now that $K_S \cong \mathcal{O}_S$. Then $h^1(L_C) = 1$ for smooth $C \in |L|$. From this it follows from (3.1) that d equals $4k$, $4k + 2$, or $4k + 4$. If S is not a $K3$-surface, then $q = 2$, and therefore since $h^1(L) = 0$ it follows that $\Gamma(L) \to \Gamma(L_C)$ is not onto. Therefore by (2.8) we have $h^0(L) \geq 2k + 3$. Thus using $h^1(\mathcal{O}_S) = 2$ and $h^1(L) = 0$, we have $h^0(L_C) = h^0(L) - 1 + 2 \geq 2k + 4$. This gives the absurdity that $d = 2g - 2 = 2h^0(L_C) - 2 \geq 4k + 6$. Q.E.D.

The following result is proved in [3, 4].

THEOREM 3.3. *Let L be a k-very ample line bundle on a smooth, projective surface, S. If $k \geq 2$ then $h^0(L) \geq k + 5$ with the exception when $k = 2$ and either $(S, L) \cong (\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(2))$, or (S, L) is the intersection of 3 quadrics in $\mathbf{P}^5$.*

LEMMA 3.4 *If $d \leq 2h^0(L) - 4$ then either $d = 2h^0(L) - 4$, and S is a $K3$ surface, i.e., $K_S \cong \mathcal{O}_S$ with $q = 0$, or $\kappa(S) = -\infty$.*

Proof. Assume that $\kappa(S) \geq 0$. Then $d \geq 2h^0(L) - 4$. Therefore $d = 2h^0(L) - 4$. This implies that $tK_S \cong \mathcal{O}_S$ for some minimum $t \geq 1$. Letting C be a smooth element in $|L|$, we conclude from, $0 \to K_S \to K_S + L \to K_C \to 0$, that $g = h^0(K_S + L) + q - h^2(\mathcal{O}_S)$. But this gives:

$$\begin{aligned} 2h^0(L) - 4 = d = d + K_S \cdot L &= 2g - 2 \\ &\geq 2h^0(K_S + L) - 2\chi(\mathcal{O}_S) = 2h^0(L) - 2\chi(\mathcal{O}_S). \end{aligned}$$

Thus $\chi(\mathcal{O}_S) \geq 2$ which implies that $t = 1$ and S is a $K3$ surface. Q.E.D.

COROLLARY 3.5. *If $d \leq \max\{10, 4k\}$, then either $\kappa(S) = -\infty$, or S is a $K3$ surface satisfying (k, d) equal either (2,10), or $(k, 4k)$.*

Proof. This is immediate from Corollary (3,2) and Lemma (3.4). Q.E.D.

The following useful consequence of Theorem (2.3) was given by the referee.

PROPOSITION 3.6. *Let L be a k-very ample line bundle on a smooth connected surface S and assume that $h^1(L_C) \geq 2$ for a smooth $C \in |L|$. Then $d \geq 5k$ if $k \geq 3$, and $d \geq 12$ if $k = 2$. In particular, if $d \leq 4k + 4$, then $(k, d) = (2,12)$, $(3,15)$, $(3,16)$, or $(4,20)$.*

Proof. By the Riemann-Roch theorem and (2.3), we obtain $d \geq k + 2h^0(L_C) - 2$. By (2.13) and (3.3), we have $h^0(L_C) \geq 2k + 1$ if $k \geq 3$ and $h^0(L_C) \geq 6$ if $k \geq 2$. Therefore $d \geq 5k$ if $k \geq 3$ and $d \geq 12$ if $k = 2$. Q.E.D.

LEMMA 3.7. *Assume that L is k-very ample with $k \geq 2$ on a smooth surface S. Assume that $d \leq 4k + 4$, and that $h^1(L_C) \neq 0$ for a smooth $C \in |L|$. Then $g \leq 2d - 3h^0(L) + 7$.*

Proof. By (3.1), it follows that $d - 2 \geq 2(h^0(L) - 3)$. By Castelnuovo's inequality, [11][8, (0.2)], the Lemma follows if we show that $d - 2 < 3(h^0(L) - 3)$. Indeed if this was false then $d \geq 3h^0(L) - 7$. If $k \leq 3$, then (3.3) gives the absurdity $4k + 4 \geq d \geq 3h^0(L) - 7 \geq 3k + 8$. If $k \geq 4$ then (2.13) gives the absurdity, $4k + 4 \geq d \geq 3h^0(L) - 7 \geq 6k - 1$. Q.E.D.

There is a useful result on Castelnuovo curves as hyperplane sections, see [11].

THEOREM 3.8. *Let L be a very ample line bundle on a smooth, projective surface, S. If there is a smooth $C \in |L|$ such that $g(L)$ equals the upper bound given by Castelnuovo's bound for the embedding of C by the linear system $|L|$, then L is arithmetically normal, and $h^2(\mathcal{O}_S) = \sum_{t=1}^{\infty} h^1(L_C^t)$. In particular, $q = 0$, $h^1(L) = 0$, and if $h^1(L_C) = 0$ for a smooth $C \in |L|$, then $h^2(\mathcal{O}_S) = 0$.*

The case when $d = 2k + 4$. It will be convenient to de the classification for the case $d = 2k + 4$ before proceeding any further.

LEMMA 3.9 *Let L be a k-very ample line bundle on a smooth, connected projective surface, S. Assume that $d = 2k + 4$ and $k \geq 2$. Then $k = 2$ and*

1. *(S, L) is the intersection of 3 quadrics in $\mathbf{P}^5$;*
2. *$(S, L) \cong (\mathbf{P}^1 \times \mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1 \times \mathbf{P}^1}(2,2))$; or*
3. *S is a Del Pezzo surface with $K_S^2 = 2$ and $L \cong -2K_S$.*

Proof. By Theorem (3.3) and Lemma (3.4) we see that either (S, L) is the intersection of 3 quadrics in $\mathbf{P}^5$ or $\kappa(S) = -\infty$.

If $h^1(L_C) \neq 0$ for some smooth $C \in |L|$, then by (3.7), we get that $g \leq 2(2k+4) - (6k+6) + 7 = 9 - 2k$ for $k \geq 3$, and using (3.3), $g \leq 2(2k+4) - 3(k+5) + 7 = k = 2$ for $k = 2$ with the exception of case 1). Also we have $g \geq 2k+1$ for all $k \geq 2$ by (2.1). Thus $2k+1 \leq g \leq 3$ for all $k \geq 2$. This contradicts $k \geq 2$.

