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TWISTOR THEORY OF MANIFOLDS WITH GRASSMANNIAN STRUCTURES

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Abstract. As a generalization of the conformal structure of type $(2, 2)$, we study Grassmannian structures of type (n, m) for $n, m \geq 2$. We develop their twistor theory by considering the complete integrability of the associated null distributions. The integrability corresponds to global solutions of the geometric structures.

A Grassmannian structure of type (n, m) on a manifold M is, by definition, an isomorphism from the tangent bundle TM of M to the tensor product $V \otimes W$ of two vector bundles V and W with rank n and m over M respectively. Because of the tensor product structure, we have two null plane bundles with fibres $P^{m-1}(\mathbb{R})$ and $P^{n-1}(\mathbb{R})$ over M. The tautological distribution is defined on each two bundles by a connection. We relate the integrability condition to the half flatness of the Grassmannian structures. Tanaka's normal Cartan connections are fully used and the Spencer cohomology groups of graded Lie algebras play a fundamental role.

Besides the integrability conditions corrsponding to the twistor theory, the lifting theorems and the reduction theorems are derived. We also study twistor diagrams under Weyl connections.

Introduction

An aspect of the twistor theory of R. Penrose is to know the relations and correspondences between geometric structures defined by a double fibration

$$
\begin{array}{ccc}
 & F & \\
\swarrow & & \searrow \\
P & & M
\end{array}
$$

for three spaces F, P and M.

As a flat model, we take the spaces F, P and M to be the homogeneous spaces of a fixed Lie group G. The group G is considered as the automorphism group of each suitably defined geometric structure. Then the maps

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in the double fibration have geometric meanings. As a curved analogue, we take the spaces F, P and M to be the manifolds with corresponding geometric structures. The twistor correspondence between P and M is given by choosing the moduli space of the orbits of the distribution which is naturally defined from the geometric structure.

Penrose himself treated the case where M is the Grassmann manifold of 2-planes in $\mathbb{C}^4$ and $P = P^3(\mathbb{C})$. Between curved manifolds, the correspondence is given when M is half conformally flat ([W-W]).

As a real version of these structures, we can consider the real 4-dimensional space-time $M = S^2 \times S^2$ of $(2,2)$-type metric and $P = P^3(\mathbb{R})$. We put on $\mathbb{C}^4$ a Hermitian inner product with type $(2,2)$. By the restriction to the null spaces of the Hermitian inner product, we get $M = S^3 \times S^1$ and $P = S^3 \times S^2$. The corresponding geometric structures are 4-dimensional Lorentzian geometry and 5-dimensional CR geometry with Levi signature $(1,1)$. We have studied the twistor theory for real space-times of $(3,1)$-type metric, i.e., Lorentzian metric in [Ma-Sa]. For real space-times of $(2,2)$-type metric, i.e., neutral metric, see [K-M].

In this paper, as a generalization of conformal structures of type $(2,2)$, we study Grassmannian structures of type (n,m). The case where $m = 2$ is more interesting, since we can define the notion of half flatness meaningfully and the geometric structure of the twistor partner is a different projective structure from a Grassmannian structure. By N. Tanaka's theory [T1], the normal Cartan connection is uniquely defined on some principal bundle Q associated with G over a manifold M with a Grassmannian structure of type (n,m). By this connection, we define the notion of half flatness for the Grassmannian structures of type (n,m). Furthermore, it is important to consider the harmonic part HK of the curvature function K of the normal Cartan connection, which is the fundamental invariant and is generated by the nonzero generators in the 2-dimensional generalized Spencer cohomology.

A Grassmannian structure of type (n,m) on M is defined by an isomorphism from the tangent bundle TM of M to the tensor product $V \otimes W$ of two vector bundles V and W with rank n and m over M respectively. Considering a set of all the null n-planes with forms $\{V_x \otimes w \mid w \in W_x\}$ in T_xM at each point $x \in M$, we have a null n-plane bundle F_L with fibre $P^{m-1}(\mathbb{R})$ and the projection $\varpi_L : F_L \to M$ over M. Similarly, considering a set of all the null m-planes with forms $\{v \otimes W_x \mid v \in V_x\}$ in T_xM, we have a null m-plane bundle F_R with fibre $P^{n-1}(\mathbb{R})$ and the projection $\varpi_R : F_R \to M$ over M. By the normal Cartan connection, the tautological distribution D_L of null n-planes on F_L over M and D_R of null m-planes on F_R over M are defined respectively. We have the following result.

THEOREM 5.1, 6.1. *Let M be a manifold with a Grassmannian structure of type (n, m) and equipped with the normal Cartan connection ω. Then*

(1) *the tautological distribution D_L on the null n-plane bundle F_L over M is completely integrable if and only if the Grassmannian structure on M is*

1. *if $n, m \geq 3$, right-half torsion-free, i.e., $HK^0_R = 0$,*
2. *if $n \geq 3$, $m = 2$, torsion-free, i.e., $K^0 = HK^0 = 0$,*
3. *if $n = 2$, $m = 2$, right-half Grassmannian flat, i.e., $HK^1_R = 0$,*

(2) *the tautological distribution D_R on the null m-plane bundle F_R over M is completely integrable if and only if the Grassmannian structure on M is*

1. *if $n, m \geq 3$, left-half torsion-free, i.e., $HK^0_L = 0$,*
2. *if $n \geq 3$, $m = 2$, left-half Grassmannian flat, i.e., $HK^1 = 0$,*
3. *if $n = 2$, $m = 2$, left-half Grassmannian flat, i.e., $HK^1_L = 0$.*

The result has some overlap with Chapter 7 of recently published book by Akivis-Goldberg ([A-G1, Theorems 7.4.4, 7.4.5 and 7.4.6], cf. [A-G2], [A-G3], [A-G4]). We have obtained the results independently in the framework of the twistor theory using Tanaka's Cartan connection [T1].

Our result gives a construction of the global completely integrable distribution. This corresponds to a construction of a global solution more than the construction of a local solution of the differential equation which defines the null distribution.

The null n-plane bundle F_L and the null m-plane bundle F_R over M have also geometric structures, which we call co-Grassmannian structures. A co-Grassmannian structure of type (k, l) on a manifold R is defined by a pair (E, F) consisting of transversal, completely integable distributions of dimensions k and l on the tangent bundle TR of R such that (i) $TR = D + [D, D]$ and (ii) $\operatorname{rank} TR/D = \operatorname{rank} E \cdot \operatorname{rank} F$ $(= kl)$ for $D = E \oplus F$. Tanaka settled the equivalence problem of the system of ordinary differential equations of second order [T3]. On that occation, he defined a pseudo-projective systems in the sense of Tanaka ([T2]). We consider a co-Grassmannian structure of type (k, l) as the extension. When

$l = 2$, it coincides with a Tanaka's pseudo-projective system. Note that a Grassmannian structure on M is defined by an isomorphism of TM to a tensor product of two vector bundles. By the Tanaka theory, we have the normal Cartan connections on F_L and F_R. Then we have the following lifting theorem.

Theorem 5.2, 6.2. (Lifting Theorem) *Let M be a manifold with a Grassmannian structure of type (n, m) and equipped with the normal Cartan connection ω.*

(1) *Suppose that an n-dimensional tautological distribution D_L of null n-planes on the null n-plane bundle F_L over M is completely integrable. Then a pair $(D_L, E_L = \mathrm{Ker}(\varpi_L)_*)$ defines a co-Grassmannian structure of type $(n, m-1)$ on F_L. Moreover the normal Cartan connection (Q, ω) of a Grassmannian structure of type (n, m) induces the normal Cartan connection $(Q, \overline{\omega})$ of the co-Grassmannian structure of type $(n, m-1)$ on F_L.*

(2) *Suppose that an m-dimensional tautological distribution D_R of null m-planes on the null m-plane bundle F_R over M is completely integrable. Then a pair $(E_R = \mathrm{Ker}(\varpi_R)_*, D_R)$ defines a co-Grassmannian structure of type $(n-1, m)$ on F_R. Moreover the normal Cartan connection (Q, ω) of a Grassmannian structure of type (n, m) induces the normal Cartan connection $(Q, \overline{\omega})$ of the co-Grassmannian structure of type $(n-1, m)$ on F_R.*

A co-Grassmannian structure induces a Grassmannian structure on one of two leaf spaces defined by the structure under the vanishing of some part of the harmonic part HK of the curvature function K. We have the following reduction theorem.

Theorem 7.1. (Reduction Theorem) *Let F be a manifold with a co-Grassmannian structure of type (k, l) by a pair (D_2, D_1) and equipped with the normal Cartan connection (Q, ω). Put $M_1 = F/D_1, M_2 = F/D_2$ with the canonical projections $\nu : F \to M_1$, $\mu : F \to M_2$. Then*

1. *if $l \geq 3$, we have $HK = (HK^0)_1 + (HK^0)_2$, and*
   (i) *$(HK^0)_1 = 0 \Longleftrightarrow (Q, \omega)$ is reduced to (Q_1, ω_1) over M_1,*
   (ii) *$(HK^0)_2 = 0 \Longleftrightarrow (Q, \omega)$ is reduced to (Q_2, ω_2) over M_2,*
2. *if $l = 2$,*
   (a) *for $k \geq 3$, we have $HK = (HK^0)_1 + (HK^0)_2 + (HK^0)_3$, and*
      (i) *$(HK^0)_1 = (HK^0)_2 = 0 \Longleftrightarrow (Q, \omega)$ is reduced to (Q_1, ω_1) on M_1,*
      (ii) *$(HK^0)_3 = 0 \Longleftrightarrow (Q, \omega)$ is reduced to (Q_2, ω_2) on M_2,*

(b) *for* $k = 2$, *we have* $HK = (HK^0)_1 + (HK^0)_2 + (HK^0)_3 + (HK^0)_4$, *and*

(i) $(HK^0)_1 = (HK^0)_2 = 0 \Longleftrightarrow (Q, \omega)$ *is reduced to* (Q_1, ω_1) *on* M_1,

(ii) $(HK^0)_3 = (HK^0)_4 = 0 \Longleftrightarrow (Q, \omega)$ *is reduced to* (Q_2, ω_2) *on* M_2,

3. *if* $l = 1$,

(a) *for* $k \geq 3$, *we have* $HK = HK^1 + HK^2$, *and*

(i) $HK^1 = 0 \Longleftrightarrow (Q, \omega)$ *is reduced to* (Q_1, ω_1) *on* M_1,

(ii) $HK^2 = 0 \Longleftrightarrow (Q, \omega)$ *is reduced to* (Q_2, ω_2) *on* M_2,

(b) *for* $k = 2$, *we have* $HK = HK^0 + HK^1 + HK^2$, *and*

(i) $HK^1 = 0 \Longleftrightarrow (Q, \omega)$ *is reduced to* (Q_1, ω_1) *on* M_1,

(ii) $HK^2 = HK^0 = 0 \Longleftrightarrow (Q, \omega)$ *is reduced to* (Q_2, ω_2) *on* M_2.

Here, (Q_1, ω_1) on M_1 and (Q_2, ω_2) on M_2 have Grassmannian structures, especially in 3 (a) (ii), (b) (ii) (Q_2, ω_2) has a projective structure.

We show that the normal Cartan connection on F_L (resp. F_R) over M_1 (resp. M_2) is induced from a Grassmannian structure of the moduli space M_2 (resp. M_1) of orbits of the distribution D_L (resp. D_R) only if the Grassmannian structure on M_1 (resp. M_2) is flat. Indeed we have the following twistor theorem.

THEOREM 7.2, 7.3. (Twistor Theorem)

1. *Let* M_1 *be a manifold with a right-half torsion-free Grassmannian structure of type* (n, m). *Then, if the structure on* M_1 *induces a Grassmannian structure of type* $(n+1, m-1)$ *on* $M_2 = F_L/D_L$, *the Grassmannian structure of type* (n, m) *on* M_1 *is flat.*

2. *Let* M_2 *be a manifold with a left-half torsion-free Grassmannian structure of type* $(n+1, m-1)$. *Then, if the structure on* M_2 *induces a Grassmannian structure of type* (n, m) *on* $M_1 = F_R/D_R$, *the Grassmannian structure of type* $(n+1, m-1)$ *on* M_2 *is flat.*

In particular, assume that $m = 2$.

1. *Let* M_1 *be a* $2n$*-dimensional manifold with a right-half Grassmannian flat Grassmannian structure of type* $(n, 2)$. *Then, if the structure on* M_1 *induces a projective structure on* M_2, *the Grassmannian structure of type* $(n, 2)$ *on* M_1 *is flat.*

2. *Let* M_2 *be an* $(n+1)$*-dimensional manifold with a projective structure. Then, if the structure on* M_2 *induces a Grassmannian structure of type* $(n, 2)$ *on the orbit space* M_1 *of the geodesic flow, the projective structure on* M_2 *is flat.*

Here, in 1 F_L denotes the null n-plane bundle on M_1, and in 2 F_R denotes the null $(m-1)$-plane bundle on M_2.

By the theorem above, we know that the flat models play important roles in the twistor diagrams together with the geometric structures. Now, we consider Weyl connections associated with conformal structures on a linear frame bundle in place of a normal Cartan connection on a frame bundle of second order. We study geometric structures related to the geodesic flows of the Weyl connections. Then, even for some non-flat spaces the twistor diagrams work well. We have the following.

THEOREM 8.1, 8.2. *Let P be an $(n+1)$-dimensional manifold with a Weyl structure with constant curvature. Then the structure on P induces a right-half Grassmannian flat Grassmannian structure of type $(n,2)$ on the orbit space M of the geodesic flow.*

In particular, assume that $n=2$. Let P be a 3-dimensional manifold with an Einstein-Weyl structure. Then the structure on P induces a self-dual conformal Hermitian structure of type $(2,2)$ on the orbit space M of the geodesic flow.

This paper is organized as follows:

In Section 1, we define a Grassmannian structure of type (n,m) and consider its structure group as a geometric structure. Typical examples are Grassmann manifolds, which are the flat models. We give some non-flat examples too. A topological obstruction to the existence of a Grassmannian structure of type $(n,2)$ is described. As a consequence, the sphere S^{2n} and the quaternionic projective space $P^m(\mathbb{H})$ $(n=2m)$ admit no Grassmannian structures of type $(n,2)$. In the case $n=2$, we remark that the notion of a Grassmannian structure of type $(2,2)$ is equivalent to that of a conformal structure of type $(2,2)$. (See also [A-G1, Table 7.4.1] and [A-G3], [A-G4].)

In Section 2, we regard a Grassmannian structure of type (n,m) as a geometric structure related to a simple graded Lie algebra of first kind. We apply the Tanaka theory which induces the existence of a unique normal Cartan connection. As a condition of the curvature, we give the definition of half flatness due to the decomposition of two invariant subspaces of the space of 2-forms Λ^2.

In Section 3, we review the Tanaka theory including the generalized Spencer cohomology, the harmonic theory and the existence of a normal Cartan connection. We explicitly write down the normal Cartan connection for a Grassmannian structure of type (n,m). We indicate the nonzero generators as $\mathfrak{g}_0$-module in the 2-dimensional generalized Spencer cohomology H^2, which make use of the fundamental invariant HK of the curvature function K.

