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ON THE DIMENSION AND MULTIPLICITY OF LOCAL COHOMOLOGY MODULES

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Abstract. This paper is concerned with a finitely generated module M over a (commutative Noetherian) local ring R. In the case when R is a homomorphic image of a Gorenstein local ring, one can use the well-known associativity formula for multiplicities, together with local duality and Matlis duality, to produce analogous associativity formulae for the local cohomology modules of M with respect to the maximal ideal. The main purpose of this paper is to show that these formulae also hold in the case when R is universally catenary and such that all its formal fibres are Cohen–Macaulay.

These formulae involve certain subsets of the spectrum of R called the pseudo-supports of M; these pseudo-supports are closed in the Zariski topology when R is universally catenary and has the property that all its formal fibres are Cohen–Macaulay. However, examples are provided to show that, in general, these pseudo-supports need not be closed. We are able to conclude that the above-mentioned associativity formulae for local cohomology modules do not hold over all local rings.

§0. Introduction

Let M be a finitely generated module over a (commutative Noetherian) local ring $(R, \mathfrak{m})$. It is well known that, for each non-negative integer i, the i-th local cohomology module $H^i_{\mathfrak{m}}(M)$ is Artinian.

D. Kirby [7, Proposition 2] showed that there is an analogue for an Artinian R-module A of the Hilbert-Samuel polynomial for a Noetherian R-module. Let $\mathfrak{q}$ be an $\mathfrak{m}$-primary ideal of R, so that, for each positive integer n, the R-module $(0 :_A \mathfrak{q}^n)$ has finite length $\ell_R(0 :_A \mathfrak{q}^n)$. Kirby proved that there is a rational polynomial $\Theta^{\mathfrak{q}}_A \in \mathbb{Q}[X]$ such that

$$\Theta^{\mathfrak{q}}_A(n) = \ell_R(0 :_A \mathfrak{q}^{n+1}) \quad \text{for all } n >> 0.$$

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Kirby (who was actually working in a more general situation) remarked [7, p. 55] that the existence of this polynomial suggests definitions of dimension and multiplicity for Artinian R-modules. R. N. Roberts [11] found three equivalent formulations for the dimension $\dim A$ of A, one of which is the degree $\deg \Theta_A^{\mathfrak{q}}$.

We call $\Theta_A^{\mathfrak{q}}$ the *Hilbert-Samuel polynomial of A with respect to* $\mathfrak{q}$. When A is non-zero and of dimension d, so that $\Theta_A^{\mathfrak{q}}$ has degree d, then we define the *multiplicity* $e'(\mathfrak{q}, A)$ *of A with respect to* $\mathfrak{q}$ in such a way that $e'(\mathfrak{q}, A)/d!$ is the leading coefficient of $\Theta_A^{\mathfrak{q}}$. These definitions are analogues of ones in the standard classical theory for Noetherian R-modules, as expounded in [4, §4.5], for example.

One can use the classical 'associativity formula' for 'Noetherian' multiplicities (see [4, 4.6.8], for example), in conjunction with local duality and Matlis duality, to produce quickly an 'associativity formula' for the multiplicity with respect to $\mathfrak{q}$ of a non-zero local cohomology module $H^i_{\mathfrak{m}}(M)$ in the case when R is a homomorphic image of a Gorenstein local ring. To describe this associativity formula, we define (without any assumption about the local ring R) the *i-th pseudo-support* $\mathrm{Psupp}^i(M)$ of M by

$$\mathrm{Psupp}^i(M) := \left\{\mathfrak{p} \in \mathrm{Spec}(R) : H^{i-\dim R/\mathfrak{p}}_{\mathfrak{p}R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \neq 0\right\}$$

and the *i-th pseudo-dimension* $\mathrm{psd}^i(M)$ of M by

$$\mathrm{psd}^i(M) = \sup\left\{\dim R/\mathfrak{p} : \mathfrak{p} \in \mathrm{Psupp}^i(M)\right\}.$$

Our associativity formula in the case when R is a homomorphic image of a Gorenstein local ring states that

$$e'\left(\mathfrak{q}, H^i_{\mathfrak{m}}(M)\right) = \sum_{\substack{\mathfrak{p}\in\mathrm{Psupp}^i(M)\\ \dim R/\mathfrak{p}=\mathrm{psd}^i(M)}} \ell_{R_{\mathfrak{p}}}\left(H^{i-\dim R/\mathfrak{p}}_{\mathfrak{p}R_{\mathfrak{p}}}(M_{\mathfrak{p}})\right)e(\mathfrak{q}, R/\mathfrak{p});$$

in this case, all the pseudo-supports of M are closed (in the Zariski topology), and so the sum on the right-hand side of the above equation is taken over finitely many prime ideals.

The main aims of this paper are to establish that all the pseudo-supports of M are closed and the above associativity formula is still valid when R is universally catenary and all its formal fibres are Cohen–Macaulay rings. In the final section, we shall give an example of a universally catenary local domain which (itself) has a non-closed pseudo-support, and also an example of a local domain all of whose formal fibres are Cohen–Macaulay but which, again, has a non-closed pseudo-support.

The notation used in this Introduction will be maintained throughout the paper.

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§1. Results over a homomorphic image of a Gorenstein local ring

Our purpose in this section is to establish the promised associativity formula in the case when R is a homomorphic image of a Gorenstein local ring.

NOTATION 1.1. The following additional notation will be in force throughout this section.

We assume that R is a homomorphic image of an n'-dimensional Gorenstein local ring $(R', \mathfrak{m}')$ under a surjective ring homomorphism $f : R' \longrightarrow R$. Let $\mathfrak{q}$ be an $\mathfrak{m}$-primary ideal of R, and let M be a non-zero finitely generated R-module. Use ℓ to denote length of modules. Recall that the *Hilbert-Samuel polynomial of M with respect to* $\mathfrak{q}$ is the polynomial $\Sigma^{\mathfrak{q}}_M \in \mathbb{Q}[X]$ of degree $\dim M$, such that $\Sigma^{\mathfrak{q}}_M(n) = \ell_R(M/\mathfrak{q}^{n+1}M)$ for all $n >> 0$.

We use E to denote the injective envelope $E_R(R/\mathfrak{m})$, and D to denote the Matlis duality functor $\operatorname{Hom}_R(\,\bullet\,, E)$.

We shall, for an integer i, denote the R-module $\operatorname{Ext}^{n'-i}_{R'}(M, R')$ by K^i_M. The Local Duality Theorem (see [3, 11.2.6], for example) yields that $H^i_{\mathfrak{m}}(M) \cong D(K^i_M)$.

PROPOSITION 1.2. *Use the notation of* 1.1*, and let i be a non-negative integer. Then*

(i) $\Theta^{\mathfrak{q}}_{H^i_{\mathfrak{m}}(M)} = \Sigma^{\mathfrak{q}}_{K^i_M}$;

(ii) $H^i_{\mathfrak{m}}(M) \neq 0$ *if and only if* $K^i_M \neq 0$*, and, when this is the case,* $e'\left(\mathfrak{q}, H^i_{\mathfrak{m}}(M)\right) = e(\mathfrak{q}, K^i_M)$;

(iii) $\operatorname{Psupp}^i(M) = \operatorname{Supp}(K^i_M)$*, and so is a closed subset of* $\operatorname{Spec}(R)$;

(iv) *a prime ideal* $\mathfrak{p}$ *of* R *is a minimal member of* $\operatorname{Psupp}^i(M) = \operatorname{Supp}(K^i_M)$ *if and only if the length of the* $R_{\mathfrak{p}}$*-module* $H^{i-\dim R/\mathfrak{p}}_{\mathfrak{p}R_{\mathfrak{p}}}(M_{\mathfrak{p}})$ *is non-zero* *and finite; furthermore, when this is the case,*

$$\ell_{R_\mathfrak{p}}\big(H^{i-\dim R/\mathfrak{p}}_{\mathfrak{p}R_\mathfrak{p}}(M_\mathfrak{p})\big) = \ell_{R_\mathfrak{p}}\big((K^i_M)_\mathfrak{p}\big)\,;$$