If $h^1(L_C) = 0$ for some smooth $C \in |L|$, then by (2.12) $d \geq 2k + g + 1$, which gives $2k + 4 \geq 2k + g + 1$, or $3 \geq g$. Using Proposition (5.1) of [6] we are done. Q.E.D.

4. Results for special classes of surfaces

The case of $\mathbf{P}^1$*-bundles.* Assume that S is a $\mathbf{P}^1$-bundle, $p : S \to Y$ over a smooth curve, Y. Assume that L is k-very ample. Let f denote a fiber of the map, p, and let E denote a section with minimal self-intersection, $-e$. Numerically $L = aE + bf$ and $L \cdot L = a(2b - ae)$. Note that $-q \leq e$ where q is the genus of the base curve. Necessary conditions for L to be k-very ample are that

$$L \cdot f = a \geq k \tag{3}$$

$$L \cdot E = b - ae \geq \mu(q, k) \tag{4}$$

where $\mu(q, k)$ is the minimum degree of a k-very ample line bundle on a curve of arithmetic genus q. Note that from the lemmas in [8, §1] it follows that if $k \geq 1$, $\mu(q, k) \geq k$ with $\mu(q, k) \geq k + 2$ if $q \geq 1$. From Lemma (2.7) it follows for $k \geq 2$ and $2 \leq q \leq 4$ that $\mu(q, k) \geq q + k + 2$. We will use these lower bounds for $\mu(q, k)$ without further notice.

Writing $\delta := 2b - ae$, we have from the equation (4) that

$$-ae \geq 2\mu(q, k) - \delta. \tag{5}$$

All the cases in the following result are shown to exist in ([8]).

THEOREM 4.1 *Assume that S is a $\mathbf{P}^1$-bundle, $p : S \to Y$ over a smooth curve, Y. If L is k-very ample with $k \geq 2$, and $d := L \cdot L \leq 4k + 4$, then*
1. $(S, L) \cong (\mathbf{P}^1 \times \mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1 \times \mathbf{P}^1}(a, b))$ *with* $k = 2$ *and* (a, b) *either* (2,2), (2,3), *or* (3.2);
2. $(S, L) \cong (\mathbf{F}_1, 2E + 4f)$ *where $k = 2$ amd $\mathbf{F}_1$ is the unique $\mathbf{P}^1$-bundle over $\mathbf{P}^1$ with a section E of self-intersection,* $-e = -1$;

3. $k = 2$, $q = 1$, $e = -1$, *and* $L \sim 2E + 2f$.

Proof. First note that $L \sim aE + bf$ where $a \geq k$ by equation (3). If $q = 0$, then a straightforward calculation using $a \geq k$ and $b - ae \geq k$ gives the two possibilities $e = 0, 1$ with a, b, k as in the statement of the Theorem.

We thus have $q \geq 1$ and $\mu(q, k) \geq k + 2$. If $e \geq 0$ then $b - ae = L \cdot E \geq k + 2$. Thus $d = a(2b - ae) \geq a(2k + 4) \geq k(2k + 4) \geq 4k + 5$.

If $q = 1$ then the only remaining cases are with $e < 0$. Since $e \geq -q = -1$ it follows from the last paragraph that $e = -1$. Then from [8, Proposition (2.2)] and equation (3) it follows that $d = a(2b + a)$ where $a \geq k$, $a + b \geq k + 2$, and $a + 2b \geq k + 2$. This gives $d \geq 4k + 5$ unless $d = 12$, $a = b = k = 2$. From here on we can assume without loss of generality that

$$q \geq 2 \quad \text{and} \quad -q \leq e < 0. \tag{6}$$

We claim that

$$\text{if } q = 2 \quad \text{then} \quad a \geq k + 3. \tag{7}$$

To see this note that from equations (5) and (6) that $2a \geq -ae \geq 2(q + k + 2) - \frac{d}{a} = 8 + 2k - \frac{d}{a}$. Using $d \leq 4k + 4$ and $a \geq k$ we get the inequality (7).

Recall Hartshorne's formula [8, (2.5.6)]. Let $D \sim aE + bf$ be an effective divisor with $a \geq 1$. Then with $g(D)$ defined by $g(D) := \frac{(K_S + D) \cdot D}{2} + 1$, we have

$$2g(D) - 2 = a2(q - 1) + \frac{a - 1}{a} D \cdot D. \tag{8}$$

We now break into two separate cases deoending on whether there is at least one smooth $C \in |L|$, with $h^1(L_C) = 0$ or $\neq 0$.

$h^1(L_C) = 0$. By (2.11), $d \geq 2k + g + 1$. Using Hartshorne's formula, this fact, and the fact that $a \geq k$, we get $(k - 1)d + 2(q - 1)k^2 \leq k(2d - 4k - 4)$. This gives $2(q - 1)k^2 \leq k(d - 4k - 4) + d$, or $(q - 1)k^2 \leq 2k + 2$. This is impossible unless $k = 2$. In this case $q = 2$. Going over the argument with the fact that $a \geq k + 3$ by (7) and $k = 2$ gives $50 \leq 5(d - 12) + d$ or $d \geq 19$ which contradicts $d \leq 12$.

$h^1(L_C) \neq 0$. First we analyze the case $k = 2$. We have from (3.7), (3.3), and (3.8) that $g \leq 2d - 15$. In particular

$$g \leq 9. \tag{9}$$

Hartshorne's formula with $a \geq k$ gives $d + 8(q-1) \leq 4(2d-16)$. This gives $8q + 56 < 7d$. Using $q \geq 2$ from equation (6) and $d \leq 4k+4$, we get that the only possibilities are $(d, q) = (11,2), (12,2), (12,3)$.

If $d = 11$ then since $d = a(2b - ae)$ with $a \geq 2$, we conclude that $a = 11$. From Hartshorne's formula we obtain that $g = 17$ which contradicts (9).