In Section 4, we define a co-Grassmannian structure of type (k,l). Typical examples are some generalized flag manifolds, which are the flat models. By considering a graded Lie algebra of second kind of type (k,l) co-Grassmann (abbreviated to a type (k,l) CGR), we apply the Tanaka theory which induces the existence of the normal Cartan connection. We indicate the nonzero generators in H^2 associated with a graded Lie algebra of type (k,l) CGR.

In Section 5, we consider the null n-plane bundle F_L to be the set of all null n-planes for a manifold M with a Grassmannian structure of type (n,m). The space F_L is a fibre bundle with fibre $P^{m-1}(\mathbb{R})$ over M. We define a tautological n-dimensional distribution D_L on F_L over M using the normal Cartan connection. We prove the theorem that the distribution D_L on F_L over M is completely integrable if and only if the Grassmannian structure on M is right-half flat. We give a non-flat half-flat example. Next, under complete integrability of D_L on F_L, we mention that a co-Grassmannian structure of type $(n,m-1)$ is defined on F_L.

In Section 6, in the same way as in Section 5, we consider the null m-plane bundle F_R to be the set of all null m-planes for M. The space F_R is a fibre bundle with fibre $P^{n-1}(\mathbb{R})$ over M. We define the tautological m-dimensional distribution D_R on F_R over M, and give the condition for D_R to be completely integrable, that is, left-half flatness. Next, under complete integrability of D_R on F_R, we mention that a co-Grassmannian structure of type $(n-1,m)$ on F_R is defined on F_R. We describe a projective structure that is a Grassmannian structure of type $(n,1)$.

In Section 7, we interpret the results above under the diagram of the twistor theory. We explain the twistor diagrams of Grassmannian structures in terms of the Dynkin diagrams. We show the reduction theorem from co-Grassmannian structures down to Grassmannian structures under the vanishing of some part of HK. Furthermore we show that the flat models play important roles in the twistor diagrams together with the geometric structures.

In Section 8, after preparing the notions of Einstein-Weyl structure, Lie contact structure, geodesic flow and Jacobi field, we study twistor diagrams under Weyl connections. We show that the Weyl connections with constant curvature work well in the twistor diagrams.

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§1. Definitions and examples of Grassmannian structures

1.1. Definitions

We define Grassmannian structures of type (n, m).

Let M be an l-dimensional real manifold. In this paper we study only the real category and not the complex category.

A Grassmannian structure of type (n, m) on M is defined by an isomorphism σ from the tangent bundle TM of M to the tensor product $V \otimes W$ of two vector bundles V and W with rank n and m $(n, m \geq 2)$ over M respectively:

$$\sigma : TM \xrightarrow{\cong} V \otimes W.$$

Note that there are various names and different but essentially equivalent definitions. (e.g., almost Grassmannian structure [Mi], [A-G1], Grassmannian spinor structure [Ma], tensor product structure [Ha], [I], paraconformal structure [B-E], generalized conformal structure [G].) If M has a Grassmannian structures of type (n, m), the dimension l of M is equal to mn.

Consider the mn-dimensional vector space $\mathbb{R}^{mn} = \mathbb{R}^n \otimes \mathbb{R}^m$. The group $GL(n, \mathbb{R})$ acts on $\mathbb{R}^n$ in the usual way and $GL(m, \mathbb{R})$ acts by inverse from the right on $\mathbb{R}^m$. The combined action $gr(n, m) \subset GL(mn, \mathbb{R})$ is the $(n^2 +$

m^2-1)-dimensional tensor product linear Lie group $GL(n,\mathbb{R})\otimes GL(m,\mathbb{R})$. We have the natural projection

$$\rho : GL(m,\mathbb{R})\times GL(n,\mathbb{R}) \longrightarrow gr(n,m) = GL(n,\mathbb{R})\otimes GL(m,\mathbb{R})$$

which defines a fibre bundle with fibre $\mathbb{R}^*$ by the scalar multiplication.

Let M be a manifold with a Grassmannian structure of type (n,m). Then the structure group of TM is reduced to $gr(n,m)$.

Let $S(GL(m,\mathbb{R})\times GL(n,\mathbb{R}))$ be the subgroup of $SL(m+n,\mathbb{R})$ consisting of matrices of the form

$$\begin{pmatrix} A & O \\ O & B \end{pmatrix}, \quad A\in GL(m,\mathbb{R}),\ B\in GL(n,\mathbb{R}).$$

Restricting the homomorphism ρ to the subgroup $S(GL(m,\mathbb{R})\times GL(n,\mathbb{R}))$, we have the homomorphism

$$\begin{array}{ccc} h : S(GL(m,\mathbb{R})\times GL(n,\mathbb{R})) & \longrightarrow & gr(n,m) = GL(n,\mathbb{R})\otimes GL(m,\mathbb{R}) \\ \cap & & \cap \\ SL(m+n,\mathbb{R}) & & GL(mn,\mathbb{R}). \end{array}$$

If we restrict the above $GL(m,\mathbb{R})$ and $GL(n,\mathbb{R})$ to $GL_+(m,\mathbb{R}) = \{A\in GL(m,\mathbb{R}) \mid \det(A)>0\}$ and $GL_+(n,\mathbb{R}) = \{B\in GL(n,\mathbb{R}) \mid \det(B)>0\}$ respectively, the homomorphism h is surjective. We define a Lie group $GR(n,m)$ by the image of h.

Identify $\mathbb{R}^{mn} = \mathbb{R}^n\otimes\mathbb{R}^m$ with the set of matrices of the form

$$\begin{pmatrix} I_m & O \\ X & I_n \end{pmatrix}, \quad X\in \mathrm{Mat}(n\times m,\mathbb{R}).$$

Then, for $g\in S(GL(m,\mathbb{R})\times GL(n,\mathbb{R}))$, $h(g)$ is expressed by the adjoint action of g:

$$h(g)(x) = Ad(g)(x) \quad \text{for } x\in\mathbb{R}^{mn}.$$

Therefore $GR(n,m)$ is the image of the linear isotropy representation of $S(GL(m,\mathbb{R})\times GL(n,\mathbb{R}))$.

A spin Grassmannian structure of type (n,m) on a manifold M is a lifting for h of the structure group $GR(n,m)$ of TM to $S(GL(m,\mathbb{R})\times GL(n,\mathbb{R}))$. From now on, we consider manifolds with spin Grassmannian structure of type (n,m).

Let M be a manifold with a Grassmannian structure of type (n, m). Put

$$SGR(n, m) = GR(n, m) \cap SL(mn, \mathbb{R}).$$

Suppose that the structure group of TM is reduced to $SGR(n, m)$. The restriction of h on $SL(m, \mathbb{R}) \times SL(n, \mathbb{R})$ is a surjective covering

$$\begin{array}{ccc} h : SL(m, \mathbb{R}) \times SL(n, \mathbb{R}) & \longrightarrow & SGR(n, m) \\ \cap & & \cap \\ S(GL(m, \mathbb{R}) \times GL(n, \mathbb{R})) & & GR(n, m). \end{array}$$

Then M is called a manifold with a scaled Grassmannian structure of type (n, m).

1.2. Typical examples

Typical examples of manifolds with Grassmannian structures are Grassmann manifolds.

Let $G_{m,n+m}$ be a Grassmann manifold consisting of all m-dimensional subspaces in the $(n+m)$-dimensional real vector space $\mathbb{R}^{n+m}$. Then $G_{m,n+m}$ is of dimension mn.

The group $G = SL(m + n, \mathbb{R})$ acts transitively on $G_{m,n+m}$. Let G' be the isotropy group at the base point. Then we have

$$G_{m,n+m} \cong G/G'.$$

Let $U_{m,n+m}$ be the universal bundle over $G_{m,n+m}$. Since the fibres are m-dimensional subspaces in $\mathbb{R}^{n+m}$, there is a natural bundle map from $U_{m,n+m}$ into the trivial bundle $G_{m,n+m} \times \mathbb{R}^{n+m}$. Denoting by V the quotient bundle of $U_{m,n+m}$ in $G_{m,n+m} \times \mathbb{R}^{n+m}$, we obtain the following exact sequence

$$0 \longrightarrow U_{m,n+m} \longrightarrow G_{m,n+m} \times \mathbb{R}^{n+m} \longrightarrow V \longrightarrow 0.$$

Let $TG_{m,n+m}$ be the tangent bundle of $G_{m,n+m}$. Then we have

$$\begin{aligned} TG_{m,n+m} &\cong \mathrm{Hom}(U_{m,n+m}, V) \\ &\cong V \otimes U^*_{m,n+m}. \end{aligned}$$

Putting $W = U^*_{m,n+m}$, we have

$$TG_{m,n+m} \cong V \otimes W.$$

Therefore the Grassmann manifold $G_{m,n+m}$ has a Grassmannian structure of type (n, m).

1.3. Nontrivial examples

We describe two kinds of nontrivial examples.

(1) Let M be an n-dimensional differentiable manifold and let TM be the tangent bundle of M. Denote by π the natural projection of TM onto M. Taking a linear connection on M, we can decompose the tangent space T_vTM at each point v of TM into the n-dimensional horizontal space $H_v = T^H_{\pi(v)}M \cong T_{\pi(v)}M$ and the n-dimensional vertical space $V_v = T^V_{\pi(v)}M \cong T_{\pi(v)}M$. Then the tangent bundle TTM of TM is decomposed as follows:

$$\begin{aligned} TTM &= H \oplus V \\ &\cong \pi^*TM \oplus \pi^*TM \\ &\cong \pi^*TM \otimes 2_{TM}, \end{aligned}$$

where π^*TM is the induced vector bundle of TM by $\pi : TM \to M$, and 2_{TM} is the trivial bundle with rank 2 over TM. Therefore the $2n$-dimensional manifold TM has a Grassmannian structure of type $(n, 2)$.

Let F^rM be the r-frame bundle of M. We mean by an r-frame a set of linearly independent r tangent vectors at a point of M. In the case $r = 1$, F^1M is nothing but the tangent bundle TM of M. In the case $r = n$, F^nM is nothing but the linear frame bundle FM of M. Denote by $\overline{\pi}$ the natural projection of F^rM onto M. Take a linear connection on M. An r-frame ξ is regarded as an into-isomorphism of $\mathbb{R}^r$ to $T_{\overline{\pi}(\xi)}M$. We denote by $\{e_1, e_2, \dots, e_r\}$ the basis of $\mathbb{R}^r$. Then we can define an isomorphism of the tangent space $T_\xi F^rM$ at ξ to $r+1$ direct sum $T^H_{\overline{\pi}(\xi)}M \oplus T^V_{\overline{\pi}(\xi)}M \oplus T^V_{\overline{\pi}(\xi)}M \oplus \cdots \oplus T^V_{\overline{\pi}(\xi)}M$ as follows:

$$X \longmapsto \big((\overline{\pi}_*X)^H, (\xi(e_1))^V, (\xi(e_2))^V, \dots, (\xi(e_r))^V\big).$$

Therefore

$$\begin{aligned} TF^rM &\cong \overline{\pi}^*TM \oplus \overline{\pi}^*TM \oplus \overline{\pi}^*TM \oplus \cdots \oplus \overline{\pi}^*TM \\ &\cong \overline{\pi}^*TM \otimes (r+1)_{F^rM}, \end{aligned}$$

where $\overline{\pi}^*TM$ is the induced vector bundle of TM by $\overline{\pi} : F^rM \to M$, and $(r+1)_{F^rM}$ is the trivial bundle with rank $r+1$ over F^rM. Thus the $n(r+1)$-dimensional manifold F^rM has a Grassmannian structure of type $(n, r+1)$.

(2) Let h be a Hermitian inner product of type $(m+1, m)$ on the complex $(2m+1)$-dimensional vector space $\mathbb{C}^{m+1,m}$. Put $n = 2m$. It follows that the quadric hypersurface N defined by $h(z, z) = 1$ is the (real) $(2n+1)$-dimensional pseudo-hyperbolic space $H^{n+1,n}$ of type $(n+1, n)$ with negative constant curvature. On $H^{n+1,n}$, $U(m+1, m)$ acts transitively and $S^1 = \{e^{i\theta}\}$ acts freely by $z \mapsto e^{i\theta} z$. The base space M of the principal bundle $H^{n+1,n}$ with structure group S^1 is nothing but the complex pseudo-hyperbolic space $H^{m,m}(\mathbb{C})$ of type (m, m):

$$\begin{tikzcd} \mathbb{C}^{m+1,m} \supset H^{n+1,n} \arrow[d, "\pi"'] & S^1 \arrow[l] \\ H^{m,m}(\mathbb{C}) & \end{tikzcd} .$$

The group $U(m+1, m)$ acts transitively on $H^{m,m}(\mathbb{C})$ and the space $H^{m,m}(\mathbb{C})$ has the form

$$H^{m,m}(\mathbb{C}) \cong U(m+1, m)/U(1) \times U(m, m)$$

as a symmetric space.

The Lie algebra $\mathfrak{g} = \mathfrak{u}(m+1, m)$ has the canonical decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ $([\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m},\ [\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h})$ as follows:

$$\begin{aligned} \mathfrak{g} &= \mathfrak{u}(m+1, m), \\ \mathfrak{h} &= \mathfrak{u}(1) + \mathfrak{u}(m, m) \\ &= \left\{ \begin{pmatrix} i\lambda & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\} + \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & A & B \\ 0 & {}^t\overline{B} & C \end{pmatrix} \right\} \\ &\qquad (\lambda \in \mathbb{R}; A, B, C \in \mathfrak{gl}(m, \mathbb{C}), A = -{}^t\overline{A}, C = -{}^t\overline{C}) \\ \mathfrak{m} &= \left\{ \begin{pmatrix} 0 & -{}^t\overline{\mathbf{x}} & {}^t\overline{\mathbf{y}} \\ \mathbf{x} & 0 & 0 \\ \mathbf{y} & 0 & 0 \end{pmatrix} \right\} \quad (\mathbf{x}, \mathbf{y} \in \mathbb{C}^m). \end{aligned}$$

The adjoint action of $H = U(1) \times U(m, m)$ on $\mathfrak{m}$ is the form

$$Ad \begin{pmatrix} e^{i\theta} & 0 & 0 \\ 0 & A & B \\ 0 & C & D \end{pmatrix} \begin{pmatrix} 0 & -{}^t\overline{\mathbf{x}} & {}^t\overline{\mathbf{y}} \\ \mathbf{x} & 0 & 0 \\ \mathbf{y} & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -e^{i\theta t}\overline{\mathbf{x}}A' + e^{i\theta t}\overline{\mathbf{y}}C' & -e^{i\theta t}\overline{\mathbf{x}}B' + e^{i\theta t}\overline{\mathbf{y}}D' \\ A\mathbf{x}e^{-i\theta} + B\mathbf{y}e^{-i\theta} & 0 & 0 \\ C\mathbf{x}e^{-i\theta} + D\mathbf{y}e^{-i\theta} & 0 & 0 \end{pmatrix},$$

$$\left(\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in U(m,m), \ \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1}.\right)$$

We identify $\mathfrak{m} \left(\ni \begin{pmatrix} 0 & -{}^t\overline{\mathbf{x}} & {}^t\overline{\mathbf{y}} \\ \mathbf{x} & 0 & 0 \\ \mathbf{y} & 0 & 0 \end{pmatrix} \right)$ with $\mathbb{C}^n \left(\ni \binom{\mathbf{x}}{\mathbf{y}} \right)$. Therefore the action of

$$\left(e^{i\theta}, \begin{pmatrix} A & B \\ C & D \end{pmatrix} \right) \in U(1) \times U(m,m)$$

on $\mathbb{C}^n = \mathbb{C}^{m,m} = \mathbb{C}^{m,m} \otimes \mathbb{C}^1$ is given as follows:

$$\left(e^{i\theta}, \begin{pmatrix} A & B \\ C & D \end{pmatrix} \right) \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} e^{-i\theta}.$$

Since $\binom{\mathbf{x}}{\mathbf{y}} \in \mathbb{C}^{m,m} \otimes \mathbb{C}^1$ is regarded as $\begin{pmatrix} x & x' \\ y & y' \end{pmatrix} \in \mathbb{R}^n \otimes \mathbb{R}^2$ ($\mathbf{x}_i = x_i + \sqrt{-1}\, x_i'$, $\mathbf{y}_i = y_i + \sqrt{-1}\, y_i' \in \mathbb{C}$; $x, x', y, y' \in \mathbb{R}^n$) and $U(1) \cong SO(2)$,

$$U(m,m) \otimes U(1) \ni \begin{pmatrix} A & B \\ C & D \end{pmatrix} \otimes \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

acts on $\mathbb{R}^n \otimes \mathbb{R}^2$. Therefore $H^{m,m}(\mathbb{C})$ has a Grassmannian structure of type $(n,2)$. Remark that $H^{m,m}(\mathbb{C})$ also has a canonical pseudo-Riemannian structure of type (n,n). (See [K-M] for $H^{1,1}(\mathbb{C})$.)