(v) *if* $H^i_\mathfrak{m}(M) \neq 0$*, then*

$$e'\left(\mathfrak{q}, H^i_\mathfrak{m}(M)\right) = \sum_{\substack{\mathfrak{p}\in \operatorname{Psupp}^i(M)\\ \dim R/\mathfrak{p} = \operatorname{psd}^i(M)}} \ell_{R_\mathfrak{p}}\big(H^{i-\dim R/\mathfrak{p}}_{\mathfrak{p}R_\mathfrak{p}}(M_\mathfrak{p})\big)e(\mathfrak{q}, R/\mathfrak{p}).$$

Proof. (i),(ii) These, essentially, follow from the fact that $H^i_\mathfrak{m}(M) \cong D(K^i_M)$, because that isomorphism yields that

$$\Big(0 :_{H^i_\mathfrak{m}(M)} \mathfrak{q}^{n+1}\Big) \cong \Big(0 :_{D(K^i_M)} \mathfrak{q}^{n+1}\Big) \cong D(K^i_M/\mathfrak{q}^{n+1}K^i_M) \quad \text{for all } n \geq 0.$$

Note that $K^i_M/\mathfrak{q}^{n+1}K^i_M$ and its Matlis dual have the same length, because $D(R/\mathfrak{m}) \cong R/\mathfrak{m}$.

(iii), (iv) Let $\mathfrak{p} \in \operatorname{Spec}(R)$ and set $t = \dim R/\mathfrak{p}$, and let $\mathfrak{p}' := f^{-1}(\mathfrak{p})$, the contraction of $\mathfrak{p}$ back to R' under f. Now $R'_{\mathfrak{p}'}$ is a Gorenstein local ring, and $\dim R'/\mathfrak{p}' = t$. Since R' is Gorenstein, we have

$$\dim R'_{\mathfrak{p}'} = \dim R' - \dim R'/\mathfrak{p}' = n' - t.$$

Let $f' : R'_{\mathfrak{p}'} \longrightarrow R_\mathfrak{p}$ be the surjective ring homomorphism for which $f'(r'/s') = f(r')/f(s')$ for all $r' \in R'$, $s' \in R' \setminus \mathfrak{p}'$. There is an $R_\mathfrak{p}$-isomorphism $\operatorname{Ext}^{n'-i}_{R'_{\mathfrak{p}'}}(M_\mathfrak{p}, R'_{\mathfrak{p}'}) \cong \Big(\operatorname{Ext}^{n'-i}_{R'}(M, R')\Big)_\mathfrak{p} = (K^i_M)_\mathfrak{p}$.

By the Local Duality Theorem and the above comments, we have

$$\begin{aligned} H^{i-t}_{\mathfrak{p}R_\mathfrak{p}}(M_\mathfrak{p}) &\cong \operatorname{Hom}_{R_\mathfrak{p}}\big(\operatorname{Ext}^{n'-i}_{R'_{\mathfrak{p}'}}(M_\mathfrak{p}, R'_{\mathfrak{p}'}), E_{R_\mathfrak{p}}(R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p})\big) \\ &\cong \operatorname{Hom}_{R_\mathfrak{p}}\big((K^i_M)_\mathfrak{p}, E_{R_\mathfrak{p}}(R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p})\big) \end{aligned}$$

as $R_\mathfrak{p}$-modules. It follows that $H^{i-t}_{\mathfrak{p}R_\mathfrak{p}}(M_\mathfrak{p}) \neq 0$ if and only if $(K^i_M)_\mathfrak{p} \neq 0$.

Hence $\mathfrak{p}$ is a minimal member of $\operatorname{Psupp}^i(M)$ if and only if $\mathfrak{p}$ is a minimal member of $\operatorname{Supp}(K^i_M)$, that is, if and only if $(K^i_M)_\mathfrak{p}$ is a non-zero $R_\mathfrak{p}$-module of finite length. Now the Matlis dual of $(K^i_M)_\mathfrak{p}$, over the local ring $R_\mathfrak{p}$, is isomorphic to $H^{i-t}_{\mathfrak{p}R_\mathfrak{p}}(M_\mathfrak{p})$. All the claims follow from this and the observation that an $R_\mathfrak{p}$-module has finite length if and only if its Matlis dual (over $R_\mathfrak{p}$) has finite length, and that, then, the two lengths are equal.

(v) This result now follows from the formula

$$e(\mathfrak{q}, K_M^i) = \sum_{\substack{\mathfrak{p}\in \operatorname{Supp}(K_M^i) \\ \dim R/\mathfrak{p} = \dim K_M^i}} \ell_{R_\mathfrak{p}}\big((K_M^i)_\mathfrak{p}\big)\, e(\mathfrak{q}, R/\mathfrak{p})$$

(see [4, 4.6.8]) used in conjunction with parts (ii), (iii) and (iv). □

§2. Extension to the case of a universally catenary local ring all of whose formal fibres are Cohen–Macaulay

The main aim of this section is to show that the formula for the multiplicity $e'\left(\mathfrak{q}, H^i_\mathfrak{m}(M)\right)$ given by Proposition 1.2(v), under the assumption that R is a homomorphic image of a Gorenstein local ring, is also valid in the case when the local ring R is universally catenary and has all its formal fibres Cohen–Macaulay. Along the way, we establish (in 2.1) a result, about a flat local homomorphism of local rings which has Cohen–Macaulay fibre over the maximal ideal, which may be of interest in its own right.

THEOREM 2.1. *Let $h : (R, \mathfrak{m}) \longrightarrow (B, \mathfrak{n})$ be a flat local homomorphism of local rings such that $B/\mathfrak{m}B$ is Cohen–Macaulay of dimension d. Then, for every R-module N, and for all integers j, we have that*

$$H^{d+j}_\mathfrak{n}(N \otimes_R B) \cong H^d_\mathfrak{n}\left(H^j_\mathfrak{m}(N) \otimes_R B\right)$$

and $H^{d+j}_\mathfrak{n}(N \otimes_R B) \neq 0$ if and only if $H^j_\mathfrak{m}(N) \neq 0$.

Proof. There is a spectral sequence $E_2^{ij} = H^i_\mathfrak{n}\Big(H^j_{\mathfrak{m}B}(N \otimes_R B)\Big) \Rightarrow E^{i+j} = H^{i+j}_{\mathfrak{n}+\mathfrak{m}B}(N \otimes_R B) = H^{i+j}_\mathfrak{n}(N \otimes_R B)$. Note that $H^j_{\mathfrak{m}B}(N \otimes_R B) \cong H^j_\mathfrak{m}(N) \otimes_R B$ as B-modules, by the Flat Base Change Theorem for local cohomology. We show now that $H^i_\mathfrak{n}\Big(H^j_\mathfrak{m}(N) \otimes_R B\Big) = 0$ for all integers i, j with $i \neq d$. It will then follow that the spectral sequence collapses and that $H^{d+j}_\mathfrak{n}(N \otimes_R B) \cong H^d_\mathfrak{n}\Big(H^j_\mathfrak{m}(N) \otimes_R B\Big)$ for all integers j.

Let L be a non-zero R-module of finite length. Then the non-zero B-module $L \otimes_R B$ has depth and dimension both equal to d. Hence $H^i_\mathfrak{n}(L \otimes_R B) = 0$ if and only if $i \neq d$. Since the formation of local cohomology modules and tensor products commute with direct limits, and since a finitely generated submodule of $H^j_\mathfrak{m}(N)$ has finite length, it follows that $H^i_\mathfrak{n}\Big(H^j_\mathfrak{m}(N) \otimes_R B\Big) = 0$ for all integers i, j with $i \neq d$.