If $d = 12$ then from $12 = d = a(2b - ae)$ we conclude that $a = 2, 3, 6$. If $q = 2$, then from inequality (7) we see that only $a = 6$ is possible. In this case we have from Hartshorne's formula that $g = 12$ which contradicts the bound $g \leq 9$ above. If $q = 3$ then equations (5) and (6) imply that $3a \geq -ae \geq 2(q+k+2) - \dfrac{d}{a} = 14 - \dfrac{12}{a}$. From this we see that $a = 6$. Hartshorne's formula implies that $g = 18$, which contradicts (9).

From here on we can assume that $k \geq 3$. We have from (3.7), (2.13), and (3.8) that $g \leq 2b - 6k$. Note that $a \leq k+3$. To see this assume that $a \geq k+4$. By Hartshorne's formula we have $2(q-1)(k+4)^2 \leq d + (k+4)(3d - 12k - 2)$. Using $d \leq 4k+4$ and $q \geq 2$ we obtain the contradiction $k \leq 2$.

We claim that $q = 2$ can't occur. If it did, then by the last paragraph and inequality (7) we see that $a = k+3$. By Hartshorne's formula we have $2(k+3)^2 \leq d + (k+3)(3d - 12k - 2)$. Since a divides d and $d \leq 4k+4$ we conclude that $d \leq 3(k+3)$, which gives the absurdity $(k+3)(5k-1) \leq d \leq 3(k+3)$.

We claim that $k = q = 3$. To see this apply Hartshorne's formula with $q \geq 3$, $a \geq k \geq 3$, and $g \leq 2d - 6k$. We obtain $(k-1)d + 2(q-1)k^2 \leq k(4d - 12k - 2)$, i.e., $k((2q-2)k - 10) \leq d$. If $k \geq 4$ this gives the contradiction $6k \leq d$. If $k = 3$ and $q \geq 4$ we obtain the contradiction $24 \leq d$.

Since $q = k = 3$ the equations (5) and (6) imply that $3a \geq -ae \geq 2(q+k+2) - \dfrac{d}{a} = 16 - \dfrac{d}{a}$. This implies that $a \geq 4$ with equality implying $d = 16$. Hartshorne's inequality with $d = 16$, $a = 4$, $q = 3$ gives $g = 15$ which contradicts $g \leq 2d - 6k = 14$. Thus $a \geq 5$. Hartshorne's formula with $a \geq 5$ and $q = 3$ gives the contradiction $50 \leq d$. Q.E.D.

The case of k-conic bundles. The following is a useful lower bound.

THEOREM 4.2. *Let L be a k-very ample line bundle on a smooth, connected, projective surface, S. If (S, L) is a k-conic bundle, then it follows that $(kK_S + L)\cdot L = 2k\delta$ where $\delta \geq 1$. If $k \geq 2$ and $\delta = 1$, then $d := L \cdot L \geq 4k+4$ with equality* *only if* $k = 2$, $d = 12$, $K_S^2 = 1$, *and* $K_S \cdot L = -4$; *this case is described, and shown to exist, in* [6, *Proposition* (5.3.4)].

Proof. Since there is a morphism $p : S \to Y$ with connected fibers from S to Y and such that $kK_S + L \cong p^* H$ for some ample H on Y, it follows that $(kK_S + L) \cdot L = 2k \deg H$.

We must now consider the case when $\deg H = 1$. If $\deg H = 1$, it follows from the fact that $kK_S + L$ is spanned, that $Y \cong \mathbf{P}^1$. From this it follows that $q = 0$ and S is rational. We have from $0 = k^2 K_S^2 + 2kK_S \cdot L + d = k^2 K_S^2 + 4k - d$ that $d = 4k + k^2 K_S^2$. Since $d \geq 2k + 4$, we conclude that $K_S^2 \geq 0$. Since S is rational, this implies that $-K_S$ has a non-trivial section and thus that $L \cdot K_S \leq -(k+2)$. Therefore $2k = (kK_S + L) \cdot L \leq -k(k+2) + d$ or $d \geq (k+4)k$. Since $d \leq 4k + 4$, we conclude that $k = 2$, $d = 12$, $K_S \cdot L = -4$, and $K_S^2 = 1$. This example is described and shown to exist in [6, Proposition (5.3.4)]. In [6] the weaker concept of k-spannedness is used, but because their basic criterion for k-spannedness is shown in [7] to hold for k-very ampleness, the results apply with no change to the current situation. Q.E.D.

COROLLARY 4.3. *Let L be a k-very ample line bundle on a smooth, connected, projective surface, S. Assume that (S, L) is a k-conic bundle with $k \geq 2$ and $d \leq 4k + 4$. Then $h^1(\mathcal{O}_S) = 0$, and either*:

1. $k = 2$, $K_S^2 = 1$, *and* $K_S \cdot L = -4$ (*this case is described, and shown to exist, in* [6, *Proposition* (5.3.4)];
2. $k = 2$, $K_S^2 = -1$, *and* $K_S \cdot L = -2$; *or*
3. $k = 3$, $d = 15$, $K_S^2 = -1$, *and* $K_S \cdot L = -1$.

Proof. Assume that (S, L) is not the case in the conclusion of the Corollary.

Now let us first assume that $K_S \cdot L \leq 0$. By (4.2), we can assume that $(kK_S + L) \cdot L = 2k\delta$ with $\delta \geq 2$. This gives that $d = (2\delta - K_S \cdot L)k \geq (4 - K_S \cdot L)k$. From $k^2 K_S^2 + 2kK_S \cdot L + d = 0$, we see that d is divisible by k and if moreover k is even, then d is divisible by $2k$. From this we see that we are reduced to the following cases:

1. $K_S \cdot L = -2$, $k = 2$, $d = 12$, $\delta = 2$;
2. $K_S \cdot L = -1$, $k = 3$, $\delta = 2$, $d = 15$;
3. $K_S \cdot L = 0$, $d = -k^2 K_S^2$.

Consider the equation $k^2 K_S^2 + 2kK_S \cdot L + d = 0$. If $K_S \cdot L = -2$, we conclude from the above list, that $K_S^2 = -1$. Noting that (S, L) has no lines and looking at the main result of [18], we see that $\mathcal{L} := K_S + L$ is very ample. Note that

$h^0(\mathscr{L}) = g(L) - q = 6 - q$. If $q = 1$, then noting that $g(\mathscr{L}) = 3$, and using the double point formula, [12, page 434], we get the absurdity, $7(7-5) - 10(3-1) + 0 = -2$. If $q \geq 2$, we get the absurdity that $q \neq 0$ and S is embedded in $\mathbf{P}^{5-q}$. Thus $q = 0$, and we get the possible case 2) of the Theorem.