1.4. Topological obstructions

There are topological obstructions for admitting a Grassmannian structure of type $(n,2)$. If M has a Grassmannian structure of type $(n,2)$, then we have

$$TM \cong V \otimes W,$$

where V and W are vector bundles with rank n (≥ 2) and 2 over M respectively.

Now assume that $H^2(M;\mathbb{Z}) = 0$. Then any vector bundle with rank 2 over M is trivial. Therefore it follows that

$$TM \cong V \oplus V.$$

Let M be the $2n$-dimensional sphere S^{2n}. Since the homotopy set $[S^{2n}; B_{SO(n)}]$ ($\cong \pi_{2n-1}(SO(n))$) from S^{2n} to the classifying space $B_{SO(n)}$ of

$SO(n)$ is 0 mod torsion, the vector bundle V with rank n over S^{2n} is trivial. So $V \oplus V$ is trivial. By the way, the Euler number of the tangent bundle TS^{2n} of S^{2n} is equal to 2. This is a contradiction. Therefore S^{2n} admits no Grassmannian structures of type $(n, 2)$.

Let M be the quaternionic projective space $P^m(\mathbb{H})$ $(n = 2m)$. The total Pontryagin classes $\{p_i\}$ are given by

$$\frac{(1+u)^{2m+2}}{1+4u} = 1 + p_1 u + p_2 u^2 + \cdots,$$

where u is the generator of $H^4(P^m(\mathbb{H}); \mathbb{Z}) \cong \mathbb{Z}$ (see e.g. [Mi-St]). It follows that $p_1 = 2(m-1)$, $p_2 = 2m^2 - 5m + 9$. For example, in the case of $P^2(\mathbb{H})$, $p_1 = 2$, $p_2 = 7$ hold. On the other hand, for the vector bundle $V \oplus V$ over M, up to mod torsion,

$$\begin{aligned} p_1(V \oplus V) &= p_1(V) + p_1(V), \\ p_2(V \oplus V) &= p_1(V) \cdot p_1(V). \end{aligned}$$

Putting $p_1(V) = x$, we have

$$2x = 2(m-1), \quad x^2 = 2m^2 - 5m + 9.$$

This is a contradiction. Therefore $P^m(\mathbb{H})$ admits no Grassmannian structures of type $(n, 2)$.

Let M be the Cayley projective space $P^2(Ca)$. Then it is known that $p_2 = 6$, $p_4 = 39$. In a similar way to $P^m(\mathbb{H})$, $P^2(Ca)$ admits no Grassmannian structures of type $(n, 2)$.

If we let M be the complex projective space $P^n(\mathbb{C})$, $H^2(P^n(\mathbb{C}); \mathbb{Z}) \cong \mathbb{Z}$ holds. We do not use the argument above. But if M is $P^2(\mathbb{C})$, $P^2(\mathbb{C})$ admits no Grassmannian structures of type $(2, 2)$. In fact, according to the following 1.5, the notion of Grassmannian structures of type $(2, 2)$ is equivalent to the notion of conformal structures of type $(2, 2)$. By the way, $P^2(\mathbb{C})$ admits no conformal structures of type $(2, 2)$ (cf. [K-M]). Therefore $P^2(\mathbb{C})$ admits no Grassmannian structures of type $(2, 2)$.

1.5. Grassmannian structures of type $(2, 2)$

Let us see that in 4-dimensinal case the notion of Grassmannian structures of type $(2, 2)$ is equivalent to the notion of conformal structures of type $(2, 2)$. Let M be a 4-dimensional manifold and let $x \in M$. Denote by U the tangent space $T_x M$ at x.

Now suppose that M has a Grassmannian structure of type $(2,2)$. Then U is represented by $U = V \otimes W$, where V and W are 2-dimensional vector spaces. As V and W are 2-dimensional, there exist canonical (conformal) symplectic forms ω_V and ω_W respectively. We take symplectic basis $\{e_1, e_2\}$ of V and $\{f_1, f_2\}$ such that $\omega_V(e_1, e_2) = 1$ and $\omega_W(f_1, f_2) = 1$ respectively. Note that $\{e_i \otimes f_j\ (1 \le i, j \le 2)\}$ is a null basis of U and $\Pi_1 = \mathrm{span}\{e_1 \otimes f_1, e_2 \otimes f_1\}$, $\Pi_2 = \mathrm{span}\{e_1 \otimes f_2, e_2 \otimes f_2\}$ are null 2-planes. (See 5.1.)

A (conformal) inner product $(\,\cdot\,,\,\cdot\,)$ of type $(2,2)$ on U is defined as follows:

$$(e_i \otimes f_j, e_k \otimes f_l) = \omega_V(e_i, e_k) \cdot \omega_W(f_j, f_l).$$

Extending it to the whole U linearly, a conformal structure of type $(2,2)$ is defined on M. Note that with respect to the inner product $(\,\cdot\,,\,\cdot\,)$, $e_i \otimes f_j$ $(1 \le i, j \le 2)$ is a null vector and Π_1, Π_2 are totally null planes.

Conversely suppose that M has a conformal structure of type $(2,2)$. With respect to a (conformal) inner product $(\,\cdot\,,\,\cdot\,)$ of type $(2,2)$, take a basis $\{s_1, s_2, t_1, t_2\}$ such that

$$\begin{aligned}
&(s_i, s_i) = 1, \quad (s_1, s_2) = 0,\\
&(t_i, t_i) = -1, \quad (t_1, t_2) = 0,\\
&(s_i, t_j) = 0 \quad (1 \le i, j \le 2).
\end{aligned}$$

Then $S = \mathrm{span}\{s_1, s_2\}$ and $T = \mathrm{span}\{t_1, t_2\}$ are definite planes. For $U = S \oplus T$, we define a mapping f from $S \oplus T$ to $\mathrm{Mat}(2, \mathbb{R})$ as follows:

$$\begin{aligned}
f : S \oplus T &\longrightarrow \mathrm{Mat}(2, \mathbb{R})\\
(a_i, b_j) &\longmapsto \begin{pmatrix} a_1 + b_1 & -a_2 + b_2 \\ a_2 + b_2 & a_1 - b_1 \end{pmatrix}.
\end{aligned}$$

Then the mapping f is an isomorphism as 4-dimensional vector spaces. Furthermore we have a linear isometry from $(U, (\,\cdot\,,\,\cdot\,))$ to $(\mathrm{Mat}(2, \mathbb{R}), \det)$. There exist 2-dimensional vector spaces V, W and bases $\{e_1, e_2\}$ of V, $\{f_1, f_2\}$ of W such that

$$\begin{aligned}
\mathrm{Mat}(2, \mathbb{R}) &\xrightarrow{\cong} V \otimes W\\
\begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} &\longmapsto \sum_{i,j=1}^{2} m_{ij} e_i \otimes f_j.
\end{aligned}$$

Using a well-known two-to-one mapping

$$\phi : SL(2, \mathbb{R}) \otimes SL(2, \mathbb{R}) \longrightarrow SO_0(2, 2),$$

we have the following commutative diagram: for $g\otimes h \in SL(2,\mathbb{R})\otimes SL(2,\mathbb{R})$,

$$\begin{tikzcd} V \otimes W \arrow[r, "g\otimes h"] \arrow[d, "f^{-1}"] & V \otimes W \arrow[d, "f^{-1}"] \\ S \oplus T \arrow[r, "\phi(g\otimes h)"'] & S \oplus T. \end{tikzcd}$$

Therefore, independently of bases, a Grassmannian structure of type $(2,2)$ is canonically defined on M.

§2. Connection and curvature of Grassmannian structures

2.1. Decomposition of the space of 2-forms

Let M be a manifold with a Grassmannian structure of type (n,m). Then the tensor spaces and the tensor fields on M give rise to decompositions with respect to the structure.

Denote by Λ^2 the space of 2-forms $\Lambda^2(TM)$ or $\Lambda^2(T^*M)$. Then Λ^2 is decomposed as follows:

$$\Lambda^2 = S^2(V)\otimes\Lambda^2(W)\oplus\Lambda^2(V)\otimes S^2(W).$$

Here we identify TM with $V\otimes W$ under σ. The decomposition is invariant under the group $GR(n,m)$. Put

$$\begin{aligned} \Lambda^2_L &= S^2(V)\otimes\Lambda^2(W), \\ \Lambda^2_R &= \Lambda^2(V)\otimes S^2(W). \end{aligned}$$

The dimensions are

$$\dim\Lambda^2_L = \frac{n(n+1)m(m-1)}{4} \quad \text{and} \quad \dim\Lambda^2_R = \frac{n(n-1)m(m+1)}{4}.$$

Especially, in the case $n=m=2$, the decomposition corresponds to the decomposition of the self-dual part and the anti-self-dual part for a conformal structure of type $(4,0)$ or $(2,2)$.

Let us write down the components of Λ^2_L and Λ^2_R explicitly. Let $\{e_i\}$ $(1\le i\le n)$ be local basis vector fields on V and $\{f_j\}$ $(1\le j\le m)$ local basis vector fields on W. Then $\{e_i\otimes f_j\}$ $(1\le i\le n,\ 1\le j\le m)$ are local basis vector fields on $V\otimes W$.

A vector u belonging to $V \otimes W$ is represented by

$$u = \sum_{i,j} \alpha_{ij} e_i \otimes f_j = (\alpha_{ij})_{1 \le i \le n,\, 1 \le j \le m}.$$

Put $x_{ik} = e_i \otimes f_k$. The space Λ^2_L has a basis of forms $\frac{1}{2}(x \wedge y + y \wedge x)$, that is to say,

$$\begin{aligned} a_{i,kl} &= x_{ik} \wedge x_{il}, \quad \dim = n\frac{m(m-1)}{2}, \\ b_{ij,kl} &= \frac{1}{2}(x_{ik} \wedge x_{jl} + x_{jk} \wedge x_{il}), \quad \dim = \frac{n(n-1)}{2}\frac{m(m-1)}{2}. \end{aligned}$$

Here, for example,

$$\begin{aligned} a_{1,12} &= x_{11} \wedge x_{12} \\ &= (e_1 \otimes f_1) \wedge (e_1 \otimes f_2) = (e_1 \odot e_1) \otimes (f_1 \wedge f_2), \\ b_{12,12} &= \frac{1}{2}(x_{11} \wedge x_{22} + x_{21} \wedge x_{12}) \\ &= \frac{1}{2}\big((e_1 \otimes f_1) \wedge (e_2 \otimes f_2) + (e_2 \otimes f_1) \wedge (e_1 \otimes f_2)\big) \\ &= (e_1 \odot e_2) \otimes (f_1 \wedge f_2). \end{aligned}$$

The space Λ^2_R has a basis of forms $\frac{1}{2}(x \wedge y - y \wedge x)$, that is to say,

$$\begin{aligned} c_{ij,k} &= x_{ik} \wedge x_{jk}, \quad \dim = \frac{n(n-1)}{2}m, \\ d_{ij,kl} &= \frac{1}{2}(x_{ik} \wedge x_{jl} - x_{jk} \wedge x_{il}), \quad \dim = \frac{n(n-1)}{2}\frac{m(m-1)}{2}. \end{aligned}$$

Here, for example,

$$\begin{aligned} c_{12,1} &= x_{11} \wedge x_{21} \\ &= (e_1 \otimes f_1) \wedge (e_2 \otimes f_1) = (e_1 \wedge e_2) \otimes (f_1 \odot f_1) \\ d_{12,12} &= \frac{1}{2}(x_{11} \wedge x_{22} - x_{21} \wedge x_{12}) \\ &= \frac{1}{2}\big((e_1 \otimes f_1) \wedge (e_2 \otimes f_2) - (e_2 \otimes f_1) \wedge (e_1 \otimes f_2)\big) \\ &= (e_1 \wedge e_2) \otimes (f_1 \odot f_2). \end{aligned}$$

2.2. Prolongation of $GR(n,m)$

If we prolongate the group $GR(n,m)$ or the Lie algebra, the second prolongation of it becomes trivial. Therefore it is of finite type of order 2 (cf. [Ko], [S]). According to Kobayashi-Nagano [K-Na], it has a structure of the following graded Lie algebra of first kind:

$$\begin{aligned}
\mathfrak{g} &= \mathfrak{sl}(m+n,\mathbb{R}) \\
&= \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \\
&= \left\{ \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix} \right\} \oplus \left\{ \begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix} \right\} \oplus \left\{ \begin{pmatrix} 0 & D \\ 0 & 0 \end{pmatrix} \right\}, \\
&\qquad (A \in \mathrm{Mat}(n \times m, \mathbb{R}),\ B \in \mathrm{Mat}(m,\mathbb{R}),\ C \in \mathrm{Mat}(n,\mathbb{R}), \\
&\qquad D \in \mathrm{Mat}(m \times n, \mathbb{R}),\ \mathrm{trace}\, B + \mathrm{trace}\, C = 0),
\end{aligned}$$

$$[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}.$$

Here $\mathfrak{g}$ is the Lie algebra of $G = SL(m+n,\mathbb{R})$, $\mathfrak{g}_0$ the Lie algebra of $G_0 = GR(n,m)$ and $\mathfrak{g}_1$ the Lie algebra of the first prolongation of $\mathfrak{g}_0$. We denote by G' the Lie group of a Lie subalgebra $\mathfrak{g}_0 \oplus \mathfrak{g}_1$ of $\mathfrak{g}$. We put $\mathfrak{m} = \mathfrak{g}_{-1}$.