Let j be a non-negative integer such that $H^j_{\mathfrak{m}}(N) \neq 0$. An argument similar to that used in the last paragraph shows that $H^{d-1}_{\mathfrak{n}}(G \otimes_R B) = 0$ for every homomorphic image G of $H^j_{\mathfrak{m}}(N)$; therefore, since $H^j_{\mathfrak{m}}(N)$ has a simple submodule S and $H^d_{\mathfrak{n}}(S \otimes_R B) \neq 0$, it follows that $H^d_{\mathfrak{n}}\left(H^j_{\mathfrak{m}}(N) \otimes_R B\right) \neq 0$. All the claims have now been proved. □

We intend to use 2.1 to study the pseudo-supports of the finitely generated R-module M. However, before we do so, we present one constructive result about pseudo-supports which holds whenever the local ring R is catenary.

LEMMA 2.2. *Assume that the local ring R is catenary. Let i be a non-negative integer. Then* $\operatorname{Psupp}^i_R(M)$ *is closed under specialization.*

Proof. Let $\mathfrak{p}, \mathfrak{q} \in \operatorname{Spec}(R)$ with $\mathfrak{p} \subseteq \mathfrak{q}$ and $\mathfrak{p} \in \operatorname{Psupp}^i_R(M)$. Therefore $H^{i-\dim R/\mathfrak{p}}_{\mathfrak{p}R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \neq 0$. Note that $R_{\mathfrak{p}}$ is isomorphic to the localization of the local ring $R_{\mathfrak{q}}$ at the prime ideal $\mathfrak{p}R_{\mathfrak{q}}$.

Now the non-zero Artinian $R_{\mathfrak{p}}$-module $H^{i-\dim R/\mathfrak{p}}_{\mathfrak{p}R_{\mathfrak{p}}}(M_{\mathfrak{p}})$ must have an attached prime ideal. We can use the Weak General Shifted Localization Principle [12, Theorem 4.8] on the local ring $R_{\mathfrak{q}}$ to deduce that $H^{i-\dim R/\mathfrak{p}+\operatorname{ht}\mathfrak{q}/\mathfrak{p}}_{\mathfrak{q}R_{\mathfrak{q}}}(M_{\mathfrak{q}})$ (has an attached prime ideal and so) is non-zero. Since R is catenary, $\dim R/\mathfrak{p} - \operatorname{ht}\mathfrak{q}/\mathfrak{p} = \dim R/\mathfrak{q}$; hence $H^{i-\dim R/\mathfrak{q}}_{\mathfrak{q}R_{\mathfrak{q}}}(M_{\mathfrak{q}}) \neq 0$ and $\mathfrak{q} \in \operatorname{Psupp}^i_R(M)$. □

PROPOSITION 2.3. *Let $i \in \mathbb{Z}$ with $i \geq 0$, let $\mathfrak{p} \in \operatorname{Spec}(R)$, and let $\mathfrak{P} \in \operatorname{Spec}(\widehat{R})$ be such that $\mathfrak{P} \cap R = \mathfrak{p}$. Let $h' : R_{\mathfrak{p}} \longrightarrow \widehat{R}_{\mathfrak{P}}$ be the flat local homomorphism induced by the inclusion homomorphism $R \longrightarrow \widehat{R}$. Assume that the fibre ring of h' over the maximal ideal $\mathfrak{p}R_{\mathfrak{p}}$ of $R_{\mathfrak{p}}$ is Cohen–Macaulay, and that R is universally catenary. Then $\mathfrak{P} \in \operatorname{Psupp}^i_{\widehat{R}}(M \otimes_R \widehat{R})$ if and only if $\mathfrak{p} \in \operatorname{Psupp}^i_R(M)$.*

Proof. The fibre ring of h' over $\mathfrak{p}R_{\mathfrak{p}}$ has dimension equal to $\dim \widehat{R}_{\mathfrak{P}} - \dim R_{\mathfrak{p}} = \operatorname{ht}\mathfrak{P} - \operatorname{ht}\mathfrak{p}$. It therefore follows from 2.1 that $H^{i-\dim R/\mathfrak{p}}_{\mathfrak{p}R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \neq 0$ if and only if $H^{i-\dim R/\mathfrak{p}+\operatorname{ht}\mathfrak{P}-\operatorname{ht}\mathfrak{p}}_{\mathfrak{P}\widehat{R}_{\mathfrak{P}}}\left(M_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}} \widehat{R}_{\mathfrak{P}}\right) \neq 0$. Since $M_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}} \widehat{R}_{\mathfrak{P}} \cong (M \otimes_R \widehat{R})_{\mathfrak{P}}$ as $\widehat{R}_{\mathfrak{P}}$-modules, the result follows from the equality $\dim R/\mathfrak{p} + \operatorname{ht}\mathfrak{p} = \dim \widehat{R}/\mathfrak{P} + \operatorname{ht}\mathfrak{P}$ which can be established with the aid of L. J. Ratliff's Theorem [9, Theorem 31.7]. □

THEOREM 2.4. *Assume that the local ring R is universally catenary, and that all the formal fibres of R are Cohen–Macaulay. Let $i \in \mathbb{Z}$ with $i \geq 0$, and let $\mathfrak{q}$ be an $\mathfrak{m}$-primary ideal of R.*

(i) *Let $\mathfrak{p} \in \operatorname{Spec}(R)$, and let $\mathfrak{P}$ be a minimal prime of $\mathfrak{p}\widehat{R}$. The following statements are equivalent:*

(a) *$\mathfrak{p}$ is a minimal member of $\operatorname{Psupp}^i_R(M)$;*

(b) *$\mathfrak{P}$ is a minimal member of $\operatorname{Psupp}^i_{\widehat{R}}(M \otimes_R \widehat{R})$;*

(c) *$\ell_{R_\mathfrak{p}}\big(H^{i-\dim R/\mathfrak{p}}_{\mathfrak{p}R_\mathfrak{p}}(M_\mathfrak{p})\big)$ is non-zero and finite.*

Furthermore, when these conditions are satisfied, we have

$$\ell_{\widehat{R}_\mathfrak{P}}\big(H^{i-\dim \widehat{R}/\mathfrak{P}}_{\mathfrak{P}\widehat{R}_\mathfrak{P}}((M \otimes_R \widehat{R})_\mathfrak{P})\big) = \ell_{R_\mathfrak{p}}\big(H^{i-\dim R/\mathfrak{p}}_{\mathfrak{p}R_\mathfrak{p}}(M_\mathfrak{p})\big)\ell_{\widehat{R}_\mathfrak{P}}\big(\widehat{R}_\mathfrak{P}/\mathfrak{p}\widehat{R}_\mathfrak{P}\big).$$

(ii) *The subset $\operatorname{Psupp}^i_R(M)$ of $\operatorname{Spec}(R)$ is closed, and its dimension $\operatorname{psd}^i(M)$ is equal to the dimension of the Artinian R-module $H^i_\mathfrak{m}(M)$.*

(iii) *Suppose that $H^i_\mathfrak{m}(M) \neq 0$. Then the multiplicity $e'\left(\mathfrak{q}, H^i_\mathfrak{m}(M)\right)$ of the Artinian module $H^i_\mathfrak{m}(M)$ with respect to $\mathfrak{q}$ satisfies*

$$e'\left(\mathfrak{q}, H^i_\mathfrak{m}(M)\right) = \sum_{\substack{\mathfrak{p} \in \operatorname{Psupp}^i(M) \\ \dim R/\mathfrak{p} = \operatorname{psd}^i(M)}} \ell_{R_\mathfrak{p}}\left(H^{i-\dim R/\mathfrak{p}}_{\mathfrak{p}R_\mathfrak{p}}(M_\mathfrak{p})\right) e(\mathfrak{q}, R/\mathfrak{p}).$$

Proof. (i) Note that $\mathfrak{P} \cap R = \mathfrak{p}$, so that $\mathfrak{p} \in \operatorname{Psupp}^i_R(M)$ if and only if $\mathfrak{P} \in \operatorname{Psupp}^i_{\widehat{R}}(M \otimes_R \widehat{R})$ by 2.3.