If $K_S \cdot L = -1$, then $g = 8$ and $K_S^2 = -1$. If $q = 0$, then we have the possible second case of the Theorem. Therefore we can assume that $q \geq 1$. Since $L \cdot \mathscr{E} \geq 3$ for all curves $\mathscr{E}$ on S, we see that S has no lines relative to L. It follows from the main result of [18] that $K_S + L$ is very ample. Similarly $g(K_S + L) = 6$, and we can conclude again from the main theorem of [18] that $\mathscr{L} := 2K_S + L$ is very ample. We have that $h^0(\mathscr{L}) = g(K_S + L) - q = 6 - q$. If $q = 1$, we use the double point formula to obtain the contradiction $7(7-5) - 10(3-1) = -2$. If $q \geq 2$, we get the absurdity that S is embedded into $\mathbf{P}^{5-q}$.

If $K_S \cdot L = 0$, then we have $d = -k^2K_S^2$. Moreover $h^1(L_C) = 0$ for a smooth $C \in |L|$ or by (1.1), we have the absurdity, that $K_S \cong \mathcal{O}_S$. By (2.11), we conclude that $d = 4k + 4$. Thus we have that $k = 2$, $d = 12 = 4(-K_S^2)$, which implies that $K_S^2 = -3$. Here $\chi(\mathcal{O}_S) = 0, 1$. Since $K_S \cdot L = 0$ and since $K_S \not\cong \mathcal{O}_S$, we know from (1.1) that $h^1(L_C) = 0$. Thus $h^0(L_C) = \dfrac{d - K_S \cdot L}{2} + h^1(L_C) = 6$. Thus by (3.3) we conclude that $h^0(L) = 7$. Following Andreatta, [1], we use the result of Le Barz [14, page 45, 59] to rule this possibility out.

Now assume that $K_S \cdot L > 0$. By (2.11), we see that

$$h^1(L_C) \neq 0$$

for smooth $C \in |L|$. Since $h^1(L_C) \neq 0$ and $h^2(\mathcal{O}_S) = 0$, we conclude from (3.8), that the Castelnuovo bound given on the genus for the embedding of $C \in |L|$ given by $|L|$ cannot be taken on. Thus we conclude from (3.7), (2.8), and (2.13) that:

$$(10) \qquad g \leq 2d - 6k \quad \text{and} \quad K_S \cdot L \leq 3d - 12k - 2 \quad \text{for} \quad k \geq 3$$

$$(11) \qquad g \leq 2d - 15 \quad \text{and} \quad K_S \cdot L \leq 3d - 32 \quad \text{for} \quad k = 2$$

Using $0 < K_S \cdot L$ with these two equations, we see that $d \geq 4k + 1$ for $k \geq 3$ and $d \geq 11$ for $k = 2$. Recall that k divides d and that $2k$ divides d when k is even. Since $d \leq 4k + 4$, we get $(k, d) = (2,12)$ or $(3,15)$.

If $(k, d) = (3,15)$, we conclude from the inequality (10), we conclude that $K_S \cdot L \leq 7$. Since $K_S \cdot L$ and d have the same parity, we conclude that $K_S \cdot L = 1, 3, 5, 7$. $(3K_S + L)^2 = 0$ gives $3K_S^2 + 2K_S \cdot L + 5 = 0$. We get divisibility contradictions unless $K_S \cdot L = 5$. In this case $K_S^2 = -5$. Note that $h^0(L_C) \geq 7$ by (3.3).

Therefore from Riemann-Roch on C we get that $h^1(L_C) \geq 2$. By (2.3) we conclude that $h^1(L_C) = 2$. In this case $\chi(\mathcal{O}_S) = 0, 1$. Using [14, pg. 45, pg. 59] we see that these cases don't occur.

If $(k, d) = (2,12)$, we enumerate the cases exactly as in the last paragraph for $(k, d) = (3,15)$, to get $d = 12$, $K_S^2 = -3 - K_S \cdot L$ with $K_S \cdot L = 2, 4$. It is easy to check using (2.3) and (2.13), that $h^1(L_C) = 1$ if $K_S \cdot L = 2$ and $h^1(L_C) = 2$ if $K_S \cdot L = 4$. In both cases $h^0(L) = 7$. Again $\chi(\mathcal{O}_S) = 0, 1$. Using [14, pg. 45, pg. 59] we see that these cases don't occur. Q.E.D.

A bound for surfaces with $\kappa(S) \geq 0$. Given a smooth projective surface, S of non-negative Kodaira dimension, with minimal model S', Let $\gamma(S) := e(S) - e(S')$ where $e(Y)$, for a space, Y, denotes the topological Euler characteristic of Y. Note that $\gamma(S) \geq 0$ with equality of and only if $S \cong S'$ under the map of S to its minimal model.

THEOREM 4.4. *Let L be a k-very ample line bundle on a smooth projective surface, S, with $\kappa(S) \geq 0$.*

1. *If $\kappa(S) = 0$, then $K_S \cdot L \geq k\gamma$.*
2. *If $\chi(\mathcal{O}_S) \geq 2$, then $K_S \cdot L \geq k\gamma + k + 2$.*
3. *If $\kappa(S) \geq 1$, then $K_S \cdot L \geq k\gamma + \dfrac{k+2}{2}$.*

Proof. Since we only need the result when $k \geq 2$, we leave the minor modifications for the case $k = 1$ to the reader.

Let $\pi : S \rightarrow S'$ be the map of S onto its minimal model, S'. Note that $K_S \cong \pi^* K_{S'} + \sum_{i=1}^{\delta(S)} \lambda_i E_i$ where the λ_i are positive integers, and each E_i is a rational curve. Since $E_i \cdot L \geq k$ by the k-very ampleness of L, it suffices to give a lower bound for $L \cdot \pi^* K_{S'}$. Since $K_{S'}$ is nef, we see that the case of $\kappa(S) = 0$ is trivial. If $K_{S'}$ has a non-trivial section, s, that isn't everywhere non-zero, then since the zero set of s isn't a smooth rational curve, $K_{S'} \cdot L' \geq k + 2$. This takes care of the case when $\chi(\mathcal{O}_S) \geq 2$.