The flat model, which is a flat homogeneous space ([O]), is the Grassmann manifold $G_{m,n+m}$ consisting of all m-dimensional subspaces in the $(n+m)$-dimensional real vector space $\mathbb{R}^{n+m}$:

$$G_{m,n+m} \cong G/G'.$$

The isotropy group G' is regarded as a subgroup of the group $G^2(mn)$ consisting of frames of second order at the origin o in $\mathbb{R}^{mn}$. Therefore we can regard G as a G'-structure of second order on G/G' (cf. [A-G1, p. 274], [A-G2, p. 26] and [A-G3, p. 195]). The group $G_0 \subset G'$ is the linear isotropy group.

2.3. Normal Cartan connection and half flatness

Let M be a manifold with a Grassmannian structure of type (n,m). Let P be the $GR(n,m)$-structure on M, namely, P is a linear frame bundle with structure group $GR(n,m)$ on M. Note that a $GR(n,m)$-connection on P generally has a torsion (cf. [O]). Let Q be the G'-structure of second order on M, namely, Q is the frame bundle of second order with structure group G' on M. From 2.2, G/G' is the flat model associated with the graded Lie algebra $\mathfrak{g}$ of first kind.

A Cartan connection ω of type G/G' on Q is a $\mathfrak{g} = \mathfrak{sl}(m+n,\mathbb{R})$-valued 1-form such that

(i) $\omega(X) \neq 0 \quad (X \in TQ,\, X \neq 0)$,
(ii) $\omega(A^*) = A, \;\; A \in \mathfrak{g}'$,
(iii) ${R_a}^* \omega = Ad(a^{-1})\omega, \;\; a \in G'$.

We have the following theorem due to Tanaka ([T1]).

THEOREM 2.1. *Under the assumption above, there exists a unique normal Cartan connection of type G/G' on Q.*

The normality condition is explained in Section 3.

We have two decompositions: the one is the space $\Lambda^2 = \Lambda^2_L \oplus \Lambda^2_R$ of 2-forms, the other the graded Lie algebra $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ of first kind. According to the decompositions, the torsion part Ω_{-1} of the curvature form Ω, which is a $\mathfrak{g}_{-1}$-valued 2-form, is decomposed as follows:

$$\Omega_{-1} = \mathfrak{g}_{-1} \otimes \Lambda^2_L \oplus \mathfrak{g}_{-1} \otimes \Lambda^2_R.$$

And the curvature part Ω_0 of Ω, which is a $\mathfrak{g}_0$-valued 2-form, is decomposed as follows:

$$\Omega_0 = \mathfrak{g}_0 \otimes \Lambda^2_L \oplus \mathfrak{g}_0 \otimes \Lambda^2_R.$$

If the components $\mathfrak{g}_{-1} \otimes \Lambda^2_L$ and $\mathfrak{g}_0 \otimes \Lambda^2_L$ are 0, a Grassmannian structure of type (n, m) is called *left-half Grassmannian flat*. If the components $\mathfrak{g}_{-1} \otimes \Lambda^2_R$ and $\mathfrak{g}_0 \otimes \Lambda^2_R$ are 0, it is called *right-half Grassmannian flat*. Both are called half Grassmannian flat.

In particular, if $n = m = 2$, the torsion of the normal Cartan connection vanishes for any manifold with a Grassmannian structure of type $(2, 2)$. The left-half Grassmannian flatness and the right-half Grassmannian flatness correspond to anti-self-duality and self-duality of a conformal structure of type $(4, 0)$ or $(2, 2)$ respectively. Both are called half conformally flat.

§3. Tanaka theory for Grassmannian structures

3.1. Review of Tanaka theory

Let us recall the definition of the normal Cartan connections in the Tanaka theory ([T1], cf. [Ko], [O]).

Let $\mathfrak{g}$ be a simple graded Lie algebra of μ-th kind. Consider the subalgebra $\mathfrak{m} = \sum_{p<0} \mathfrak{g}_p$ of $\mathfrak{g}$. The Lie algebra $\mathfrak{g}$ is regarded as the $\mathfrak{m}$-module with respect to the adjoint representation $ad : \mathfrak{m} \to \mathfrak{gl}(\mathfrak{g})$. Then we have the Lie algebra cohomology called the generalized Spencer cohomology.

First, we define C^q by

$$C^q = \mathfrak{g} \otimes \Lambda^q(\mathfrak{m}^*) = \mathrm{Hom}(\Lambda^q(\mathfrak{m}), \mathfrak{g}),$$

and the operator ∂ is given by, for $c \in C^q$, $X_1, \dots, X_{q+1} \in \mathfrak{m}$,

$$\begin{aligned}
&(\partial c)(X_1 \wedge \cdots \wedge X_{q+1}) \\
&\quad = \sum_i (-1)^{i+1} [X_i, c(X_1 \wedge \ldots \wedge \check{X}_i \wedge \cdots \wedge X_{q+1})] \\
&\qquad + \sum_{i<j} (-1)^{i+j} c([X_i, X_j] \wedge X_1 \wedge \cdots \wedge \check{X}_i \wedge \cdots \wedge \check{X}_j \wedge \cdots \wedge X_{q+1}).
\end{aligned}$$

Then we obtain a complex $\{C^q, \partial\}$:

$$\cdots \longrightarrow C^q \xrightarrow{\partial} C^{q+1} \longrightarrow \cdots.$$

Therefore we can define the cohomology group $H^q(\mathfrak{m}, \mathfrak{g})$ of this cochain complex. The group $H^q(\mathfrak{m}, \mathfrak{g})$ is the Lie algebra cohomology of the nilpotent Lie algebra $\mathfrak{m}$ with respect to the adjoint representation $ad : \mathfrak{m} \to \mathfrak{gl}(\mathfrak{g})$.

Next, we define the adjoint operator ∂^* of the operator ∂. With respect to the Killing form B, a subalgebra $\sum_{p>0} \mathfrak{g}_p$ of $\mathfrak{g}$ can be identified with the dual space $\mathfrak{m}^*$ of $\mathfrak{m}$. Let $\{e_1, \dots, e_m\}$ be a basis of $\mathfrak{m}$. The dual basis $\{e_1^*, \dots, e_m^*\}$ of $\mathfrak{m}^* = \sum_{p>0} \mathfrak{g}_p$ with $B(e_i, e_j^*) = \delta_{ij}$ is determined. Then the operator $\partial^* : C^{q+1} \to C^q$ is defined as follows: for $c \in C^{q+1}$, $X_1, \dots, X_q \in \mathfrak{m}$,

$$\begin{aligned}
&(\partial^* c)(X_1 \wedge \cdots \wedge X_q) \\
&\quad = \sum_j [e_j^*, c(e_j \wedge X_1 \wedge \cdots \wedge X_q)] \\
&\qquad + \frac{1}{2} \sum_{i,j} (-1)^{i+1} c([e_j^*, X_i]_- \wedge e_j \wedge X_1 \wedge \cdots \wedge \check{X}_i \wedge \cdots \wedge X_q),
\end{aligned}$$

where $[e_j^*, X_i]_-$ is an $\mathfrak{m}$-component of $[e_j^*, X_i]$ with respect to the decomposition $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{g}'$. Let ρ be an involutive automorphism of $\mathfrak{g}$ such that $\rho\mathfrak{g}_p = \mathfrak{g}_{-p}$ and $B(X, \rho X) < 0$ $(X \neq 0)$. Then we define a positive definite inner product $(\,\cdot\,,\,\cdot\,)$ in $\mathfrak{g}$ by

$$(X, Y) = -B(X, \rho Y), \quad X, Y \in \mathfrak{g}.$$

We have

$$(\partial c, c') = (c, \partial^* c'), \quad c \in C^q,\ c' \in C^{q+1}.$$

The operator Δ called the Laplacian is defined as usual:

$$\Delta = \partial^*\partial + \partial\partial^* : C^q \longrightarrow C^q.$$

If $\Delta c = 0$ for $c \in C^q$, c is called harmonic. Evidently c is harmonic if and only if $\partial c = \partial^* c = 0$. We denote by H^q the set of all harmonic forms in C^q. It is well-known that

$$H^q \cong H^q(\mathfrak{m}, \mathfrak{g}).$$

Since $\mathfrak{m} = \sum_{j<0} \mathfrak{g}_j$, the space $\Lambda^q(\mathfrak{m}^*)$ is decomposed as follows:

$$\Lambda^q(\mathfrak{m}^*) = \sum_{r_1,\dots,r_q<0} \mathfrak{g}^*_{r_1} \wedge \cdots \wedge \mathfrak{g}^*_{r_q}.$$

Furthermore, we define the subspace $\Lambda^q_i(\mathfrak{m}^*)$ as follows:

$$\Lambda^q_i(\mathfrak{m}^*) = \sum_{\substack{r_1+\cdots+r_q=i \\ r_1,\dots,r_q<0}} \mathfrak{g}^*_{r_1} \wedge \cdots \wedge \mathfrak{g}^*_{r_q}.$$

Then we have

$$\Lambda^q(\mathfrak{m}^*) = \sum_i \Lambda^q_i(\mathfrak{m}^*) \quad \text{(direct sum)}.$$

Since $\mathfrak{g} = \sum_j \mathfrak{g}_j$, the space $C^q = \mathfrak{g} \otimes \Lambda^q(\mathfrak{m}^*)$ is decomposed as follows:

$$C^q = \mathfrak{g} \otimes \Lambda^q(\mathfrak{m}^*) = \sum_{i,j} \mathfrak{g}_j \otimes \Lambda^q_i(\mathfrak{m}^*).$$

We define the subspace $C^{p,q}$ by

$$C^{p,q} = \sum_j \mathfrak{g}_j \otimes \Lambda^q_{j-p-q+1}(\mathfrak{m}^*).$$

In particular, we have

$$\begin{aligned} C^{p,0} &= \mathfrak{g}_{p-1}, \\ C^{p,1} &= \sum_{j<p} \mathfrak{g}_j \otimes \mathfrak{g}^*_{j-p}. \end{aligned}$$

Furthermore, we have

$$C^q = \sum_p C^{p,q} \quad \text{(direct sum)}$$

and

$$\begin{aligned} \partial C^{p,q} &\subset C^{p-1,q+1}, \\ \partial^* C^{p,q} &\subset C^{p+1,q-1}, \\ \Delta C^{p,q} &\subset C^{p,q}. \end{aligned}$$

We denote by $H^{p,q}$ the set of all harmonic forms in $C^{p,q}$. Then we have

$$H^q = \sum_p H^{p,q} \quad \text{(direct sum)}.$$

The group G_0 linearly acts on C^q as follows: for $c \in C^q$ and $a \in G_0$,

$$(ac)(X_1 \wedge \cdots \wedge X_q) = Ad(a^{-1})c(Ad(a)X_1 \wedge \cdots \wedge Ad(a)X_q),$$

where $X_1, \ldots, X_q \in \mathfrak{m}$. It follows that $\mathfrak{g}_j \otimes \Lambda^q_i(\mathfrak{m}^*)$ is G_0 invariant and $a(\partial c) = \partial(ac)$, $a(\partial^* c) = \partial^*(ac)$. Hence $C^{p,q}$ and $H^{p,q}$ are G_0 invariant subspaces of C^q and H^q respectively.

The case $q = 2$ is important.

Let G/G' be a homogeneous space associated with the simple graded Lie algebra $\mathfrak{g}$. Let M be a manifold with $\dim M = \dim G/G'$. Let Q be a G'-principal bundle over M and ω a Cartan connection of type G/G' on Q. It is a $\mathfrak{g}$-valued 1-form. Let Ω be the curvature form on Q. It is a $\mathfrak{g}$-valued 2-form. Then the curvature function $K : Q \to C^2 = \mathfrak{g} \otimes \Lambda^2(\mathfrak{m}^*)$ on Q is defined:

$$\Omega = \frac{1}{2} K(\omega_- \wedge \omega_-),$$

where ω_- is an $\mathfrak{m}$-component with respect to the decomposition $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{g}'$ of ω.

Corresponding to the decomposition $\mathfrak{g} = \sum_j \mathfrak{g}_j$, ω and Ω are decomposed as follows:

$$\omega = \sum_j \omega_j, \quad \Omega = \sum_j \Omega_j.$$

The curvature function K is decomposed as follows:

$$K = \sum_j K_j = \sum_p K^{p,2}.$$

We abbreviate $K^{p,2}$ to K^p.

A Cartan connection ω is called *normal* if

(1) $K^p = 0 \quad (p < 0)$,
(2) $\partial^* K^p = 0 \quad (p \geq 0)$.
Take $M = G/G'$. Then the Maurer-Cartan form is a Cartan connection ω which is a $\mathfrak{g}$-valued 1-form on $Q = G$. As $K = 0$ holds, ω is normal.

The space C^2 is orthogonally decomposed into

$$C^2 = H^2 + \Delta C^2.$$

Denote by H the orthogonal projection $C^2 \to H^2$. We have

$$HC^{p,2} = H^{p,2}.$$

The function $HK : Q \to H^2$ on Q is the harmonic part of K. As $K = \sum_p K^p$,

$$HK = \sum_p HK^p.$$

We remark that HK gives the fundamental invariant of the normal Cartan connection for the geometric structure subordinate to type G/G' associated with the simple graded Lie algebra $\mathfrak{g}$. Namely, we have the following theorem due to Tanaka ([T1]).

THEOREM 3.1. *We have*

$$K = 0 \Longleftrightarrow HK = 0.$$

Moreover we have the following theorem.

THEOREM 3.2. *We have, for some $p \geq 0$,*

$$K^q = 0 \text{ for all } q < p \Longrightarrow K^p = HK^p.$$

3.2. Tanaka theory for Grassmannian structure

Let M be a manifold with a Grassmannian structure of type (n, m). The flat model is the Grassmann manifold $G_{m,n+m} \cong G/G'$ associated with the graded Lie algebra $\mathfrak{g} = \mathfrak{sl}(m + n, \mathbb{R})$ of first kind:

$$\begin{aligned}
\mathfrak{g} &= \mathfrak{sl}(m+n, \mathbb{R}) \\
&= \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \\
&= \left\{ \begin{pmatrix} O_m & 0 \\ A & O_n \end{pmatrix} \right\} \oplus \left\{ \begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix} \right\} \oplus \left\{ \begin{pmatrix} O_m & D \\ 0 & O_n \end{pmatrix} \right\}, \\
&\qquad (A \in \mathrm{Mat}(n \times m, \mathbb{R}),\ B \in \mathrm{Mat}(m, \mathbb{R}),\ C \in \mathrm{Mat}(n, \mathbb{R}), \\
&\qquad D \in \mathrm{Mat}(m \times n, \mathbb{R}),\ \operatorname{trace} B + \operatorname{trace} C = 0),
\end{aligned}$$

$[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$.