Suppose $\mathfrak{p}$ is a minimal member of $\operatorname{Psupp}^i_R(M)$, that $\mathfrak{Q} \in \operatorname{Psupp}^i_{\widehat{R}}(M \otimes_R \widehat{R})$ and that $\mathfrak{Q} \subset \mathfrak{P}$ (we reserve '$\subset$' to denote strict inclusion). Then $\mathfrak{Q} \cap R \in \operatorname{Psupp}^i_R(M)$ by 2.3, and since $\mathfrak{Q} \cap R \neq \mathfrak{p}$ because $\mathfrak{P}$ is a minimal prime of $\mathfrak{p}\widehat{R}$, we must have $\mathfrak{Q} \cap R \subset \mathfrak{p}$. This contradicts the fact that $\mathfrak{p}$ is a minimal member of $\operatorname{Psupp}^i_R(M)$. Hence $\mathfrak{P}$ is a minimal member of $\operatorname{Psupp}^i_{\widehat{R}}(M \otimes_R \widehat{R})$.

Now suppose $\mathfrak{P}$ is a minimal member of $\operatorname{Psupp}^i_{\widehat{R}}(M \otimes_R \widehat{R})$, and that there exists $\mathfrak{q} \in \operatorname{Psupp}^i_R(M)$ with $\mathfrak{q} \subset \mathfrak{p}$. Because the induced ring homomorphism $h' : R_\mathfrak{p} \longrightarrow \widehat{R}_\mathfrak{P}$ is faithfully flat, there exists $\mathfrak{Q} \in \operatorname{Spec}(\widehat{R})$ such that $\mathfrak{Q} \subseteq \mathfrak{P}$ and $\mathfrak{Q} \cap R = \mathfrak{q}$. Since $\mathfrak{q} \subset \mathfrak{p}$, we must have $\mathfrak{Q} \subset \mathfrak{P}$. Also,

$\mathfrak{Q} \in \operatorname{Psupp}^i_{\widehat{R}}(M \otimes_R \widehat{R})$, by 2.3. This is a contradiction. Therefore $\mathfrak{p}$ is a minimal member of $\operatorname{Psupp}^i_R(M)$.

It is immediate from 1.2(iv) and the fact that $\widehat{R}$ is a homomorphic image of a regular local ring (by Cohen's Structure Theorem) that $\mathfrak{P}$ is a minimal member of $\operatorname{Psupp}^i_{\widehat{R}}(M \otimes_R \widehat{R})$ if and only if $\ell_{\widehat{R}_{\mathfrak{P}}}\big(H^{i-\dim \widehat{R}/\mathfrak{P}}_{\mathfrak{P}\widehat{R}_{\mathfrak{P}}}((M \otimes_R \widehat{R})_{\mathfrak{P}})\big)$ is non-zero and finite. Also, we can use Ratliff's Theorem [9, Theorem 31.7] to see that $\dim \widehat{R}/\mathfrak{P} = \dim R/\mathfrak{p}$; note also that $\sqrt{\mathfrak{p}R_{\mathfrak{p}}\widehat{R}_{\mathfrak{P}}} = \mathfrak{P}\widehat{R}_{\mathfrak{P}}$. It therefore follows from the Flat Base Change Theorem for local cohomology that there are $\widehat{R}_{\mathfrak{P}}$-isomorphisms

$$\begin{aligned} H^{i-\dim \widehat{R}/\mathfrak{P}}_{\mathfrak{P}\widehat{R}_{\mathfrak{P}}}((M \otimes_R \widehat{R})_{\mathfrak{P}}) &\cong H^{i-\dim \widehat{R}/\mathfrak{P}}_{\mathfrak{P}\widehat{R}_{\mathfrak{P}}}(M_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}} \widehat{R}_{\mathfrak{P}}) \\ &\cong H^{i-\dim R/\mathfrak{p}}_{\mathfrak{p}R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \otimes_{R_{\mathfrak{p}}} \widehat{R}_{\mathfrak{P}}. \end{aligned}$$

Since h' is faithfully flat, it follows that $\ell_{\widehat{R}_{\mathfrak{P}}}\big(H^{i-\dim \widehat{R}/\mathfrak{P}}_{\mathfrak{P}\widehat{R}_{\mathfrak{P}}}((M \otimes_R \widehat{R})_{\mathfrak{P}})\big)$ is non-zero and finite if and only if $\ell_{R_{\mathfrak{p}}}\big(H^{i-\dim R/\mathfrak{p}}_{\mathfrak{p}R_{\mathfrak{p}}}(M_{\mathfrak{p}})\big)$ is non-zero and finite, and that, when this is the case,

$$\ell_{\widehat{R}_{\mathfrak{P}}}\big(H^{i-\dim \widehat{R}/\mathfrak{P}}_{\mathfrak{P}\widehat{R}_{\mathfrak{P}}}((M \otimes_R \widehat{R})_{\mathfrak{P}})\big) = \ell_{R_{\mathfrak{p}}}\big(H^{i-\dim R/\mathfrak{p}}_{\mathfrak{p}R_{\mathfrak{p}}}(M_{\mathfrak{p}})\big)\ell_{\widehat{R}_{\mathfrak{P}}}\big(\widehat{R}_{\mathfrak{P}}/\mathfrak{p}\widehat{R}_{\mathfrak{P}}\big).$$

(ii) Since $\operatorname{Psupp}^i_R(M)$ is closed under specialization (by 2.2), it is enough, in order to show that $\operatorname{Psupp}^i_R(M)$ is closed, for us to show that $\operatorname{Psupp}^i_R(M)$ has only finitely many minimal members. So let $\mathfrak{p}$ be a minimal member of $\operatorname{Psupp}^i_R(M)$. Let $\mathfrak{P}$ be a minimal prime of $\mathfrak{p}\widehat{R}$ (so that $\mathfrak{P} \cap R = \mathfrak{p}$). By part (i), $\mathfrak{P}$ is a minimal member of $\operatorname{Psupp}^i_{\widehat{R}}(M \otimes_R \widehat{R})$. Since $\widehat{R}$ is a homomorphic image of a regular local ring, we can now deduce from 1.2(iii) that $\operatorname{Psupp}^i_{\widehat{R}}(M \otimes_R \widehat{R})$ is a closed subset of $\operatorname{Spec}(\widehat{R})$, so that it has only finitely many minimal members; hence $\operatorname{Psupp}^i_R(M)$ has only finitely many minimal members.

Use of 2.3 now enables us to deduce that $\operatorname{psd}^i_R(M) = \operatorname{psd}^i_{\widehat{R}}(M \otimes_R \widehat{R})$. We can use 1.2(iii) again to see that $\operatorname{psd}^i_{\widehat{R}}(M \otimes_R \widehat{R}) = \dim_{\widehat{R}}\big(K^i_{M \otimes_R \widehat{R}}\big)$. Now $H^i_{\mathfrak{m}}(M)$ has a natural structure as an $\widehat{R}$-module, and, when considered as such, $H^i_{\mathfrak{m}}(M) \cong H^i_{\mathfrak{m}}(M) \otimes_R \widehat{R} \cong H^i_{\widehat{\mathfrak{m}}}(M \otimes_R \widehat{R})$ (where $\widehat{\mathfrak{m}}$ denotes the maximal ideal of $\widehat{R}$). It follows, on use of 1.2(i), that

$$\Theta^{\mathfrak{q}}_{H^i_{\mathfrak{m}}(M)} = \Theta^{\mathfrak{q}\widehat{R}}_{H^i_{\widehat{\mathfrak{m}}}(M \otimes_R \widehat{R})} = \Sigma^{\mathfrak{q}\widehat{R}}_{K^i_{M \otimes_R \widehat{R}}};$$

we can now deduce that $\dim_R(H^i_{\mathfrak{m}}(M)) = \dim_{\widehat{R}}\left(H^i_{\widehat{\mathfrak{m}}}(M \otimes_R \widehat{R})\right) = \dim_{\widehat{R}}\left(K^i_{M\otimes_R \widehat{R}}\right) = \operatorname{psd}^i_R(M)$.