Assume now that $\kappa(S) = 1$. Let $\phi : S' \rightarrow B$ denote the canonical fibration, and let F denote a generic fiber of the map. There is a possibly empty set of multiple fibers, $\{m_i F_i \mid i \in I, m_i \geq 2\}$. The canonical bundle formula says in this case that $K_{S'}$ is numerically equal to $\left(\chi(\mathcal{O}_{S'}) - 2\chi(\mathcal{O}_B) + \sum_{i \in I} \dfrac{m_i - 1}{m_i}\right) F$. By renumbering if necessary we can assume that if I is non-empty, $m_1 \leq \cdots \leq m_{|I|}$, where $|I|$ denotes the cardinality of I. If $\chi(\mathcal{O}_{S'}) - 2\chi(\mathcal{O}_B) > 0$ then since $L \cdot$

$\pi^{-1}(F) \geq k+2$ we are done. If $\chi(\mathcal{O}_{S'}) - 2\chi(\mathcal{O}_B) = 0$, then since $\kappa(S) = 1$, we know that $\left(\sum_{i\in I}\frac{m_i - 1}{m_i}\right)F$ is numerically non-trivial. Thus there is a multiple fiber F_i. Thus letting F_i' denote the pullback of F_i we see that $K_S = \sum_{i=1}^{\delta(S)}\lambda_i E_i + (m_i - 1)F_i'$. Since the arithmetic genus of F_i' is 1, we conclude that $L\cdot K_S \geq \gamma(S)k + (m_i - 1)(k+2) \geq \gamma(S)k + (k+2)$, which would prove the lemma in this case. Thus we can assume that $\chi(\mathcal{O}_{S'}) - 2\chi(\mathcal{O}_B) = -1, -2$.

First let $\chi(\mathcal{O}_{S'}) - 2\chi(\mathcal{O}_B) = -1$. Since $\kappa(S) = 1$ and $K_{S'}$ is not numerically trivial, $\sum_{i\in I}\left(\frac{m_i - 1}{m_i}\right) > 1$. Following the argument of the last paragraph, we will be done if we can show that $\left(\sum_{i\in I}\left(\frac{m_i - 1}{m_i}\right) - 1\right)m_j \geq \frac{1}{2}$ where m_j is the largest of the integers m_j. Note that if the cardinality of I was 1, the expression, $\left(\sum_{i\in I}\left(\frac{m_i - 1}{m_i}\right) - 1\right)$, could not be positive. If I has cardinality 2, then again using the positivity of $\left(\frac{m_1 - 1}{m_1} + \frac{m_2 - 1}{m_2} - 1\right)$, we see that at least one of the m_i is ≥ 3. Thus $\left(\sum_{i\in I}\left(\frac{m_i - 1}{m_i}\right) - 1\right)m_j \geq \frac{m_j}{6} \geq \frac{1}{2}$. In the case of cardinality of I at least 3, it is easily seen that $\left(\sum_{i\in I}\left(\frac{m_i - 1}{m_i}\right) - 1\right)m_j \geq \frac{m_j}{2} \geq 1$.

Now turn to the case when $\chi(\mathcal{O}_{S'}) - 2\chi(\mathcal{O}_B) = -2$. Since $\chi(\mathcal{O}_{S'}) \geq 0$, we conclude that $\chi(\mathcal{O}_{S'}) = 0$ and $B \cong \mathbf{P}^1$. To prove the Theorem in this case it suffices to show that $\left(\sum_{i\in I}\left(\frac{m_i - 1}{m_i}\right) - 2\right)m_j \geq \frac{1}{2}$ with m_j the largest of the multiplicities. Using the fact that $K_{S'}$ is not numerically trivial, and thus that $\left(\sum_{i\in I}\left(\frac{m_i - 1}{m_i}\right) - 2\right)$ is positive, we see that the cardinality of I must be at least 3. If it is more than 3, then it is easy to see that $\left(\sum_{i\in I}\left(\frac{m_i - 1}{m_i}\right) - 2\right)m_j \geq \frac{1}{2}$. Assume now that we are in the case when the cardinality of I is 3. By renaming if necassary we can assume that $m_1 \leq m_2 \leq m_3$. By a theorem of Katsura-Ueno ([13, Theorem (3.3)]; see also [16, Prop. 1.3]), we know that the m_i satisfy the strong condition that each m_i divides the least common multiple of the other two multiplicities. This is equivalent to $m_1 = \mu xy$, $m_2 = \mu xz$, and $m_3 = \mu yz$ where μ is the least common divisor of all three m_i, the integers x, y, z are pairwise relatively prime, and $x \leq y \leq z$. Thus the fact that $K_{S'}$ is numerically non-trivial is equivalent to $1 - \frac{1}{\mu xy} - \frac{1}{\mu xz} - \frac{1}{\mu yz} > 0$. We need to show that $\Big(1 - \frac{1}{\mu xy} -$

$\frac{1}{\mu xz} - \frac{1}{\mu yz}\Big) \mu yz \geq \frac{1}{2}$. Assume this is false. Then we have $0 < \Big(1 - \frac{1}{\mu xy} - \frac{1}{\mu xz} - \frac{1}{\mu yz}\Big) \mu yz < \frac{1}{2}$. Multiplying through by x we get $0 < \mu xyz - z - y - x < \frac{x}{2}$ or $x + y + z < \mu xyz < x + y + z + \frac{x}{2}$. Thus we have that $\frac{x}{2} > 1$, i.e., $x \geq 3$. Note that no two of the x, y, z can be equal because this would imply that μ was not the greatest common divisor of m_1, m_2, m_3. Thus $y \geq 4$, and $z \geq 5$. Thus we get the contradiction, $12 \leq \mu xy < \frac{x+y+z}{z} + \frac{x}{2z} < 3 + \frac{1}{2}$.