We put their bases and coordinates as follows:

$\mathbf{e}_{ij} = \begin{pmatrix} O_m & 0 \\ E_{ij} & O_n \end{pmatrix}$,

($E_{ij} \in \mathrm{Mat}(n\times m), \mathbb{R})$: matrix unit, i.e., (i,j)-component $=1$, otherwise $=0$),

$\mathbf{g}_{ij} = \begin{pmatrix} E_{ij} & 0 \\ 0 & O_n \end{pmatrix}$, $\mathbf{h}_{ij} = \begin{pmatrix} O_m & 0 \\ 0 & E_{ij} \end{pmatrix}$, $\mathbf{e}^*_{ij} = {}^t\mathbf{e}_{ji}$,

($E_{ij} \in \mathrm{Mat}(m, \mathbb{R})$, $E_{ij} \in \mathrm{Mat}(n, \mathbb{R})$: matrix units respectively)

and

$$\begin{pmatrix} g_{ij} & d_{ij} \\ x_{ij} & h_{ij} \end{pmatrix} = \sum x_{ij}\mathbf{e}_{ij} + \sum g_{ij}\mathbf{g}_{ij} + \sum h_{ij}\mathbf{h}_{ij} + \sum d_{ij}\mathbf{e}^*_{ij}.$$

Let Q be the frame bundle of second order with structure group G' on M. By 2.3, there exists a $\mathfrak{g} = \mathfrak{sl}(m+n, \mathbb{R})$-valued normal Cartan connection of type G/G'. As $\mathfrak{g}$ is of first kind, we have

$$\begin{aligned} C^2 &= C^{0,2} \oplus C^{1,2} \oplus C^{2,2} \\ &= \mathfrak{g}_{-1} \otimes \Lambda^2_{-2} \oplus \mathfrak{g}_0 \otimes \Lambda^2_{-2} \oplus \mathfrak{g}_1 \otimes \Lambda^2_{-2}. \end{aligned}$$

The curvature function K of the connection ω has

$$\begin{aligned} K\,(= K_{-1} + K_0 + K_1) &= K^{0,2} + K^{1,2} + K^{2,2} \\ &= K^0 + K^1 + K^2. \end{aligned}$$

We can view K^0 as the torsion part and K^1 as the curvature part.

It follows that K satisfies the condition (1) of normality as a consequence. Let us write down the condition (2) of normality:

(i) $\partial^* K^0 = 0$,

(ii) $\partial^* K^1 = 0$.

Since $\partial^* : C^{2,2} \to C^{3,1} = 0$, we have $\partial^* K^2 = 0$.

In particular, in the case $m = 2$, i.e., a Grassmannian structure of type $(n, 2)$, let us see it explicitly. Put $\mathbf{a}_i = \mathbf{e}_{i1}$, $\mathbf{b}_i = \mathbf{a}_{n+i} = \mathbf{e}_{i2}$. The dual basis of $\mathfrak{g}_1$ is $\mathbf{a}^*_1 = {}^t\mathbf{a}_1, \ldots, \mathbf{a}^*_n = {}^t\mathbf{a}_n$, $\mathbf{a}^*_{n+1} = {}^t\mathbf{b}_1, \ldots, \mathbf{a}^*_{2n} = {}^t\mathbf{b}_n$.

We investigate the case (i):

$$\begin{aligned} \partial^* : C^{0,2} = \mathfrak{g}_{-1} \otimes \Lambda^2_{-2} &\longrightarrow C^{1,1} = \mathfrak{g}_0 \otimes \Lambda^1_{-1} \\ K^0 &\longmapsto \partial^* K^0. \end{aligned}$$

For $X \in \mathfrak{m} = \mathfrak{g}_{-1}$,

$$\begin{aligned}(\partial^* K^0)(X) &= \sum_{j=1}^{2n} [\mathbf{a}_j^*, K^0(\mathbf{a}_j \wedge X)] + \frac{1}{2} \sum_{j=1}^{2n} K^0([\mathbf{a}_j^*, X]_- \wedge \mathbf{a}_j) \\ &= \sum_{j=1}^{2n} [\mathbf{a}_j^*, K^0(\mathbf{a}_j \wedge X)],\end{aligned}$$

where the second equality holds from $[\mathbf{a}_j^*, X]_- = 0$ for $[\mathbf{a}_j^*, X] \in \mathfrak{g}_0$.

The Lie bracket of $\mathfrak{g}_1$ and $\mathfrak{g}_{-1}$ are given by, for $i, j = 1, \ldots, n$,

$$\begin{aligned} &[\mathbf{a}_i^*, \mathbf{a}_j] = \delta_{ij}\mathbf{g}_{11} - \mathbf{h}_{ji}, \quad && [\mathbf{b}_i^*, \mathbf{b}_j] = \delta_{ij}\mathbf{g}_{22} - \mathbf{h}_{ji}, \\ &[\mathbf{a}_i^*, \mathbf{b}_j] = \delta_{ij}\mathbf{g}_{12}, && [\mathbf{b}_i^*, \mathbf{a}_j] = \delta_{ij}\mathbf{g}_{21}. \end{aligned}$$

If we assume that $\partial^* K^0 = 0$, the following relations hold in $\mathfrak{g}_0$:

$$\begin{aligned}
\mathbf{g}_{11}\text{-component} = 0 &\iff \sum_{i=1}^{n} K_i^0(\mathbf{a}_i \wedge X) = 0, \\
\mathbf{g}_{12}\text{-component} = 0 &\iff \sum_{i=1}^{n} K_{n+i}^0(\mathbf{a}_i \wedge X) = 0, \\
\mathbf{g}_{21}\text{-component} = 0 &\iff \sum_{i=1}^{n} K_i^0(\mathbf{a}_{n+i} \wedge X) = 0, \\
\mathbf{g}_{22}\text{-component} = 0 &\iff \sum_{i=1}^{n} K_{n+i}^0(\mathbf{a}_{n+i} \wedge X) = 0, \\
\mathbf{h}_{i,j}\text{-component} = 0 &\iff K_i^0(\mathbf{a}_j \wedge X) + K_{n+i}^0(\mathbf{a}_{n+j} \wedge X) = 0.
\end{aligned}$$

Here K_i^0 and K_{n+i}^0 are $\mathbf{a}_i$ and $\mathbf{a}_{n+i} = \mathbf{b}_i$ components in $\mathfrak{g}_{-1}$ respectively.

From $\partial^* K^0 = 0$ of the normality, we have the following.

Proposition 3.1. *Let M be a manifold with a Grassmannian structure of type $(n, 2)$. Then, for the normal Cartan connection ω of type G/G' on Q, the $\mathfrak{g}_{-1} \otimes \Lambda_L^2$-component of the torsion part Ω_{-1} is 0.*

Proof. We use the relations

$$K_i^0(\mathbf{a}_j \wedge X) = -K_{n+i}^0(\mathbf{a}_{n+j} \wedge X).$$

Putting $X = \mathbf{a}_{n+j}$, we obtain

$$K_i^0(\mathbf{a}_j \wedge \mathbf{a}_{n+j}) = 0.$$

Putting $X = \mathbf{a}_j$, we obtain

$$K_{n+i}^0(\mathbf{a}_{n+j} \wedge \mathbf{a}_j) = 0.$$

Putting $X = \mathbf{a}_{n+k}$, we have

$$\begin{aligned} K_i^0(\mathbf{a}_j \wedge \mathbf{a}_{n+k}) &= -K_{n+i}^0(\mathbf{a}_{n+j} \wedge \mathbf{a}_{n+k}) \\ &= K_{n+i}^0(\mathbf{a}_{n+k} \wedge \mathbf{a}_{n+j}) \\ &= -K_i^0(\mathbf{a}_k \wedge \mathbf{a}_{n+j}). \end{aligned}$$

Thus we obtain

$$K_i^0(\mathbf{a}_j \wedge \mathbf{a}_{n+k} + \mathbf{a}_k \wedge \mathbf{a}_{n+j}) = 0.$$

Putting $X = \mathbf{a}_k$, we similarly obtain

$$K_{n+i}^0(\mathbf{a}_j \wedge \mathbf{a}_{n+k} + \mathbf{a}_k \wedge \mathbf{a}_{n+j}) = 0.$$

Therefore it follows that $\mathfrak{g}_{-1} \otimes \Lambda_L^2$-component is 0. □

PROPOSITION 3.2. *Let M be a 4-dimensional manifold with a Grassmannian structure (or a conformal structure) of type $(2,2)$. Then, for the normal Cartan connection ω, the torsion part Ω_{-1} is 0.*

Proof. A basis of $\mathfrak{g}_{-1}$ is $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3 = \mathbf{b}_1$, $\mathbf{a}_4 = \mathbf{b}_2$.
We have

$$\begin{aligned} &K_i^0(\mathbf{a}_1 \wedge \mathbf{a}_2) = 0, \quad K_{2+i}^0(\mathbf{a}_1 \wedge \mathbf{a}_2) = 0 \quad (i = 1, 2), \\ &K_i^0(\mathbf{b}_1 \wedge \mathbf{b}_2) = 0, \quad K_{2+i}^0(\mathbf{b}_1 \wedge \mathbf{b}_2) = 0 \quad (i = 1, 2). \end{aligned}$$

Moreover we have

$$\begin{aligned} &K_i^0(\mathbf{a}_1 \wedge \mathbf{b}_2) = 0, \quad K_{2+i}^0(\mathbf{a}_1 \wedge \mathbf{b}_2) = 0 \quad (i = 1, 2), \\ &K_i^0(\mathbf{a}_2 \wedge \mathbf{b}_1) = 0, \quad K_{2+i}^0(\mathbf{a}_2 \wedge \mathbf{b}_1) = 0 \quad (i = 1, 2). \end{aligned}$$

Thus it follows that $\mathfrak{g}_{-1} \otimes \Lambda_R^2$-component is 0. Therefore the torsion part Ω_{-1} is 0. □

We investigate the case (ii):

$$\begin{aligned}\partial^* : C^{1,2} = \mathfrak{g}_0 \otimes \Lambda^2_{-2} &\longrightarrow C^{2,1} = \mathfrak{g}_1 \otimes \Lambda^1_{-1} \\ K^1 &\longmapsto \partial^* K^1.\end{aligned}$$

We will adopt the notation and the argument of (i) similarly. For $X \in \mathfrak{m} = \mathfrak{g}_{-1}$,

$$(\partial^* K^1)(X) = \sum_{j=1}^{2n} [\mathbf{a}_j^*, K^1(\mathbf{a}_j \wedge X)].$$

The Lie bracket of $\mathfrak{g}_1$ and $\mathfrak{g}_0$ are given by, for $i, j = 1, \ldots, n$,

$$\begin{aligned}&[\mathbf{a}_i^*, \mathbf{g}_{11}] = -\mathbf{a}_i^*,\ \ [\mathbf{a}_i^*, \mathbf{g}_{12}] = 0,\ \ [\mathbf{a}_i^*, \mathbf{g}_{21}] = -\mathbf{a}_{n+i}^*,\ \ [\mathbf{a}_i^*, \mathbf{g}_{22}] = 0,\\ &[\mathbf{a}_{n+i}^*, \mathbf{g}_{11}] = 0,\ \ [\mathbf{a}_{n+i}^*, \mathbf{g}_{12}] = -\mathbf{a}_i^*,\ \ [\mathbf{a}_{n+i}^*, \mathbf{g}_{21}] = 0,\ \ [\mathbf{a}_{n+i}^*, \mathbf{g}_{22}] = -\mathbf{a}_{n+i}^*,\\ &[\mathbf{a}_i^*, \mathbf{h}_{jk}] = \delta_{ij}\mathbf{a}_k^*,\ \ [\mathbf{a}_{n+i}^*, \mathbf{h}_{jk}] = \delta_{ij}\mathbf{a}_{n+k}^*.\end{aligned}$$

If we assume that $\partial^* K^1 = 0$, the following relations hold in $\mathfrak{g}_1$: for $i = 1, \ldots, n$,

$$\begin{aligned}&\mathbf{a}_i\text{-component} = 0\\ &\qquad \Longleftrightarrow K_x^1(\mathbf{a}_i \wedge X) + K_y^1(\mathbf{a}_{n+i} \wedge X) = \sum_{j=1}^{n} K_{ji}^1(\mathbf{a}_j \wedge X),\\ &\mathbf{a}_{n+i} = \mathbf{b}_i\text{-component} = 0\\ &\qquad \Longleftrightarrow K_z^1(\mathbf{a}_i \wedge X) + K_w^1(\mathbf{a}_{n+i} \wedge X) = \sum_{j=1}^{n} K_{ji}^1(\mathbf{a}_{n+j} \wedge X).\end{aligned}$$

Here K_x^1, K_y^1, K_z^1, K_w^1 are $\mathbf{g}_{11}$, $\mathbf{g}_{12}$, $\mathbf{g}_{21}$, $\mathbf{g}_{22}$ components and K_{ij}^1 are $\mathbf{h}_{ij}$ components in $\mathfrak{g}_0$ respectively. The number of independent equations is $4n^2$. We write down $4n^2$ independent relations of the normality:

$$K_x^1(\mathbf{a}_i \wedge \mathbf{a}_k) + K_y^1(\mathbf{b}_i \wedge \mathbf{a}_k) = \sum_{j=1}^{n} K_{ji}^1(\mathbf{a}_j \wedge \mathbf{a}_k),$$

$$K_x^1(\mathbf{a}_i \wedge \mathbf{b}_k) + K_y^1(\mathbf{b}_i \wedge \mathbf{b}_k) = \sum_{j=1}^{n} K_{ji}^1(\mathbf{a}_j \wedge \mathbf{b}_k),$$

$$K_z^1(\mathbf{a}_i \wedge \mathbf{a}_k) + K_w^1(\mathbf{b}_i \wedge \mathbf{a}_k) = \sum_{j=1}^{n} K_{ji}^1(\mathbf{b}_j \wedge \mathbf{a}_k),$$

$$K_z^1(\mathbf{a}_i \wedge \mathbf{b}_k) + K_w^1(\mathbf{b}_i \wedge \mathbf{b}_k) = \sum_{j=1}^{n} K_{ji}^1(\mathbf{b}_j \wedge \mathbf{b}_k).$$

Next, we consider the Bianchi identities. They are given, by Lemmas 2.10, 2.11 in [T1], as follows: for $p \geq 0$,

$$\partial K^p = \Psi^{p-1}.$$

We consider $K = K^0 + K^1 + K^2$ with respect to a graded Lie algebra $\mathfrak{g}$ of first kind.

For K^0, $\partial K^0 = \Psi^{-1} \in C^{-1,3} = \mathfrak{g}_{-2} \otimes \Lambda^3_{-3} = 0$ holds. Joining $\partial^* K^0 = 0$ of normality together, we see that K^0 is harmonic, that is,

$$HK^0 = K^0 = K_{-1} \in H^{0,2}.$$

For K^1, assume that the torsion part $K^0 = K_{-1}$ is 0. Then it follows that $\partial K^1 = \Psi^0 \in C^{0,3} = \mathfrak{g}_{-1} \otimes \Lambda^3_{-3}$ is 0 from the definition of Ψ^0. Joining $\partial^* K^1 = 0$ of normality together, we see that K^1 is harmonic, that is, under the assumption $K^0 = 0$,

$$HK^1 = K^1 = K_0 \in H^{1,2}.$$

For K^2, in general $\partial K^2 = \Psi^1$ is not 0.

3.3. Nonzero generators in H^2

Before we investigate the harmonic part HK of the curvature function K, which is the fundamental invariant of a Grassmannian structure of type (n, m), we calculate $H^2 \cong H^2(\mathfrak{m}, \mathfrak{g})$ according to Yamaguchi ([Y, Proposition 5.5]).