(iii) Observe that, if N is any finitely generated R-module, then, for all $n \in \mathbb{Z}$ with $n > 0$, we have

$$\ell_R(N/\mathfrak{q}^n N) = \ell_{\widehat{R}}\big((N/\mathfrak{q}^n N) \otimes_R \widehat{R}\big) = \ell_{\widehat{R}}\big((N \otimes_R \widehat{R})/(\mathfrak{q}\widehat{R})^n(N \otimes_R \widehat{R})\big),$$

and so $e(\mathfrak{q}, R/\mathfrak{p}) = e(\mathfrak{q}\widehat{R}, \widehat{R}/\mathfrak{p}\widehat{R})$ for all $\mathfrak{p} \in \operatorname{Spec}(R)$. Furthermore, it follows from [4, 4.6.8] that

$$e(\mathfrak{q}\widehat{R}, \widehat{R}/\mathfrak{p}\widehat{R}) = \sum_{\substack{\mathfrak{P}\in \operatorname{Supp}_{\widehat{R}}\left(\widehat{R}/\mathfrak{p}\widehat{R}\right) \\ \dim \widehat{R}/\mathfrak{P} = \dim R/\mathfrak{p}}} \ell_{\widehat{R}_{\mathfrak{P}}}\big(\widehat{R}_{\mathfrak{P}}/\mathfrak{p}\widehat{R}_{\mathfrak{P}}\big) e(\mathfrak{q}\widehat{R}, \widehat{R}/\mathfrak{P}).$$

We observed above that $\Theta^{\mathfrak{q}}_{H^i_{\mathfrak{m}}(M)} = \Theta^{\mathfrak{q}\widehat{R}}_{H^i_{\widehat{\mathfrak{m}}}(M\otimes_R \widehat{R})}$; hence $e'\left(\mathfrak{q}, H^i_{\mathfrak{m}}(M)\right) = e'\left(\mathfrak{q}\widehat{R}, H^i_{\widehat{\mathfrak{m}}}(M \otimes_R \widehat{R})\right)$. Set $s := \operatorname{psd}^i(M)$, so that $s = \operatorname{psd}^i_{\widehat{R}}(M \otimes_R \widehat{R})$ by the above proof of part (ii). In view of Cohen's Structure Theorem for complete local rings, we can apply 1.2(v) to obtain that

$$\begin{aligned} & e'\big(\mathfrak{q}\widehat{R}, H^i_{\widehat{\mathfrak{m}}}(M \otimes_R \widehat{R})\big) \\ & \qquad = \sum_{\substack{\mathfrak{P}\in \operatorname{Psupp}^i_{\widehat{R}}(M\otimes_R \widehat{R}) \\ \dim \widehat{R}/\mathfrak{P} = s}} \ell_{\widehat{R}_{\mathfrak{P}}}\big(H^{i-s}_{\mathfrak{P}\widehat{R}_{\mathfrak{P}}}((M \otimes_R \widehat{R})_{\mathfrak{P}})\big) e(\mathfrak{q}\widehat{R}, \widehat{R}/\mathfrak{P}). \end{aligned}$$

We now make use of the immediately preceding paragraph in this proof, together with part (i) and 2.3, to see that, for a $\mathfrak{p} \in \operatorname{Psupp}^i_R(M)$ with $\dim R/\mathfrak{p} = s$,

$$\begin{aligned} & \sum_{\substack{\mathfrak{P}\in \operatorname{Psupp}^i_{\widehat{R}}(M\otimes_R \widehat{R}) \\ \dim \widehat{R}/\mathfrak{P} = s, \mathfrak{P}\cap R = \mathfrak{p}}} \ell_{\widehat{R}_{\mathfrak{P}}}\big(H^{i-s}_{\mathfrak{P}\widehat{R}_{\mathfrak{P}}}((M \otimes_R \widehat{R})_{\mathfrak{P}})\big) e(\mathfrak{q}\widehat{R}, \widehat{R}/\mathfrak{P}) \\ & \qquad = \sum_{\substack{\mathfrak{P}\in \operatorname{Psupp}^i_{\widehat{R}}(M\otimes_R \widehat{R}) \\ \dim \widehat{R}/\mathfrak{P} = s, \mathfrak{P}\cap R = \mathfrak{p}}} \ell_{R_{\mathfrak{p}}}\big(H^{i-s}_{\mathfrak{p}R_{\mathfrak{p}}}(M_{\mathfrak{p}})\big) \ell_{\widehat{R}_{\mathfrak{P}}}\left(\widehat{R}_{\mathfrak{P}}/\mathfrak{p}\widehat{R}_{\mathfrak{P}}\right) e(\mathfrak{q}\widehat{R}, \widehat{R}/\mathfrak{P}) \\ & \qquad = \ell_{R_{\mathfrak{p}}}\big(H^{i-s}_{\mathfrak{p}R_{\mathfrak{p}}}(M_{\mathfrak{p}})\big) e(\mathfrak{q}\widehat{R}, \widehat{R}/\mathfrak{p}\widehat{R}) = \ell_{R_{\mathfrak{p}}}\big(H^{i-s}_{\mathfrak{p}R_{\mathfrak{p}}}(M_{\mathfrak{p}})\big) e(\mathfrak{q}, R/\mathfrak{p}), \end{aligned}$$

and the result follows easily from this. ∎

Our final result in this section compares the i-th pseudo-support of M with the co-support of the Artinian R-module $H^i_{\mathfrak{m}}(M)$, as defined by L. Melkersson and P. Schenzel in [10, 7.1]: they define the *co-support* $\operatorname{Cos}_R A$ of an Artinian R-module A by

$$\operatorname{Cos}_R A := \{\mathfrak{p} \in \operatorname{Spec}(R) : \operatorname{Hom}_R(R_{\mathfrak{p}}, A) \neq 0\}.$$

We use $\operatorname{Var}(\mathfrak{a})$ to denote the *variety* $\{\mathfrak{p} \in \operatorname{Spec}(R) : \mathfrak{p} \supseteq \mathfrak{a}\}$ of an ideal $\mathfrak{a}$ of R. By [10, 7.2],

$$\operatorname{Cos}_R A = \bigcup_{\mathfrak{p} \in \operatorname{Att}_R A} \operatorname{Var}(\mathfrak{p}),$$

and so is a closed subset of $\operatorname{Spec}(R)$.

PROPOSITION 2.5. *Assume that the local ring R is universally catenary, and that all the formal fibres of R are Cohen–Macaulay. Let $i \in \mathbb{Z}$ with $i \geq 0$. Then* $\operatorname{Psupp}^i_R(M) = \operatorname{Cos}_R\left(H^i_{\mathfrak{m}}(M)\right)$, *that is, the i-th pseudo-support of M is equal to the co-support of the Artinian R-module $H^i_{\mathfrak{m}}(M)$.*

Proof. We first consider the special case in which R is a homomorphic image of a Gorenstein local ring, and, for this case, we use the notation of 1.1. Then $H^i_{\mathfrak{m}}(M) \cong D(K^i_M)$; hence, by [3, 10.2.20], we have $\operatorname{Att}_R\left(H^i_{\mathfrak{m}}(M)\right) = \operatorname{Ass}_R(K^i_M)$. Therefore, by 1.2(iii),

$$\begin{aligned}\operatorname{Psupp}^i(M) = \operatorname{Supp}(K^i_M) &= \bigcup_{\mathfrak{p} \in \operatorname{Ass}(K^i_M)} \operatorname{Var}(\mathfrak{p}) \\ &= \bigcup_{\mathfrak{p} \in \operatorname{Att}(H^i_{\mathfrak{m}}(M))} \operatorname{Var}(\mathfrak{p}) = \operatorname{Cos}_R\left(H^i_{\mathfrak{m}}(M)\right).\end{aligned}$$