In the case when $\kappa(S) = 2$, note that since $K_{S'}^2 \geq 1$ and $\chi(\mathcal{O}_{S'}) \geq 1$, there is a non-trivial divisor $D \in |2K_{S'}|$. If D is reducible, then $2L \cdot \pi^* K_{S'} \geq 2k$ which gives the result for $k \geq 2$. If D is irreducible, then since $2g(D) - 2 = 6K_{S'}^2 \geq 6$, we have that D has arithmetic genus ≥ 4. Thus using [8, Lemma (1.1)] as in Lemma (2.7), we get $L \cdot D \geq k + 2$, which finishes the proof. Q.E.D.

For some more information on k-very ampleness on elliptic surfaces see Mella and Palleschi [15].

5. The classification result for degree $\leq \max\{11, 4k + 2\}$

The following result is a corollary of [8, Theorem (3.1)].

THEOREM 5.1. *Let L be a k-very ample line bundle on a smooth projective surface, S. Assume that $k \geq 2$ and $d := L \cdot L \leq 4k + 4$. Then $tK_S + L$ is very ample for $t = 0, \ldots, k-1$, and $kK_S + L$ is spanned and big unless either*:

1. $S \cong \mathbf{P}^2$ *with* $L \cong \mathcal{O}_{\mathbf{P}^2}(a)$ *for* $3 \leq k \leq a \leq 4$ *or* $k = 2 \leq a \leq 3$;
2. $(S, L) \cong (\mathbf{P}^1 \times \mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1 \times \mathbf{P}^1}(a, b))$ *with* $k = 2$ *and* (a, b) *either* (2,2), (2,3), *or* (3.2);
3. $(S, L) \cong (\mathbf{F}_1, 2E + 4f)$ *where* $k = 2$ *and* $\mathbf{F}_1$ *is the unique* $\mathbf{P}^1$*-bundle over* $\mathbf{P}^1$ *with a section* E *if self-intersection,* $-e = -1$;
4. S *is a* $\mathbf{P}^1$*-bundle over an elliptic curve with invariant* $e = -1$, *and* $k = 2$, $q = 1$, *and* $L \sim 2E + 2f$;
5. S *is a Del Pezzo surface with* $L \cong -2K_S$ *and* $k = 2$, $2 \leq K_S^2 \leq 3$;
6. (S, L) *is a* 2*-conic bundle*:
   (*a*) $d = 12$, $h^1(\mathcal{O}_S) = 0$, $K_S^2 = 1$, $K_S \cdot L = -4$ (*this case is described, and shown to exist, in* [6, *Proposition* (5.3.4)]);
   (*b*) $d = 12$, $h^1(\mathcal{O}_S) = 0$, $K_S^2 = -1$, $K_S \cdot L = -2$; *or*
7. (S, L) *is a* 3*-conic bundle with* $h^1(\mathcal{O}_S) = 0$, $d = 15$, $K_S^2 = -1$, *and* $K_S \cdot L$

$= -1.$

Proof. This follows immediately from [8, Theorem (3.1)], Theorems (4.1) and (4.3) above. Q.E.D.

COROLLARY 5.2. *Let L be a k-very ample line bundle on a smooth, connected projective surface, S. Assume that $k \geq 2$. If $kK_S + L$ is nef and big, then $(kK_S + L) \cdot L \geq k + 4$.*

Proof. The proof of [8, Theorem (3.1)] shows that $kK_S + L$ is spanned. Thus we can choose a smooth $D \in |kK_S + L|$. Let $\alpha := L \cdot D$. We must show that $\alpha \geq k + 4$. Let $\sigma := K_S \cdot (kK_S + L)$.

Assume first that $g(D) = 0$, then $2g(D) - 2 = -2$. This implies

$$-2 = ((k+1)K_S + L) \cdot (kK_S + L) = (k+1)\sigma + \alpha.$$

By bigness of $kK_s + L$, we have that $\alpha + k\sigma > 0$. Solving for σ in terms of k and α, we get that $\alpha + k\sigma = \dfrac{\alpha - 2k}{k+1} > 0$. This gives that $\alpha \geq 3k + 1 \geq k + 5$.

A similar calculation for $g(D) = 1$ gives $\alpha \geq 2k + 2 \geq k + 4$.

Note that $\alpha \geq k$. By [8, §1], $\alpha \leq k + 3$ implies that D is isomorphic to a curve of order ≤ 4. By Castelnuovo's bound $g(D) \leq 3$. From (2.7) we conclude that $\alpha \geq k + 4$ if $g(D) = 2, 3$. Thus $\alpha \geq k + 4$ without exceptions. Q.E.D.

COROLLARY 5.3. *Let L be a k-very ample line bundle on a smooth projective surface, S. Assume that that $k \geq 2$ and $d := L \cdot L \leq 4k + 4$. Assume that $K_S \cdot L \leq -1$. Then (S, L) is one of the classes* 1) *to* 6) *of* (5.1).

Proof. By (5.1), it can be assumed without loss of generality that $kK_S + L$ is nef and big. Since $(k-1)K_S + L$ is very ample by (5.1), we also know from [18] that $kK_S + L$ is spanned.

$K_S \cdot L < 0$ implies that $\kappa(S) = -\infty$. We will first do the cases when $K_S \cdot L = -1, -2$.

Assume that $K_S \cdot L = -1, -2$. Using the Hodge index theorem and (1.2), we conclude that $K_S \cdot K_S \leq 0$. If $K_S \cdot K_S = 0$, then either S is a $\mathbf{P}^1$-bundle over an elliptic curve or a rational surface. Since the former has been covered by Theorem (4.1), we can assume that S is a rational surface. In this case $K_S \cdot K_S = 0$ implies that $-K_S$ is effective, and in particular since curves in $-K_S$ have arithmetic genus 1, $-K_S \cdot L \geq k + 2$. Thus if $K_S \cdot L = -1, -2$, it can be assumed that K_S^2

≤ -1. Since $kK_S + L$ is spanned and big, we have that $(kK_S + L)\cdot((k+1)K_S + L) = 2g(kK_S + L) - 2 \geq -2$. From this we conclude that if $K_S \cdot L = -1, -2$, then $4k + 4 \geq d \geq k(k+1) - ((2k+1)K_S \cdot L) - 2$. From this we see that $K_S \cdot L = -1$, $k = 2$, $K_S^2 = -1$, and by parity $d = 9, 11$. Note that $h^0(L_C) = d - g + 1 = \frac{d+1}{2} \geq 6$ by (3.3). Thus $d = 11$. Note that since $kK_S + L$ is nef and big, there is a unique 2-minimal model (S', L') of (S, L). To describe the 2-minimal model, note that $g(K_S + L) = 4$, and thus that $h^0(2K_S + L) = 4$. From the very ampleness of $K_S + L$, and the fact that $(2K_S + L)^2 = 3$, we conclude that the 2-minimal model of (S, L) is (S', L') with S' a cubic surface in $\mathbf{P}^3$, and thus $\chi(\mathcal{O}_S) = 1$. Note that $h^0(L) = 7$. By [14, page 45, 59] we show this case doesn't exist.