A nonzero generator in H^2 decomposed as an irreducible $\mathfrak{g}_0$-module is represented by

$$x_{\sigma(\theta)} \otimes x_{\Phi_\sigma} \in H^{p_{ij},2}$$

for $\sigma = \sigma_{ij} \in W^0(2)$. Here $\sigma = \sigma_{ij} = \sigma_i \cdot \sigma_j = \sigma_{\alpha_i} \cdot \sigma_{\alpha_j}$ ($\alpha_i, \alpha_j \in \Delta$: a fixed simple root system of $\mathfrak{g}$) is the composition of reflections by α_i, α_j, and $x_{\sigma(\theta)} \in \mathfrak{g}_{\sigma(\theta)(E)}$ is the root vector for the root by the reflection $\sigma = \sigma_{ij}$ of the highest root θ, and $x_{\Phi_\sigma} \in \Lambda^2\mathfrak{m}^*$ is the exterior product of two root vectors for $\Phi_\sigma = \Phi_{\sigma_{ij}} = \{\alpha_i, \alpha_j - \langle \alpha_j, \alpha_i \rangle \alpha_i\}$, and p_{ij} is a nonnegative integer decided by σ_{ij}. See [Y] in detail.

We consider the simple graded Lie algebra

$$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1, \quad \mathfrak{m} = \mathfrak{g}_{-1}$$

of first kind for the Lie algebra $\mathfrak{g} = \mathfrak{sl}(m+n,\mathbb{R})$ of $G = SL(m+n,\mathbb{R})$ associated with a Grassmannian structure of type (n,m). We have

$$\Lambda^2\mathfrak{m}^* = \Lambda^2\mathfrak{g}_{-1}^* = \Lambda^2_{-2} = \Lambda^2_L \oplus \Lambda^2_R,$$

where, from $TM \cong V \otimes W$,

$$\Lambda^2_L = S^2(V) \otimes \Lambda^2(W), \quad \Lambda^2_R = \Lambda^2(V) \otimes S^2(W).$$

This is a decomposition as a $\mathfrak{g}_0$-module. We remark that

$$\begin{aligned} C^{0,2} &= \mathfrak{g}_{-1} \otimes \Lambda^2_{-2} = \mathfrak{g}_{-1} \otimes \Lambda^2_L \oplus \mathfrak{g}_{-1} \otimes \Lambda^2_R, \\ C^{1,2} &= \mathfrak{g}_0 \otimes \Lambda^2_{-2} = \left(\mathfrak{g}_0^L \oplus \mathfrak{g}_0^R \oplus z\right) \otimes \left(\Lambda^2_L \oplus \Lambda^2_R\right), \\ C^{2,2} &= \mathfrak{g}_1 \otimes \Lambda^2_{-2} = \mathfrak{g}_1 \otimes \Lambda^2_L \oplus \mathfrak{g}_1 \otimes \Lambda^2_R, \end{aligned}$$

where

$$\mathfrak{g}_0^L = \mathfrak{sl}_L = \mathfrak{sl}(n,\mathbb{R}), \quad \mathfrak{g}_0^R = \mathfrak{sl}_R = \mathfrak{sl}(m,\mathbb{R})$$

and z is the trace part of $\mathfrak{g}_0$.

Assume that $n \geq m$. Then we obtain the following.

Proposition 3.3. *The components of nonzero generators as $\mathfrak{g}_0$-modules in H^2 are represented by*

1. *if $n, m \geq 3$,*
   (i) $\mathbf{e}_{n1} \otimes (\mathbf{e}_{1m-1}{}^* \wedge \mathbf{e}_{1m}{}^*) \in H^{0,2} \subset \mathfrak{g}_{-1} \otimes \Lambda^2_L$,
   (ii) $\mathbf{e}_{n1} \otimes (\mathbf{e}_{1m}{}^* \wedge \mathbf{e}_{2m}{}^*) \in H^{0,2} \subset \mathfrak{g}_{-1} \otimes \Lambda^2_R$,
2. *if $n \geq 3$, $m = 2$,*
   (i) $\mathbf{e}_{n1} \otimes (\mathbf{e}_{12}{}^* \wedge \mathbf{e}_{22}{}^*) \in H^{0,2} \subset \mathfrak{g}_{-1} \otimes \Lambda^2_R$,
   (ii) $\mathbf{h}_{n1} \otimes (\mathbf{e}_{11}{}^* \wedge \mathbf{e}_{12}{}^*) \in H^{1,2} \subset \mathfrak{g}_0^L \otimes \Lambda^2_L$,
3. *if $n = 2$, $m = 2$,*
   (i) $\mathbf{h}_{21} \otimes (\mathbf{e}_{11}{}^* \wedge \mathbf{e}_{12}{}^*) \in H^{1,2} \subset \mathfrak{g}_0^L \otimes \Lambda^2_L$,
   (ii) $\mathbf{g}_{21} \otimes (\mathbf{e}_{12}{}^* \wedge \mathbf{e}_{22}{}^*) \in H^{1,2} \subset \mathfrak{g}_0^R \otimes \Lambda^2_R$.

If $n \geq 3$, $m = 2$, from 2 (i) in the above proposition, it follows that the $\mathfrak{g}_{-1} \otimes \Lambda^2_L$-component of the torsion part is 0. See Proposition 3.1.

If $n = 2$, $m = 2$, from 3 in the above proposition, it follows that $K^0 = 0$, i.e., the torsion part is 0. See Proposition 3.2. Hence $K^1 = HK^1$ holds.

When we take $m = 1$, the geometric structure becomes a projective structure whose flat model is the n-dimensional projective space $P^n(\mathbb{R})$ ($n \geq 3$). See 6.5, 7.1. There is not the notion of half flatness. Now we remark that

$$\mathfrak{g}_{-1} = \mathrm{span}\{\mathbf{a}_i = \mathbf{e}_{ij}\}_{1\leq i\leq n, j=1}, \ \ \mathfrak{g}_0 = \mathrm{span}\{\mathbf{g}_{11}, \mathbf{h}_{ij}\}_{1\leq i,j\leq n}, \ \ \mathfrak{g}_1 = \mathfrak{g}_{-1}^*.$$

Then the components of nonzero generators as $\mathfrak{g}_0$-modules in H^2 are represented by

$$\mathbf{h}_{n1} \otimes (\mathbf{a}_1^* \wedge \mathbf{a}_2^*) \in H^{1,2} \subset \mathfrak{g}_0 \otimes \Lambda^2.$$

Therefore it follows that $K^0 = 0$, i.e., the torsion part is 0. Hence $K^1 = HK^1$ holds.

§4. Co-Grassmannian structures

4.1. Example as the flat model

Before we argue the twistor theory of Grassmannian structures in Section 7, we define and study co-Grassmannian structures that are geometric structures of the top space, that is, the incidence space of the double fibration by twistor theory of Grassmannian structures.

In the $(n+m)$-dimensional real vector space $V = \mathbb{R}^{n+m}$, let $F_{m-1,m}$ be the following generalized flag manifold:

$$F_{m-1,m} = \{(S_{m-1}, S_m) \mid S_i : i\text{-dimensional subspace of } V,\ S_{m-1} \subset S_m\}.$$

The dimension of $F_{m-1,m}$ is $mn+m-1$. The group $G = SL(m+n, \mathbb{R})$ acts transitively on $F_{m-1,m}$.

Let $\{a_1, \ldots, a_m, b_1, \ldots, b_n\}$ be a basis of $V = \mathbb{R}^{n+m}$. Choose $Z_0 = (X_0, Y_0) \in F_{m-1,m}$ such that

$$X_0 = \mathrm{span}\{a_1, \ldots, a_{m-1}\} \cong \mathrm{span}\begin{pmatrix} I_{m-1} \\ O \end{pmatrix} (\in \mathrm{Mat}((m+n) \times (m-1), \mathbb{R})),$$

$$Y_0 = \mathrm{span}\{a_1, \ldots, a_{m-1}, a_m\} \cong \mathrm{span}\begin{pmatrix} I_m \\ O \end{pmatrix} (\in \mathrm{Mat}((m+n) \times m, \mathbb{R})),$$

where $a_i = {}^t(0, \ldots, 0, \overset{i}{1}, 0, \ldots, 0) \in V = \mathbb{R}^{n+m}$. Let G' be the isotropy group of $G = SL(m+n, \mathbb{R})$ at the base point $Z_0 = (X_0, Y_0)$. Then we have

$$F_{m-1,m} \cong G/G',$$

and it is easy to check that

$$G' = \left\{ \begin{array}{cl} \begin{array}{ccc} m-1 & 1 & n \end{array} & \\ \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix} & \begin{array}{c} m-1 \\ 1 \\ n \end{array} \end{array} \in SL(m+n,\mathbb{R}) \right\}.$$

This is the flat model of manifolds with co-Grassmannian structures of type $(n, m-1)$, which are defined in the next 4.2.

The Lie algebra $\mathfrak{g} = \mathfrak{sl}(m+n,\mathbb{R})$ of $G = SL(m+n,\mathbb{R})$ has a decomposition as follows:

$$\begin{aligned}
\mathfrak{g} &= \mathfrak{sl}(m+n,\mathbb{R}) \\
&= \mathfrak{m} \oplus \mathfrak{g}' \\
&= \left\{ \begin{pmatrix} O_{m-1} & 0 & 0 \\ \mathbf{f} & 0 & 0 \\ A & \mathbf{e} & O_n \end{pmatrix} \right\} \oplus \left\{ \begin{pmatrix} B & {}^t\mathbf{h} & D \\ 0 & d & {}^t\mathbf{g} \\ 0 & 0 & C \end{pmatrix} \right\}, \\
&\qquad (A \in \mathrm{Mat}(n \times (m-1),\mathbb{R}),\ B \in \mathrm{Mat}(m-1,\mathbb{R}),\ C \in \mathrm{Mat}(n,\mathbb{R}), \\
&\qquad D \in \mathrm{Mat}((m-1) \times n,\mathbb{R}),\ d \in \mathbb{R},\ \mathbf{e},\mathbf{g} \in \mathbb{R}^n,\ \mathbf{f},\mathbf{h} \in \mathbb{R}^{m-1}, \\
&\qquad \operatorname{trace} B + d + \operatorname{trace} C = 0).
\end{aligned}$$

The subalgebra $\mathfrak{g}'$ is the Lie algebra of G', and $\mathfrak{m}$ is identified with the tangent space $T_{Z_0}F_{m-1,m}$ of $F_{m-1,m}$ at Z_0. Furthermore, $\mathfrak{g} = \mathfrak{sl}(m+n,\mathbb{R})$ has more decompositions:

$$\begin{aligned}
\mathfrak{g} &= \mathfrak{sl}(m+n,\mathbb{R}) \\
&= \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \\
&= \left\{ \begin{pmatrix} O_{m-1} & 0 & 0 \\ 0 & 0 & 0 \\ A & 0 & O_n \end{pmatrix} \right\} \oplus \left\{ \begin{pmatrix} O_{m-1} & 0 & 0 \\ \mathbf{f} & 0 & 0 \\ 0 & \mathbf{e} & O_n \end{pmatrix} \right\} \oplus \left\{ \begin{pmatrix} B & 0 & 0 \\ 0 & d & 0 \\ 0 & 0 & C \end{pmatrix} \right\} \\
&\qquad \oplus \left\{ \begin{pmatrix} O_{m-1} & {}^t\mathbf{h} & 0 \\ 0 & 0 & {}^t\mathbf{g} \\ 0 & 0 & O_n \end{pmatrix} \right\} \oplus \left\{ \begin{pmatrix} O_{m-1} & 0 & D \\ 0 & 0 & 0 \\ 0 & 0 & O_n \end{pmatrix} \right\}, \\
\mathfrak{m} &= \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}, \\
\mathfrak{g}' &= \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2,
\end{aligned}$$

and has a structure of a simple graded Lie algebra of second kind.

We consider

$$\mathfrak{e} = \left\{ \begin{pmatrix} O_{m-1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \mathbf{e} & O_n \end{pmatrix} \right\}, \quad \mathfrak{f} = \left\{ \begin{pmatrix} O_{m-1} & 0 & 0 \\ \mathbf{f} & 0 & 0 \\ 0 & 0 & O_n \end{pmatrix} \right\}.$$

Then we have

1) $\mathfrak{g}_{-1} = \mathfrak{e} \oplus \mathfrak{f}$,
2) $[\mathfrak{e}, \mathfrak{e}] = [\mathfrak{f}, \mathfrak{f}] = 0$,
3) $\mathfrak{g}_{-2} = [\mathfrak{e}, \mathfrak{f}] \cong \mathfrak{e} \otimes \mathfrak{f}$,
4) $[\mathfrak{g}_0, \mathfrak{e}] = \mathfrak{g}_0\mathfrak{e} \subset \mathfrak{e}$, $[\mathfrak{g}_0, \mathfrak{f}] = \mathfrak{g}_0\mathfrak{f} \subset \mathfrak{f}$.

Furthermore, we consider

$$\mathfrak{a} = \mathfrak{e} + \mathfrak{g}', \quad \mathfrak{b} = \mathfrak{f} + \mathfrak{g}'.$$

Then we can easily verify that

5) $Ad(G')\mathfrak{a} = \mathfrak{a}$, $Ad(G')\mathfrak{b} = \mathfrak{b}$,
6) $\mathfrak{a} \cap \mathfrak{b} = \mathfrak{g}'$,
7) both $\mathfrak{a}$ and $\mathfrak{b}$ are subalgebras of $\mathfrak{g}$.

By 5), $\mathfrak{a}$ and $\mathfrak{b}$ induce invariant differential systems E (with dimension n) and F (with dimension $m-1$) on $F_{m-1,m} = G/G'$ respectively. By 6) and 7), the pair (E, F) has

1) E and F are transversal,
2) both E and F are completely integrable,

and, by $\mathfrak{g}_{-2} = [\mathfrak{g}_{-1}, \mathfrak{g}_{-1}]$,

3) $TF_{m-1,m} = D + [D, D]$,

where we put $D = E \oplus F$. Moreover, the quotient bundle $TF_{m-1,m}/D$ has

4) $\operatorname{rank} TF_{m-1,m}/D = \operatorname{rank} E \cdot \operatorname{rank} F \ (= n(m-1))$.

Motivated by the discussion of the above example, we will give the definition of a co-Grassmannian structure in the next 4.2, 4.3.