Now we relax our conditions on the local ring R, and just assume that R is universally catenary and that all the formal fibres of R are Cohen–Macaulay. By [3, 11.3.7(iii)],

$$\operatorname{Att}_R(H^i_{\mathfrak{m}}(M)) = \left\{\mathfrak{P} \cap R : \mathfrak{P} \in \operatorname{Att}_{\widehat{R}}\left(H^i_{\widehat{\mathfrak{m}}}(M \otimes_R \widehat{R})\right)\right\},$$

where $\widehat{\mathfrak{m}}$ denotes the maximal ideal of $\widehat{R}$. By the immediately preceding paragraph (and Cohen's Structure Theorem for complete local rings), $\operatorname{Att}_{\widehat{R}}\left(H^i_{\widehat{\mathfrak{m}}}(M \otimes_R \widehat{R})\right) \subseteq \operatorname{Psupp}^i_{\widehat{R}}(M \otimes_R \widehat{R})$. We now use 2.3 to see that

$\mathrm{Att}_R(H^i_{\mathfrak{m}}(M)) \subseteq \left\{\mathfrak{P} \cap R : \mathfrak{P} \in \mathrm{Psupp}^i_{\widehat{R}}(M \otimes_R \widehat{R})\right\} = \mathrm{Psupp}^i_R(M)$. Since $\mathrm{Psupp}^i_R(M)$ is closed under specialization (by 2.2), we deduce that

$$\bigcup_{\mathfrak{p} \in \mathrm{Att}_R(H^i_{\mathfrak{m}}(M))} \mathrm{Var}(\mathfrak{p}) \subseteq \mathrm{Psupp}^i_R(M).$$

Now let $\mathfrak{p}' \in \mathrm{Psupp}^i_R(M)$, and let $\mathfrak{P} \in \mathrm{Spec}(\widehat{R})$ be such that $\mathfrak{P} \cap R = \mathfrak{p}'$. Then $\mathfrak{P} \in \mathrm{Psupp}^i_{\widehat{R}}(M \otimes_R \widehat{R})$ by 2.3. There exists $\mathfrak{Q} \in \mathrm{Att}_{\widehat{R}}\left(H^i_{\widehat{\mathfrak{m}}}(M \otimes_R \widehat{R})\right)$ such that $\mathfrak{P} \supseteq \mathfrak{Q}$, by the first paragraph of this proof. Therefore $\mathfrak{p}' = \mathfrak{P} \cap R \supseteq \mathfrak{Q} \cap R \in \mathrm{Att}_R(H^i_{\mathfrak{m}}(M))$. Hence $\mathfrak{p}' \in \bigcup_{\mathfrak{p} \in \mathrm{Att}_R(H^i_{\mathfrak{m}}(M))} \mathrm{Var}(\mathfrak{p})$. The result follows. □

The co-support of an Artinian R-module is always a closed subset of $\mathrm{Spec}(R)$. In the next section, we shall provide some examples of local rings which have (some) non-closed pseudo-supports; this means that we cannot hope for the conclusion of 2.5 to be valid in general, over every local ring.

§3. Non-closed pseudo-supports

Theorem 2.4, the main result of §2, was proved under the hypotheses that R is universally catenary and all its formal fibres are Cohen–Macaulay. It is natural to ask whether the result is still true without these additional hypotheses on R. In this section, we provide examples to show that this question has a negative answer. For instance, the local domain of Example 3.2 (considered as a module over itself) contains in its second pseudo-support infinitely many primes of dimension equal to the second pseudo-dimension of R, and in these circumstances the right-hand side of the equation in 2.4(iii) does not make sense. We also provide examples to show that, for the conclusion (in 2.4(ii)) that the i-th pseudo-support of M be closed to be valid, neither the hypotheses that R be universally catenary, nor the hypothesis that all the formal fibres of R be Cohen–Macaulay, can be dropped.

We start with an example of a universally catenary local domain which (itself) has a non-closed pseudo-support. We appeal to [2] to find a suitable example.

Example 3.1. It follows from [2, (15)] that there exists a 3-dimensional local Noetherian domain $(R, \mathfrak{m})$ with the following properties:

(i) R is a $\mathbb{Q}$-algebra such that $R/\mathfrak{p}$ is essentially of finite type over $\mathbb{Q}$ for all $\mathfrak{p} \in \operatorname{Spec}(R) \setminus \{0\}$;

(ii) the completion $\widehat{R}$ of R can be identified with $B/\mathfrak{b}$, where

$$B := \mathbb{Q}[[V_1, V_2, X, Y]], \quad \mathfrak{b} := (V_1V_2) \cap (V_1^2, V_2^2),$$

and V_1, V_2, X, Y are independent indeterminates over $\mathbb{Q}$;

(iii) with this identification, the prime ideals $\widehat{\mathfrak{p}}_1 := (V_1)/\mathfrak{b}$, $\widehat{\mathfrak{p}}_2 := (V_2)/\mathfrak{b}$, $\widehat{\mathfrak{q}} := (V_1, V_2)/\mathfrak{b}$ of $\widehat{R}$ satisfy

$$\begin{aligned} &\widehat{\mathfrak{p}}_1 \subset \widehat{\mathfrak{q}}, \quad \widehat{\mathfrak{p}}_2 \subset \widehat{\mathfrak{q}}, \quad \widehat{\mathfrak{p}}_1 + \widehat{\mathfrak{p}}_2 = \widehat{\mathfrak{q}}, \quad \operatorname{Ass} \widehat{R} = \{\widehat{\mathfrak{p}}_1, \widehat{\mathfrak{p}}_2, \widehat{\mathfrak{q}}\}, \\ &\dim \widehat{R}/\widehat{\mathfrak{p}}_1 = \dim \widehat{R}/\widehat{\mathfrak{p}}_2 = 3, \quad \dim \widehat{R}/\widehat{\mathfrak{q}} = 2. \end{aligned}$$

Then R is universally catenary, $\operatorname{Psupp}^3(R) = \operatorname{Spec}(R)$, but $\operatorname{Psupp}^2(R)$ is not closed in the Zariski topology.

Proof. Since $\widehat{\mathfrak{p}}_1$ and $\widehat{\mathfrak{p}}_2$ are the only minimal primes of $\widehat{R}$ and $\dim \widehat{R}/\widehat{\mathfrak{p}}_1 = \dim \widehat{R}/\widehat{\mathfrak{p}}_2 = 3$, it follows from [9, 31.6] that R is universally catenary. Therefore, by 2.2, all the pseudo-supports of R are closed under specialization. Since $0 \in \operatorname{Psupp}^3(R)$, it follows that $\operatorname{Psupp}^3(R) = \operatorname{Spec}(R)$.

Now let $\mathfrak{p} \in \operatorname{Spec}(R)$ and let $\widehat{\mathfrak{p}}$ be a minimal prime ideal of $\mathfrak{p}\widehat{R}$. We now aim to show that

$$(\dagger) \qquad \operatorname{depth} R_{\mathfrak{p}} = \begin{cases} \operatorname{ht}(\mathfrak{p}) & \text{if } \widehat{\mathfrak{q}} \not\subseteq \widehat{\mathfrak{p}}, \\ \operatorname{ht}(\mathfrak{p}) - 1 & \text{if } \widehat{\mathfrak{q}} \subseteq \widehat{\mathfrak{p}}. \end{cases}$$

Consider first the case where $\widehat{\mathfrak{q}} \not\subseteq \widehat{\mathfrak{p}}$. As $\widehat{\mathfrak{q}} = \widehat{\mathfrak{p}}_1 + \widehat{\mathfrak{p}}_2$, it follows that $\widehat{\mathfrak{p}}_{i'} \not\subseteq \widehat{\mathfrak{p}}$ for exactly one $i' \in \{1, 2\}$. Let i be the other member of $\{1, 2\}$. As $\operatorname{Ass} \widehat{R} = \{\widehat{\mathfrak{p}}_1, \widehat{\mathfrak{p}}_2, \widehat{\mathfrak{q}}\}$ and $\widehat{\mathfrak{p}}_i$ is the unique $\widehat{\mathfrak{p}}_i$-primary component of the zero ideal of $\widehat{R}$, it follows that there are ring isomorphisms

$$\widehat{R}_{\widehat{\mathfrak{p}}} \cong \left(\widehat{R}/\widehat{\mathfrak{p}}_i\right)_{\widehat{\mathfrak{p}}/\widehat{\mathfrak{p}}_i} \cong (B/V_iB)_{\mathfrak{P}/V_iB} \cong B_{\mathfrak{P}}/V_iB_{\mathfrak{P}},$$

where $\mathfrak{P}$ is the contraction of $\widehat{\mathfrak{p}}$ to B. Hence $\widehat{R}_{\widehat{\mathfrak{p}}}$ is Cohen–Macaulay, and, since the inclusion homomorphism $R \longrightarrow \widehat{R}$ induces a flat local homomorphism $R_{\mathfrak{p}} \longrightarrow \widehat{R}_{\widehat{\mathfrak{p}}}$, it follows that $R_{\mathfrak{p}}$ is Cohen–Macaulay and $\operatorname{depth} R_{\mathfrak{p}} = \operatorname{ht} \mathfrak{p}$.