Now assume that $K_S \cdot L \leq -3$. It follows from (5.2) that $d - 3k \geq d + kK_S \cdot L \geq k + 4$. Thus we conclude that $d \geq 4k + 4$, and since $d \leq 4k + 4$ that $K_S \cdot L = -3$ and $d = 4k + 4$. By the Hodge index theorem we conclude that $K_S \cdot K_S \leq 0$. From this we conclude $(kK_S + L)\cdot(kK_S + L) \leq 0 - 6k + 4k + 4 \leq 4 - 2k \leq 0$ which contradicts the bigness of $kK_S + L$. This finishes the proof of the Theorem. Q.E.D.

THEOREM 5.4. *Let L be a k-very ample line bundle on a smooth projective surface, S. Assume that that $k \geq 2$ and that $d := L \cdot L \leq \max\{11, 4k + 2\}$. Then (S, L) is one of the following*:

1. $(S, L) \cong (\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(a))$ *with* $2 \leq k \leq 3$ *for* $k \leq a \leq 3$ *and with* $a = k = 4$;
2. S *is a* $K3$-*surface with* $d = 4k, 4k + 2$;
3. $k = 2$ *and* $(S, L) \cong (\mathbf{P}^1 \times \mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1 \times \mathbf{P}^1}(2,2))$; *or*
4. S *is a Del Pezzo surface and* $L \cong -2K_S$ *with* $k = 2 = K_S^2$.

Proof. First assume that $h^1(L_C) = 0$ for some smooth $C \in |L|$, then by (2.12), it follows that $4k + 4 + K_S \cdot L \leq d$. In the case $d \leq 11$ it follows that $K_S \cdot L < 0$ and in the case $d \leq 4k + 2$ it follows that $K_S \cdot L \leq -2$. In both cases we have $K_S \cdot L \leq -1$. Thus Corollary (5.3) applies to cover these cases.

Therefore from now on we can assume that $h^1(L_C) \neq 0$ for smooth $C \in |L|$. We can assume that $K_S \cdot L > 0$. Otherwise we would have that $K_S \cong \mathcal{O}_S$ by (1.1). In this case by (3.1) and (3.2), we have $d = 4k, 4k + 2$ with S a $K3$-surface.

If $h^1(L_C) = 1$, then by (2.5), we conclude that $d \geq 4k + 2 + K_S \cdot L$. Thus the only possibility is $k = 2$, $d = 11$, $K_S \cdot L = 1$. By (4.4) we conclude that $\kappa(S) = -\infty$. By the Hodge index theorem. $K_S^2 \leq 0$. If it was equal to 0, then either S is a $\mathbf{P}^1$-bundle over an elliptic curve, which is covered by (4.1), or S is rational. If S

is rational and $K_S \cdot K_S = 0$, then $|-K_S|$ is non-empty, which gives the contradiction that $K_S \cdot L < 0$. Moreover since $(kK_S + L)^2 \geq 0$, we conclude that $K_S^2 \geq -3$. Therefore we conclude that $\chi(\mathcal{O}_S) = 0, 1$. By [14, page 45, 59] we see that these cases don't exist.

If $h^1(L_C) \geq 2$, then by (3.6) we are done. Q.E.D.

6. The classification result for large k

THEOREM 6.1 *Let L be a k-very ample line bundle on a smooth, connected, projective surface, S, with $d := L \cdot L \leq 4k + 4$.*

1. *If $\kappa(S) = -\infty$ then $k \leq 8$;*
2. *If $\kappa(S) \geq 0$ then if $k \geq 5$, either:*
   (*a*) *$K_S \cong \mathcal{O}_S$ with $d = 4k, 4k+2, 4k+4$, with S a $K3$-surface; or*
   (*b*) *$d = 4k + 4$ and S is an Enriques surface, i.e., $2K_S \cong \mathcal{O}_S$, $q = 0$, and $K_S \not\cong \mathcal{O}_S$.*

Proof. Note:

1. By (5.4) we can assume without loss of generality that $d \geq 4k + 3$.
2. We can assume in light of (5.1) and (4.3) that $kK_S + L$ is spanned by global sections and big, and therefore that $-2 \leq 2g(kK_S + L) - 2 = (kK_S + L) \cdot ((k+1)K_S + L)$.
3. Since the cases with K_S numerically trivial are listed we can assume that if $K_S \cdot L = 0$ then $K_S^2 < 0$ and $h^1(L_C) = 0$.

If $h^1(L_C) = 0$ for a smooth $C \in |L|$, then using (5.3) and (2.12) we conclude that $K_S \cdot L = 0$ and $d = 4k + 4$. By item 3) we can assume that $K_S^2 < 0$. By 2) we conclude that $-k(k+1) + d \geq -2$. This gives $k \leq 4$. Therefore we can assume that $h^1(L_C) \neq 0$.

If $h^1(L_C) = 1$, then $d \geq 4k + 2 + K_S \cdot L$. By 1), we must have $K_S \cdot L = 1$ and 2 respectively. By the Hodge index theorem we conclude that $K_S^2 \leq 0$.

If $K_S^2 = 0$ then either S is a $\mathbf{P}^1$-bundle over an elliptic curve, a rational surface, or $\kappa(S) \geq 0$. In the first case we are done by (4.1). If S is rational and $K_S \cdot K_S = 0$, then $|-K_S|$ is non-empty, and therefore $K_S \cdot L < 0$ giving the contradiction that $h^1(L_C) = 0$. If $\kappa(S) \geq 0$, then by (4.4) it follows that S is minimal if $k \geq 3$. Since $K_S \cdot L > 0$ and $K_S^2 = 0$, we conclude that $\kappa(S) = 1$. This implies S is an elliptic surface mapped onto a curve by some power of K_S. By (4.4), we conclude $K_S \cdot L \geq \dfrac{k+2}{2}$, which with $K_S \cdot L \leq 2$ gives $k \leq 2$.