We note the following. Fix an (positive definite) inner product on V. For $Z_0 = (X_0, Y_0) \in F_{m-1,m}$, put

$$Y_0 = X_0 \oplus \mathbb{R} \quad \text{(orthogonal sum)}.$$

We consider two orthogonal projections

$$p_1 : V = Y_0 \oplus Y_0^{\perp} \longrightarrow Y_0$$

and

$$p_2 : Y_0 = X_0 \oplus \mathbb{R} \longrightarrow X_0.$$

Let W be an open subset consisting of all $Z = (X, Y) \in F_{m-1,m}$ such that $p_1(Y) = Y_0$, $p_2(X) = X_0$. Then $Z = (X, Y) \in W$ can be regarded as the direct product of graphs of two linear mappings $T_1(Y) : Y_0 \to Y_0^{\perp}$ and $T_2(X) : X_0 \to \mathbb{R}$. Let $\{x_1, x_2, \dots, x_{m-1}\}$ be an orthonormal basis of X_0. By adding x_m, we let $\{x_1, x_2, \dots, x_{m-1}, x_m\}$ be an orthonormal basis of Y_0. For each $Z = (X, Y) \in W$, there exist a unique basis $\{u_1, u_2, \dots, u_m\}$ of Y and a unique basis $\{v_1, v_2, \dots, v_{m-1}\}$ of X such that

$$\begin{aligned}
&p_{1_*}(u_1) = x_1, p_{1_*}(u_2) = x_2, \dots, p_{1_*}(u_m) = x_m,\\
&p_{2_*}(v_1) = x_1, p_{2_*}(v_2) = x_2, \dots, p_{2_*}(v_{m-1}) = x_{m-1}, p_{2_*}(v_m) = x_m.
\end{aligned}$$

Then the equations hold:

$$\begin{aligned}
u_i &= x_i + T_1(Y)x_i \quad (i = 1, \dots, m),\\
v_i &= x_i + T_2(X)x_i \quad (i = 1, \dots, m-1).
\end{aligned}$$

Therefore, for the tangent space $T_Z F_{m-1,m}$ at $Z = (X, Y) \in F_{m-1,m}$, we have

$$\begin{aligned}
T_Z F_{m-1,m} &\cong \mathrm{Hom}(Y_0, Y_0^{\perp}) \oplus \mathrm{Hom}(X_0, \mathbb{R})\\
&\cong Y_0^{\perp} \otimes Y_0^* \oplus X_0^*.
\end{aligned}$$

Let U_{m-1}, U_m be the tautological vector bundles with rank $m-1$, m over $F_{m-1,m}$ respectively. Then $U_{m-1} \subset U_m$. Let V_{n+1}, V_n be the quotient bundles of U_{m-1}, U_m in a trivial bundle $F_{m-1,m} \times \mathbb{R}^{n+m}$ respectively. Then $V_{n+1} \supset V_n$. Then we have the following:

$$\begin{aligned}
TF_{m-1,m} &\cong V_n \otimes U_m^* \oplus U_{m-1}^*\\
&\cong V_n \otimes (U_{m-1}^* \oplus 1_{F_{m-1,m}}) \oplus U_{m-1}^*\\
&\cong V_n \oplus U_{m-1}^* \oplus V_n \otimes U_{m-1}^*\\
&= E \oplus F \oplus E \otimes F,
\end{aligned}$$

where we put $E = V_n$ and $F = U_{m-1}^*$.

4.2. Definition

Let R be an r-dimensional real manifold. Let E and F be two differential systems on R, i.e., subbundles with rank k and l $(k + l < r)$ of the tangent bundle TR of R respectively. *A co-Grassmannian structure of type* (k, l) on R is defined by the pair (E, F) which satisfies the following conditions:

(1) E and F are transversal.

(2) Both E and F are completely integable.

(3) The derived system of a differential system $D = E \oplus F$ on R coincides with TR, i.e.,

$$TR = D + [D, D].$$

(4) The quotient bundle TR/D has

$$\operatorname{rank} TR/D = \operatorname{rank} E \cdot \operatorname{rank} F\,(= kl).$$

Here the equation in (3) means the equality by taking the sheaf of germs of local sections at each point of R. The dimension r of R is $k+l+kl$. This is a subclass of pseudo-product structures in the sense of Tanaka ([T2]).

The pair (E, F) is called an almost co-Grassmannian structure of type (k, l) on R if the condition (2) is not necessarily satisfied.

We remark the following. Now, assume globally that the quotient bundle TR/D is isomorphic to the tensor product bundle $E \otimes F$ over R, i.e.,

$$TR \cong E \oplus F \oplus E \otimes F.$$

Then, the leaf space $R_E = R/E$ is of dimension $(k + 1)l$ and

$$TR_E \cong F \oplus E \otimes F = (E \oplus 1_{R_E}) \otimes F.$$

Therefore R_E has a Grassmannian structure of type $(k+1, l)$. The leaf space $R_F = R/F$ is of dimension $k(l + 1)$ and

$$TR_F \cong E \oplus E \otimes F = E \otimes (F \oplus 1_{R_F}).$$

Therefore R_F has a Grassmannian structure of type $(k, l + 1)$.

4.3. Normal Cartan connection

Let R be a manifold with a pair (E, F) that satisfies (1), (2), (3) in 4.2. For each point $x \in R$, with a differential system $D = E \oplus F$ we associate a graded algebra $\mathfrak{m}(x)$ called a symbol algebra of D ([T2]).

For $x \in R$, put

$$\begin{aligned} \mathfrak{g}_{-1}(x) &= D_x, \\ \mathfrak{g}_{-2}(x) &= T_x R / D_x, \\ \mathfrak{m}(x) &= \mathfrak{g}_{-2}(x) \oplus \mathfrak{g}_{-1}(x). \end{aligned}$$

Let ϖ be the projection of TR onto TR/D. We define a bracket operator $[\,\cdot\,,\,\cdot\,]$ in $\mathfrak{m}(x)$ by the requirement that

$$\begin{aligned} &[X_x, Y_x] = \varpi([X, Y]_x), \quad X, Y \in \Gamma(D) \text{ (local sections of } D), \\ &[\mathfrak{g}_{-2}(x), \mathfrak{g}_{-2}(x)] = [\mathfrak{g}_{-2}(x), \mathfrak{g}_{-1}(x)] = 0. \end{aligned}$$

Then we see that $[\,\cdot\,,\,\cdot\,]$ is well-defined and that $\mathfrak{m}(x)$ becomes a Lie algebra. Further $\mathfrak{m}(x)$ is a (truncated) graded Lie algebra of second kind. It is called the symbol algebra of the differential system D at the point x.

Let R be a manifold with a co-Grassmannian structure of type (k, l) equipped with a pair (E, F). The spaces E_x and F_x are subspaces of $\mathfrak{g}_{-1}(x) = D_x$:

$$\mathfrak{g}_{-1}(x) = E_x \oplus F_x.$$

Since E and F are completely integrable,

$$[E_x, E_x] = [F_x, F_x] = 0.$$

Moreover we have

$$\dim \mathfrak{g}_{-2}(x) = \dim E_x \cdot \dim F_x \, (= kl).$$

This implies that

$$\mathfrak{g}_{-2}(x) \cong [E_x, F_x] \cong E_x \otimes F_x.$$

Let $\mathfrak{m} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$ be a (truncated) graded Lie algebra of second kind such that $\mathfrak{g}_p = 0$ for all $p \geq 0$. Let $\mathfrak{e}$ and $\mathfrak{f}$ be subspaces of $\mathfrak{g}_{-1}$ and $\dim \mathfrak{e} = k$, $\dim \mathfrak{f} = l$. A triplet $\mathfrak{L} = \{\mathfrak{m}; \mathfrak{e}, \mathfrak{f}\}$ is called *a graded Lie algebra of type* (k, l) *co-Grassmann* or briefly a type (k, l) CGR if it satisfies the following conditions:

(1) $\mathfrak{g}_{-1} = \mathfrak{e} \oplus \mathfrak{f}$,
(2) $[\mathfrak{e}, \mathfrak{e}] = [\mathfrak{f}, \mathfrak{f}] = 0$,
(3) $\mathfrak{g}_{-2} = [\mathfrak{g}_{-1}, \mathfrak{g}_{-1}]$,
(4) $\dim \mathfrak{g}_{-2} = \dim \mathfrak{e} \cdot \dim \mathfrak{f}\,(= kl)$.

There exist bases $\{\mathbf{e}_1, \ldots, \mathbf{e}_k\}$ of $\mathfrak{e}$ and $\{\mathbf{f}_1, \ldots, \mathbf{f}_l\}$ of $\mathfrak{f}$ such that $\{[\mathbf{e}_i, \mathbf{f}_j]\}_{1 \le i \le k,\, 1 \le j \le l}$ of $\mathfrak{g}_{-2}$. By corresponding $[\mathbf{e}_i, \mathbf{f}_j]$ to $\mathbf{e}_i \otimes \mathbf{f}_j$, from (2), (3), (4), it follows that $\mathfrak{g}_{-2} = [\mathfrak{e}, \mathfrak{f}] \cong \mathfrak{e} \otimes \mathfrak{f}$. A graded Lie algebra of type (k, l) CGR is uniquely determined up to isomorphisms by the conditions above. A triplet $\mathfrak{L}$ is nothing but $\mathfrak{m}$, $\mathfrak{e}$, $\mathfrak{f}$ defined in 4.1 for $\mathfrak{g} = \mathfrak{sl}(m+n, \mathbb{R})$.

Let $\mathfrak{g}_0(\mathfrak{m})$ be the gradation preserving derivation algebra $\mathrm{Der}(\mathfrak{m})$ of $\mathfrak{m}$ and $\mathrm{Der}(\mathfrak{L})$ of $\mathfrak{L}$ be as follows:

$$\mathrm{Der}(\mathfrak{L}) = \{X \in \mathfrak{g}_0(\mathfrak{m}) \mid X\mathfrak{e} \subset \mathfrak{e},\ X\mathfrak{f} \subset \mathfrak{f}\}.$$

We denote $\mathrm{Der}(\mathfrak{L})$ by $\mathfrak{g}_0$. The prolongation $\mathfrak{g} = \mathfrak{g}(\mathfrak{m}, \mathfrak{g}_0)$ of $(\mathfrak{m}, \mathfrak{g}_0)$ is called the prolongation of type CGR $\mathfrak{L}$. Then $\mathfrak{g}$ is finite dimensional ([T2]). In the case $k \neq 1$, $l \neq 1$, since we can easily see that

$$\mathfrak{g}_0 = \mathfrak{g}_0(\mathfrak{m}),$$

the prolongation $\mathfrak{g}$ of type CGR $\mathfrak{L}$ becomes the prolongation $\mathfrak{g}(\mathfrak{m})$ of $\mathfrak{m}$. According to Yamaguchi ([Y, Theorem 5.3], cf. [Ka]), if $\dim \mathfrak{e} = n$, $\dim \mathfrak{f} = m - 1$, it follows that $\mathfrak{g} = \mathfrak{sl}(m+n, \mathbb{R})$. The group $G_0 = \mathrm{Aut}(\mathfrak{L})$ is the Lie group of $\mathfrak{g}_0$. And if we let G and G' be the Lie groups of $\mathfrak{g}$ and $\mathfrak{g}' = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$ respectively, we have

$$F_{m-1,m} \cong G/G'.$$

Let R be a manifold with a co-Grassmannian structure of type (k, l) equipped with a pair (E, F). Then, at each point $x \in R$, the symbol algebra $\mathfrak{m}(x)$ of a differential system $D = E \oplus F$ is isomorphic to a graded Lie algebra $\mathfrak{L} = \{\mathfrak{m}; \mathfrak{e}, \mathfrak{f}\}$ of type (k, l) CGR. Conversely, let R be a manifold with a differential system D of $\mathfrak{L} = \{\mathfrak{m}; \mathfrak{e}, \mathfrak{f}\}$ of type (k, l) CGR. Then a co-Grassmannian structure of type (k, l) equipped with a pair (E, F) on R is defined. At each point $x \in R$, the symbol algebra $\mathfrak{m}(x)$ of $D = E \oplus F$ is isomorphic to $\mathfrak{L}$.

According to Tanaka ([T1], [T2]), we have the following.

THEOREM 4.1. *Let R be a manifold with a co-Grassmannian structure of type (k, l). Then there exist a principal bundle Q with structure group G' over R and a unique normal Cartan connection ω of type G/G' on Q.*

4.4. Nonzero generators in H^2

We consider the simple graded Lie algebra

$$\begin{aligned}\mathfrak{g} &= \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2,\\ \mathfrak{m} &= \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1},\\ \mathfrak{g}_{-1} &= \mathfrak{e} \oplus \mathfrak{f}\end{aligned}$$

of second kind of type (k,l) CGR for the Lie algebra $\mathfrak{g} = \mathfrak{sl}(m+n,\mathbb{R})$ of $G = SL(m+n,\mathbb{R})$ $(m+n = k+l+1)$ associated with a co-Grassmannian structure of type (k,l). We write it down explicitly:

$$\begin{aligned}\mathfrak{g} &= \mathfrak{sl}(m+n,\mathbb{R})\\ &= \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2\\ &= \left\{\begin{pmatrix} O_l & 0 & 0\\ 0 & 0 & 0\\ A & 0 & O_k\end{pmatrix}\right\} \oplus \left\{\begin{pmatrix} O_l & 0 & 0\\ \mathbf{f} & 0 & 0\\ 0 & \mathbf{e} & O_k\end{pmatrix}\right\} \oplus \left\{\begin{pmatrix} B & 0 & 0\\ 0 & d & 0\\ 0 & 0 & C\end{pmatrix}\right\}\\ &\qquad \oplus \left\{\begin{pmatrix} O_l & {}^t\mathbf{h} & 0\\ 0 & 0 & {}^t\mathbf{g}\\ 0 & 0 & O_k\end{pmatrix}\right\} \oplus \left\{\begin{pmatrix} O_l & 0 & D\\ 0 & 0 & 0\\ 0 & 0 & O_k\end{pmatrix}\right\},\\ &\qquad (A \in \mathrm{Mat}(k\times l,\mathbb{R}),\ B \in \mathrm{Mat}(l,\mathbb{R}),\ C \in \mathrm{Mat}(k,\mathbb{R}),\\ &\qquad D \in \mathrm{Mat}(l\times k,\mathbb{R}),\ d \in \mathbb{R},\ \mathbf{e},\mathbf{g} \in \mathbb{R}^k,\ \mathbf{f},\mathbf{h} \in \mathbb{R}^l,\\ &\qquad \operatorname{trace} B + d + \operatorname{trace} C = 0),\\ \mathfrak{e} &= \left\{\begin{pmatrix} O_l & 0 & 0\\ 0 & 0 & 0\\ 0 & \mathbf{e} & O_k\end{pmatrix}\right\}, \quad \mathfrak{f} = \left\{\begin{pmatrix} O_l & 0 & 0\\ \mathbf{f} & 0 & 0\\ 0 & 0 & O_k\end{pmatrix}\right\},\\ [\mathfrak{e},\mathfrak{e}] &= [\mathfrak{f},\mathfrak{f}] = 0, \quad \mathfrak{g}_{-2} = [\mathfrak{e},\mathfrak{f}] \cong \mathfrak{e}\otimes\mathfrak{f},\\ [\mathfrak{g}_i,\mathfrak{g}_j] &\subset \mathfrak{g}_{i+j}.\end{aligned}$$