Now consider the case where $\widehat{\mathfrak{q}} \subseteq \widehat{\mathfrak{p}}$. Since $\widehat{\mathfrak{p}} \cap R = \mathfrak{p}$ and $\widehat{\mathfrak{p}}$ is a minimal prime of $\mathfrak{p}\widehat{R}$, we can use the flat local homomorphism $R_{\mathfrak{p}} \longrightarrow \widehat{R}_{\widehat{\mathfrak{p}}}$ to see that it is enough for us to show that $\operatorname{depth} \widehat{R}_{\widehat{\mathfrak{p}}} = \operatorname{ht} \widehat{\mathfrak{p}} - 1$. Note that, by [9, 17.2], $\operatorname{depth} \widehat{R}_{\widehat{\mathfrak{p}}} \leq \operatorname{ht} \widehat{\mathfrak{p}} - 1$, in view of the fact that $\widehat{R}_{\widehat{\mathfrak{p}}}$ has an embedded associated prime ideal $\widehat{\mathfrak{q}}\widehat{R}_{\widehat{\mathfrak{p}}}$. Let $\mathfrak{P}$ be the contraction of $\widehat{\mathfrak{p}}$ to B. Consider the regular local ring $C := B_{\mathfrak{P}}$, and let $\mathfrak{c} := V_1V_2C$, $\mathfrak{d} := (V_1^2, V_2^2)C$. Note that $\widehat{R}_{\widehat{\mathfrak{p}}} \cong C/\mathfrak{c} \cap \mathfrak{d}$, so that $\dim \widehat{R}_{\widehat{\mathfrak{p}}} = \dim C - 1$. Since V_1, V_2 form a C-sequence and C is a Cohen–Macaulay ring, $\operatorname{depth}_C C/\mathfrak{c} = \dim C - 1$ and

$$\operatorname{depth}_C C/\mathfrak{d} = \operatorname{depth}_C C/(V_1, V_2)^2C = \dim C - 2.$$

We can therefore use the exact sequence

$$0 \longrightarrow C/\mathfrak{c} \cap \mathfrak{d} \longrightarrow C/\mathfrak{c} \oplus C/\mathfrak{d} \longrightarrow C/(V_1, V_2)^2C \longrightarrow 0$$

(see [3, 3.2.1]) to deduce that $\operatorname{depth}_C C/\mathfrak{c} \cap \mathfrak{d} \geq \dim C - 2$. Therefore

$$\operatorname{depth} \widehat{R}_{\widehat{\mathfrak{p}}} = \operatorname{depth}_C C/\mathfrak{c} \cap \mathfrak{d} \geq \dim \widehat{R}_{\widehat{\mathfrak{p}}} - 1 = \operatorname{ht} \widehat{\mathfrak{p}} - 1,$$

as required.

We have now proved our claim (†). A consequence of this is that $\operatorname{depth} R_{\mathfrak{p}} \in \{\dim R_{\mathfrak{p}}, \dim R_{\mathfrak{p}} - 1\}$ for all $\mathfrak{p} \in \operatorname{Spec}(R)$. Since R is a catenary local domain, it follows that $\operatorname{Psupp}^i(R) = \emptyset$ for $i = 0, 1$, that

$$\begin{aligned} \operatorname{Psupp}^2(R) &= \{\mathfrak{p} \in \operatorname{Spec}(R) : \operatorname{depth} R_{\mathfrak{p}} = \dim R_{\mathfrak{p}} - 1\} \\ &= \{\mathfrak{p} \in \operatorname{Spec}(R) : R_{\mathfrak{p}} \text{ is not Cohen–Macaulay}\}, \end{aligned}$$

and that $\operatorname{ht} \mathfrak{p} \geq 2$ for all $\mathfrak{p} \in \operatorname{Psupp}^2(R)$.

Now let $x \in \mathfrak{m} \setminus \{0\}$, let $\widehat{\mathfrak{p}}$ be a minimal prime of $\widehat{\mathfrak{q}} + x\widehat{R}$, and let $\mathfrak{p} = \widehat{\mathfrak{p}} \cap R$. Then, by [8, (15.E), Lemma 4],

$$\widehat{\mathfrak{p}} \in \operatorname{Ass}_{\widehat{R}}(\widehat{R}/x\widehat{R}) = \operatorname{Ass}_{\widehat{R}}(\widehat{R} \otimes_R R/xR);$$

hence $\mathfrak{p} \in \operatorname{Ass}_R(R/xR)$ and $\widehat{\mathfrak{p}} \in \operatorname{Ass}_{\widehat{R}}(\widehat{R} \otimes_R R/\mathfrak{p}) = \operatorname{Ass}_{\widehat{R}}(\widehat{R}/\mathfrak{p}\widehat{R})$. Note that $\mathfrak{p} \neq 0$ because $x \in \mathfrak{p}$; since it follows from hypothesis (i) that the fibre ring of $R/\mathfrak{p} \longrightarrow \widehat{R}/\mathfrak{p}\widehat{R}$ over the zero ideal is Cohen–Macaulay, we can now deduce that $\widehat{\mathfrak{p}}$ is a minimal prime of $\mathfrak{p}\widehat{R}$. As $\widehat{\mathfrak{q}} \subseteq \widehat{\mathfrak{p}}$, it follows from (†) that $\operatorname{depth} R_{\mathfrak{p}} = \dim R_{\mathfrak{p}} - 1$; therefore $\mathfrak{p} \in \operatorname{Psupp}^2(R)$. As $\mathfrak{p} \in \operatorname{Ass}_R(R/xR)$, we have $\operatorname{depth} R_{\mathfrak{p}} = 1$; therefore $\operatorname{ht} \mathfrak{p} = 2$, and $\mathfrak{p}$ is a minimal member of $\operatorname{Psupp}^2(R)$. Since $x \in \mathfrak{p}$, each non-zero element of $\mathfrak{m}$ belongs to a minimal member of $\operatorname{Psupp}^2(R)$. It follows that $\operatorname{Psupp}^2(R)$ must have infinitely many minimal members, and so cannot be a closed subset of $\operatorname{Spec}(R)$. ∎

Our second example is of a non-catenary 3-dimensional Noetherian local domain all of whose formal fibres are Cohen–Macaulay but which nevertheless has non-closed third and second pseudo-supports.

EXAMPLE 3.2. It follows from [1, (8)] that there exists a 3-dimensional excellent regular Noetherian domain S which is a $\mathbb{Q}$-algebra and has precisely two maximal ideals $\mathfrak{r}$, $\mathfrak{s}$, and these are such that

(i) $\operatorname{ht}\mathfrak{r} = 2$ and $\operatorname{ht}\mathfrak{s} = 3$;

(ii) the natural maps $\mathbb{Q} \longrightarrow S/\mathfrak{r}$ and $\mathbb{Q} \longrightarrow S/\mathfrak{s}$ are isomorphisms; and

(iii) $\mathfrak{r} \cap \mathfrak{s}$ contains no non-zero prime ideal of S.