If $K_S \cdot K_S < 0$, then $\kappa(S) = -\infty$. Indeed if $\kappa(S) \geq 0$, then $K_S \cdot K_S < 0$ implies that S is not minimal, but by (4.4) we conclude then that $K_S \cdot L \geq k$, which implies that $k \leq 2$. By 2) we conclude that $8k + 8 \geq k(k+1)$, i.e., $8 \geq k$.

If $h^1(L_C) \geq 2$, (3.6) implies that $k \leq 4$. Q.E.D.

Remark 6.2. Let L be a k-very ample line bundle on a smooth projective surface, S. If $d := L \cdot L \geq 4k + 5$, then the question of the k-very ampleness of $K_S + L$ is completely treated in [8]. Looking over the list in Theorem (5.4) we see that for 1), 3), and 4) $K_S + L$ is not k-very ample. For 2), $K_S + L \cong L$ is k-very ample.

From Theorem (6.1), we see that for $k \geq 9$ the only cases where questions about the k-very ampleness of $K_S + L$ remain are where $d = 4k + 4$, and S is an Enriques surface. Here our knowledge is poor. By the result of [8], $K_S + L$ is $(k-1)$-very ample. Looking over the classical approach to k-very ampleness for $K_S + L$, for $k = 1$, we find that this case is difficult there also and in fact for $d = 8$, requires the full knowledge of the adjunction mapping acquired in the case $h^1(\mathcal{O}_S) = 0$ by the second author in [17].

We note that [14, page 45, 59] rules out all surfaces that are not $K3$ with $d \leq 12$ and K_S numerically trivial.

Remark 6.3. It would be nice to complete the classification of pairs, (S, L), with L a k-very ample line bundle on a smooth connected projective surface S, and with $L \cdot L \leq 4k + 4$.

Further calculation shows that in addition to the cases with $k = 2$, $d = 12$ already mentioned in (5.1), there is only one other possibility with $(k, d) = (2, 12)$ and K_S not numerically trivial. It has the invariants, $K_S \cdot L = 0$, $h^1(L_C) = 0$ for a smooth $C \in |L|$, $K_S^2 = -2$, $\chi(\mathcal{O}_S) = 1$.

For the case when $k = 8$, $d \leq 4k + 4$, and K_S is not numerically trivial, further calculation shows that the only possible set of invariants is $d = 36$, $K_S^2 = -1$, $K_S \cdot L = 2$, $h^1(L_C) = 1$ for a smooth $C \in |L|$, and $\chi(\mathcal{O}_S) = 1$. This surface is rational if it exists since $g(kK_S + L) = 0$.

REFERENCES

[1] M. Andreatta, "Surfaces of sectional genus ≤ 8 with no trisecant lines," Archiv der Math., **60** (1993), 85–95.

[2] E. Arbarello, M. Cornalba, P. A. Griffiths, and J. Harris, Geometry of Algebraic Curves, Volume I, Grund. der math. Wissen., **267**, Springer-Verlag, (1985).

[3] E. Ballico, "A characterization of the Veronese surface," Proc. A. M. S., **105**, (1989), 531–534.

[4] ——, "On k-spanned projective surfaces," 1988 L'Aquila Proceedings: Hyperplane sections, edited by A. J. Sommese, A. Biancofiore, and E. L. Livorni, Lecture Notes in Math., **1417** (1990), page 23, Springer-Verlag.

[5] ——, "On k-spanned embeddings of projective manifolds," Manuscripta. Math., **75** (1992), 103–107.

[6] M. C. Beltrametti and A. J. Sommese, "On k-spannedness for projective surfaces," 1988 L'Aquila Proceedings: Hyperplane sections, Lecture Notes in Math., **1417** (1990), 24–51, Springer-Verlag.

[7] ——, "Zero cycles and kth order embeddings of smooth projective surfaces," 1988 Cortona Proceedings: Projective Surfaces and their Classification, ed. by F. Catanese, Symposia Mathematica, **32**, (1992) INDAM, 33–48, Academic Press.

[8] ——, "On the preservation of k-very ampleness under adjunction," Math. Z., **212** (1993), 257–284.

[9] M. Coppens and G. Martens, "Secant spaces and Clifford's theorem," Compos, Math., **78** (1991), 193–212.

[10] W. Fulton and R. Lazarsfeld, "On the connectedness of the degeneracy loci and special divisors," Acta Math., **146** (1981), 271–283.

[11] J. Harris, "A bound on the geometric genus of projective varieties," Ann. Norm. Super. Pisa, C. Sci. Ser. 4, **8** (1981), 35–68.

[12] R. Hartshorne, Algebraic Geometry, GTM 52, Springer-Verlag, New York (1987).

[13] T. Katsura and K. Ueno, "Elliptic surfaces in characteristic p," Math. Ann., **272** (1985), 291–330.

[14] P. Le Barz, "Formules pour les trisécantes des surfaces algébriques," L'Enseign. Math. II. Sér. **33** (1987), 1–66.

[15] M. Mella and M. Palleschi, "The k-very ampleness of an elliptic quasi-bundle," Abh. Math. Sem. Univ. Hamburg, **63** (1993), 215–226.

[16] F. Serrano, "Elliptic surfaces with an ample divisor of genus two," Pac. J. Math. **152** (1992), 187–199.

[17] A. J. Sommese, "The projective classification of surfaces, I: The adjunction mapping," Duke Math. J., **46** (1979), 377–401.

[18] A. J. Sommese and A. Van de Ven, "On the adjunction mapping," Math. Ann., **278** (1987), 593–603.

Edoardo Ballico
Dipartimento di Matematica
Università di Trento
38050 Povo (TN), Italia
ballico @itncisca. bitnet
fax: Italy +461881624

Andrew J. Sommese
Department of Mathematics
University of Notre Dame
Notre Dame, Indiana 46556, U.S.A.
sommese. 1@nd. edu
fax: U.S.A. +219-631-6579