Then $R := \mathbb{Q} + (\mathfrak{r} \cap \mathfrak{s})$ is a 3-dimensional Noetherian local domain all of whose formal fibres are Cohen–Macaulay but which has non-closed third and second pseudo-supports. Furthermore, the third pseudo-support of R is not closed under specialization.

Proof. S. Greco [5, §3] called a commutative Noetherian ring A *quasi-excellent* precisely when (a) for each $\mathfrak{q} \in \operatorname{Spec}(A)$, the canonical ring homomorphism from $A_{\mathfrak{q}}$ to its completion is regular in the sense of [8, (33.A)], and (b) for each finitely generated A-algebra B, the subset $\operatorname{Reg}(B) := \{\mathfrak{q} \in \operatorname{Spec}(B) : B_{\mathfrak{q}} \text{ is regular}\}$ is an open subset of $\operatorname{Spec}(B)$ in the Zariski topology.

By [1, (14) and (16)], the ring R is a Noetherian quasi-excellent local subring of S, and S is a finite integral extension of R. Thus all the formal fibres of R are regular, and therefore Cohen–Macaulay. Note that $\mathfrak{m} := \mathfrak{r} \cap \mathfrak{s}$ is the maximal ideal of R. Note also that $\dim R = \dim S = \operatorname{ht}\mathfrak{s} = 3$, and that R and S have the same quotient field (since $\mathfrak{m}S = \mathfrak{m} \subseteq R$), so that the (integrally closed) ring S must be the integral closure of R.

Now let $\mathfrak{p} \in \operatorname{Spec}(R) \setminus \{\mathfrak{m}\}$. As $\mathfrak{m}S \subseteq R$, it follows that $R_{\mathfrak{p}} \cong (R\setminus\mathfrak{p})^{-1}S$; hence there is exactly one $\mathfrak{p}' \in \operatorname{Spec}(S)$ such that $\mathfrak{p}' \cap R = \mathfrak{p}$; note that $(R \setminus \mathfrak{p})^{-1}S \cong S_{\mathfrak{p}'}$ (again because $\mathfrak{m}S \subseteq R$). Thus contraction provides a bijective map

$$\operatorname{Spec}(S) \setminus \{\mathfrak{r}, \mathfrak{s}\} \longrightarrow \operatorname{Spec}(R) \setminus \{\mathfrak{m}\},$$

and $R_{\mathfrak{p}' \cap R} \cong S_{\mathfrak{p}'}$ for all $\mathfrak{p}' \in \operatorname{Spec}(S) \setminus \{\mathfrak{r}, \mathfrak{s}\}$.

Now let

$$U := \left\{\mathfrak{p}' \cap R : \mathfrak{p}' \in \operatorname{Spec}(S) \text{ and } 0 \neq \mathfrak{p}' \subset \mathfrak{r}\right\},$$
$$V := \left\{\mathfrak{p}' \cap R : \mathfrak{p}' \in \operatorname{Spec}(S) \text{ and } 0 \neq \mathfrak{p}' \subset \mathfrak{s}\right\}.$$

(Recall that '$\subset$' is reserved to denote strict inclusion.) Then hypothesis (iii) and our conclusion in the immediately preceding paragraph above ensure that U and V are disjoint subsets of $\operatorname{Spec}(R)$. In fact, $\operatorname{Spec}(R)$ can be written as a disjoint union $\operatorname{Spec}(R) = \{\mathfrak{m}\}\dot\cup U\dot\cup V\dot\cup\{0\}$.

We can now use the above-mentioned bijective map to see that

$$U = \{\mathfrak{p} \in \operatorname{Spec}(R) \setminus \{\mathfrak{m}, 0\} : \operatorname{ht}\mathfrak{p} = \dim R/\mathfrak{p} = 1\},$$
$$V = \{\mathfrak{p} \in \operatorname{Spec}(R) \setminus \{\mathfrak{m}, 0\} : \operatorname{ht}\mathfrak{p} + \dim R/\mathfrak{p} = 3\}.$$

Now let $\mathfrak{p} \in \operatorname{Spec}(R) \setminus \{\mathfrak{m}, 0\}$. We have seen that $R_\mathfrak{p} \cong S_{\mathfrak{p}'}$, where $\mathfrak{p}'$ is the unique prime of S which contracts to $\mathfrak{p}$. Thus $R_\mathfrak{p}$ is regular, and so $H^j_{\mathfrak{p}R_\mathfrak{p}}(R_\mathfrak{p}) = 0$ if and only if $j \neq \operatorname{ht}\mathfrak{p}$. Therefore

$$\mathfrak{p} \in U \iff \mathfrak{p} \in \operatorname{Psupp}^2(R) \qquad \text{and} \qquad \mathfrak{p} \in V \iff \mathfrak{p} \in \operatorname{Psupp}^3(R).$$

Since S is regular, $\operatorname{grade}_S(\mathfrak{r}\cap\mathfrak{s}) = 2$, so that $\operatorname{depth}_R S = \operatorname{grade}_S(\mathfrak{r}\cap\mathfrak{s}) = 2$. We now use the notation of [3, Chapter 2] for ideal transforms; in particular, $D_\mathfrak{m}$ will denote the (left exact) $\mathfrak{m}$-transform functor from the category of all R-modules to itself.

By [3, 6.2.7], we have $H^i_\mathfrak{m}(S) = 0$ for $i = 0, 1$; because $\mathfrak{m}S \subseteq R$, the R-module S/R is $\mathfrak{m}$-torsion; it therefore follows from [3, 2.2.13] that there is an isomorphism $\psi' : S \longrightarrow D_\mathfrak{m}(R)$ such that the diagram

$$\begin{array}{ccc} R & \xrightarrow{\subseteq} & S \\ & {\scriptstyle\eta_R}\searrow & \cong\big\downarrow{\scriptstyle\psi'} \\ & & D_\mathfrak{m}(R) \end{array}$$

(in which $\eta_R : R \longrightarrow D_\mathfrak{m}(R)$ is the natural map) commutes. It therefore follows from [3, 2.2.5] that $H^1_\mathfrak{m}(R) \neq 0$, so that $\mathfrak{m} \in \operatorname{Psupp}^1(R)$. It also follows from the fact that S/R is an $\mathfrak{m}$-torsion R-module that $H^2_\mathfrak{m}(R) \cong H^2_\mathfrak{m}(S)$: see [3, 2.1.7(i)]. As

$$H^2_\mathfrak{m}(S)_\mathfrak{r} \cong H^2_{\mathfrak{m}S_\mathfrak{r}}(S_\mathfrak{r}) = H^2_{\mathfrak{r}S_\mathfrak{r}}(S_\mathfrak{r}) = H^{\dim S_\mathfrak{r}}_{\mathfrak{r}S_\mathfrak{r}}(S_\mathfrak{r}) \neq 0,$$

it follows that $H^2_\mathfrak{m}(R) \neq 0$ and $\mathfrak{m} \in \operatorname{Psupp}^2(R)$.

We can now combine our various results to conclude as follows:

$$\operatorname{Psupp}^3(R) = \{0\}\dot\cup V\dot\cup\{\mathfrak{m}\}; \quad \operatorname{Psupp}^2(R) = U\dot\cup\{\mathfrak{m}\};$$
$$\operatorname{Psupp}^1(R) = \{\mathfrak{m}\}; \quad \operatorname{Psupp}^0(R) = \emptyset.$$

As $U \neq \emptyset$, we see that $\operatorname{Psupp}^3(R)$ is not closed under specialization. Also, as U contains infinitely many primes of height 1, $\operatorname{Psupp}^2(R)$ is not closed in the Zariski topology. □

In view of Lemma 2.2, the local ring in Example 3.2 is not catenary. The examples in this section raise the following question: does there exist a catenary local ring all of whose formal fibres are Cohen–Macaulay, but which is not *universally* catenary and has a non-closed pseudo-support? Although some examples of local domains, with Cohen–Macaulay formal fibres, which are catenary but not universally catenary are known (see, for example, [6, Example 28]), we have not been able to find one with a non-closed pseudo-support.

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