We put their bases and coordinates as follows:

$$\begin{aligned}\mathbf{e}_{ij} &= \begin{pmatrix} O_l & 0 & 0\\ 0 & 0 & 0\\ E_{ij} & 0 & O_k\end{pmatrix}, \quad \mathbf{e}_i = \begin{pmatrix} O_l & 0 & 0\\ 0 & 0 & 0\\ 0 & e_i & O_k\end{pmatrix}, \quad \mathbf{f}_i = \begin{pmatrix} O_l & 0 & 0\\ f_i & 0 & 0\\ 0 & 0 & O_k\end{pmatrix},\\ &(E_{ij} \in \mathrm{Mat}(k\times l,\mathbb{R}) : \text{matrix unit, i.e., } (i,j)\text{-component} = 1,\\ &\ \text{otherwise} = 0,\\ &\ e_i \in \mathbb{R}^k,\ f_i \in \mathbb{R}^l : i\text{-component} = 1, \text{ otherwise} = 0),\end{aligned}$$

$$\mathbf{g}_{ij} = \begin{pmatrix} E_{ij} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & O_k \end{pmatrix}, \quad \mathbf{h}_{ij} = \begin{pmatrix} O_l & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & E_{ij} \end{pmatrix},$$

$$(E_{ij} \in \mathrm{Mat}(l, \mathbb{R}),\ E_{ij} \in \mathrm{Mat}(k, \mathbb{R}) : \text{matrix units respectively})$$

and

$$\begin{pmatrix} g_{ij} & {}^t h_j & d_{ij} \\ y_j & d & {}^t g_i \\ x_{ij} & x_i & h_{ij} \end{pmatrix} = \sum x_{ij}\mathbf{e}_{ij} + \sum x_i \mathbf{e}_i + \sum y_j \mathbf{f}_j + \sum g_{ij}\mathbf{g}_{ij} + \sum h_{ij}\mathbf{h}_{ij} + \cdots.$$

As in 3.3, we calculate $H^2 \cong H^2(\mathfrak{m}, \mathfrak{g})$. We have

$$\begin{aligned} \mathfrak{g} &= \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2, \\ \mathfrak{g}_{-1} &= \mathfrak{g}_{-1}^R \oplus \mathfrak{g}_{-1}^L, \end{aligned}$$

where

$$\mathfrak{g}_{-1}^R = \mathfrak{f}, \quad \mathfrak{g}_{-1}^L = \mathfrak{e}.$$

Further we have

$$\Lambda^2\mathfrak{m}^* = \Lambda^2_{-4} \oplus \Lambda^2_{-3} \oplus \Lambda^2_{-2},$$

where

$$\begin{aligned} \Lambda^2_{-4} &= \Lambda^2\mathfrak{g}^*_{-2}, \\ \Lambda^2_{-3} &= \mathfrak{g}^*_{-2} \wedge \mathfrak{g}^{L*}_{-1} \oplus \mathfrak{g}^*_{-2} \wedge \mathfrak{g}^{R*}_{-1}, \\ \Lambda^2_{-2} &= \Lambda^2\mathfrak{g}^{L*}_{-1} \oplus \mathfrak{g}^{L*}_{-1} \wedge \mathfrak{g}^{R*}_{-1} \oplus \Lambda^2\mathfrak{g}^{R*}_{-1}. \end{aligned}$$

We remark that

$$C^2 = \bigoplus_{p=-1}^{5} C^{p,2},$$

where

$$\begin{aligned} C^{-1,2} &= && && \mathfrak{g}_{-2} \otimes \Lambda^2_{-2}, \\ C^{0,2} &= && \mathfrak{g}_{-2} \otimes \Lambda^2_{-3} \oplus && \mathfrak{g}_{-1} \otimes \Lambda^2_{-2}, \\ C^{1,2} &= \mathfrak{g}_{-2} \otimes \Lambda^2_{-4} \oplus && \mathfrak{g}_{-1} \otimes \Lambda^2_{-3} \oplus && \mathfrak{g}_0 \otimes \Lambda^2_{-2}, \\ C^{2,2} &= \mathfrak{g}_{-1} \otimes \Lambda^2_{-4} \oplus && \mathfrak{g}_0 \otimes \Lambda^2_{-3} \oplus && \mathfrak{g}_1 \otimes \Lambda^2_{-2}, \\ C^{3,2} &= \mathfrak{g}_0 \otimes \Lambda^2_{-4} \oplus && \mathfrak{g}_1 \otimes \Lambda^2_{-3} \oplus && \mathfrak{g}_2 \otimes \Lambda^2_{-2}, \\ C^{4,2} &= \mathfrak{g}_1 \otimes \Lambda^2_{-4} \oplus && \mathfrak{g}_2 \otimes \Lambda^2_{-3}, \\ C^{5,2} &= \mathfrak{g}_2 \otimes \Lambda^2_{-4}. \end{aligned}$$

Then we obtain the following.

Proposition 4.1. *The components of nonzero generators as $\mathfrak{g}_0$-modules in H^2 are represented by*

1. *if* $l \geq 3$,
   (i) $\mathbf{e}_{k1} \otimes (\mathbf{f}_l^* \wedge \mathbf{e}_{1l}^*) \in H^{0,2} \subset \mathfrak{g}_{-2} \otimes (\mathfrak{g}_{-2}^* \wedge (\mathfrak{g}_{-1}^R)^*)$,
   (ii) $\mathbf{e}_{k1} \otimes (\mathbf{e}_{1l}^* \wedge \mathbf{e}_1^*) \in H^{0,2} \subset \mathfrak{g}_{-2} \otimes (\mathfrak{g}_{-2}^* \wedge (\mathfrak{g}_{-1}^L)^*)$,
2. *if* $l = 2$,
   (a) *for* $k + l + 1 \geq 6$, *i.e.*, $k \geq 3$,
      (i) $\mathbf{e}_k \otimes (\mathbf{f}_1^* \wedge \mathbf{f}_2^*) \in H^{0,2} \subset \mathfrak{g}_{-1}^L \otimes \Lambda^2(\mathfrak{g}_{-1}^R)^*$,
      (ii) $\mathbf{e}_{k1} \otimes (\mathbf{e}_{1l}^* \wedge \mathbf{f}_l^*) \in H^{0,2} \subset \mathfrak{g}_{-2} \otimes (\mathfrak{g}_{-2}^* \wedge (\mathfrak{g}_{-1}^R)^*)$,
      (iii) $\mathbf{e}_{k1} \otimes (\mathbf{e}_{1l}^* \wedge \mathbf{e}_1^*) \in H^{0,2} \subset \mathfrak{g}_{-2} \otimes (\mathfrak{g}_{-2}^* \wedge (\mathfrak{g}_{-1}^L)^*)$,
   (b) *for* $k + l + 1 = 5$, *i.e.*, $k = 2$,
      (i) $\mathbf{e}_2 \otimes (\mathbf{f}_1^* \wedge \mathbf{f}_2^*) \in H^{0,2} \subset \mathfrak{g}_{-1}^L \otimes \Lambda^2(\mathfrak{g}_{-1}^R)^*$,
      (ii) $\mathbf{e}_{21} \otimes (\mathbf{e}_{12}^* \wedge \mathbf{f}_2^*) \in H^{0,2} \subset \mathfrak{g}_{-2} \otimes (\mathfrak{g}_{-2}^* \wedge (\mathfrak{g}_{-1}^R)^*)$,
      (iii) $\mathbf{e}_{21} \otimes (\mathbf{e}_{12}^* \wedge \mathbf{e}_1^*) \in H^{0,2} \subset \mathfrak{g}_{-2} \otimes (\mathfrak{g}_{-2}^* \wedge (\mathfrak{g}_{-1}^L)^*)$,
      (iv) $\mathbf{f}_1 \otimes (\mathbf{e}_1^* \wedge \mathbf{e}_2^*) \in H^{0,2} \subset \mathfrak{g}_{-1}^R \otimes \Lambda^2(\mathfrak{g}_{-1}^L)^*$,
3. *if* $l = 1$,
   (a) *for* $k + l + 1 \geq 5$, *i.e.*, $k \geq 3$,
      (i) $\mathbf{e}_k \otimes (\mathbf{e}_{1(1)}^* \wedge \mathbf{f}^*) \in H^{1,2} \subset \mathfrak{g}_{-1}^L \otimes (\mathfrak{g}_{-2}^* \wedge (\mathfrak{g}_{-1}^R)^*)$,
      (ii) $\mathbf{h}_{k1} \otimes (\mathbf{e}_{1(1)}^* \wedge \mathbf{e}_1^*) \in H^{2,2} \subset \mathfrak{g}_0 \otimes (\mathfrak{g}_{-2}^* \wedge (\mathfrak{g}_{-1}^L)^*)$,
      therefore, $K^0 = 0$, $K^1 = HK^1$,
   (b) *for* $k + l + 1 = 4$, *i.e.*, $k = 2$,
      (i) $\mathbf{f} \otimes (\mathbf{e}_1^* \wedge \mathbf{e}_2^*) \in H^{0,2} \subset \mathfrak{g}_{-1}^R \otimes \Lambda^2(\mathfrak{g}_{-1}^L)^*$,
      (ii) $\mathbf{e}_2 \otimes (\mathbf{e}_{1(1)}^* \wedge \mathbf{f}^*) \in H^{1,2} \subset \mathfrak{g}_{-1}^L \otimes (\mathfrak{g}_{-2}^* \wedge (\mathfrak{g}_{-1}^R)^*)$,
      (iii) $\mathbf{h}_{21} \otimes (\mathbf{e}_{1(1)}^* \wedge \mathbf{e}_1^*) \in H^{2,2} \subset \mathfrak{g}_0 \otimes (\mathfrak{g}_{-2}^* \wedge (\mathfrak{g}_{-1}^L)^*)$.

§5. Null n-plane bundle

5.1. Definition

In the vector space $\mathbb{R}^{mn} = \mathbb{R}^n \otimes \mathbb{R}^m$, a vector u is called *null* (simple, or decomposable) if there exist a vector v belonging to $\mathbb{R}^n$ and a vector w belonging to $\mathbb{R}^m$ such that $u = v \otimes w$.

Let $\{e_i\}$ $(1 \leq i \leq n)$ be a basis of $\mathbb{R}^n$ and $\{f_j\}$ $(1 \leq j \leq m)$ a basis of $\mathbb{R}^m$. For $v = \sum_{i=1}^n \alpha_i e_i$, $w = \sum_{j=1}^m \beta_j f_j$,

$$\begin{aligned} u = v \otimes w &= \Big(\sum_{i=1}^n \alpha_i e_i\Big) \otimes \Big(\sum_{j=1}^m \beta_j f_j\Big) \\ &= \sum_{i,j} \alpha_i \beta_j \, e_i \otimes f_j = (\alpha_i \beta_j)_{1\leq i\leq n,\, 1\leq j\leq m}. \end{aligned}$$

In particular, each vector $e_i \otimes f_j$ that makes a basis of $\mathbb{R}^{mn} = \mathbb{R}^n \otimes \mathbb{R}^m$ is null.

A k-dimensional subspace of $\mathbb{R}^{mn} = \mathbb{R}^n \otimes \mathbb{R}^m$ is called a null k-plane (or an isotropic k-plane) if each vector in the k-dimensional subspace is a null vector. Note that $k \leq \max(n, m)$ holds. We consider the set of all n-planes with forms $\{\mathbb{R}^n \otimes w \mid w \in \mathbb{R}^m\}$. It has a $P^{m-1}(\mathbb{R})$ family. Each the null n-plane is called *a null n-plane*. And moreover we consider the set of all null m-planes with forms $\{v \otimes \mathbb{R}^m \mid v \in \mathbb{R}^n\}$. It has a $P^{n-1}(\mathbb{R})$ family. Each the null m-plane is called *a null m-plane*. We remark that the intersection of each null n-plane and each null m-plane is 1-dimensional subspace.

Let M be a manifold with a Grassmannian structure of type (n, m). Considering a set of all the null n-planes in the tangent space at each point of M, we have a fibre bundle F_L with fibre $P^{m-1}(\mathbb{R})$ over M, called *a null n-plane bundle*:

$$\begin{array}{ccc} F_L & \longleftarrow & P^{m-1}(\mathbb{R}) \\ \Big\downarrow{\scriptstyle \varpi_L} & & \\ M & & . \end{array}$$

5.2. Tautological distribution

An n-dimensional distribution D on the null n-plane bundle F_L over M is called *an n-dimensional tautological distribution of null n-planes* if it satisfies the following: for the n-dimensional subspace $D_\Pi \subset T_\Pi F_L$,

$$\varpi_{L_*}(D_\Pi) = \Pi \subset T_{\varpi_L(\Pi)} M.$$

Here note that $\Pi \in F_L$ is a null n-plane in $T_{\varpi_L(\Pi)} M$.

From now, we will define an n-dimensional tautological distribution D of null n-planes on F_L. When we consider the horizontal lift of a null n-plane Π in M to the null n-plane bundle F_L by a connection, we note that

F_L, which is the bundle associated with the linear frame bundle P with structure group $GR(n,m)$ on M, does not have a canonical connection. But F_L is the bundle associated with the frame bundle Q of second order with structure group G' on M as well. We observe from 2.3 that there exists a unique Cartan connection on Q. Considering the horizontal lift of Π in M to Q and doing the reduction to F_L, we can define an n-dimensional tautological distribution D_L of null n-planes on F_L. We will describe the argument fully in the following.

Let M be a manifold with a Grassmannian structure of type (n,m). We identify TM with $V \otimes W$ under σ. For $x \in M$,

$$T_xM = V_x \otimes W_x,$$

where V_x is an n-dimensional vector space and W_x an m-dimensional vector space. Take a basis $\{e_i\}$ $(1 \le i \le n)$ of V_x and a basis $\{f_j\}$ $(1 \le j \le m)$ of W_x. A set $\{e_i \otimes f_j\}$ $(1 \le i \le n,\ 1 \le j \le m)$ is a null basis of $V_x \otimes W_x$. Therefore $\lambda_x = (e_1 \otimes f_1, e_2 \otimes f_1, \dots, e_n \otimes f_1, \dots, e_1 \otimes f_m, e_2 \otimes f_m, \dots, e_n \otimes f_m)$ belongs to P.

Put

$$\Pi_{L_x} = \mathrm{span}(e_1 \otimes f_m, e_2 \otimes f_m, \dots, e_n \otimes f_m).$$

The space Π_{L_x} is a null n-plane in T_xM. Namely, Π_{L_x} belongs to F_L. Then a mapping $p_L : P \to F_L$ is defined by

$$p_L : \lambda_x \longmapsto \Pi_{L_x}.$$

A subgroup of $G_0 = GR(n,m)$ which leaves the null n-plane Π_{L_x} invariant is

$$H_{0L} = S(\{B\} \times GL(n,\mathbb{R})),$$

where

$$B = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m-1} & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m-1} & a_{2m} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m-11} & a_{m-12} & \cdots & a_{m-1m-1} & a_{m-1m} \\ 0 & 0 & \cdots & 0 & a_{mm} \end{pmatrix} \in GL(m,\mathbb{R}).$$

Consequently we define a principal bundle $P(F_L, H_{0L}, p_L)$. The null n-plane bundle F_L over M is the fibre bundle with fibre $G_0/H_{0L} \cong P^{m-1}(\mathbb{R})$ associated with P:

$$F_L = P \times_{G_0} G_0/H_{0L} = P/H_{0L}.$$

For a linear frame bundle P with structure group $G_0 = GR(n,m)$ over M, let us consider the frame bundle Q of second order with structure group G'. Let π_P be a canonical projection $Q \to P$. Then the null n-plane bundle F_L over M is the fibre bundle with fibre $G'/H'_L \cong P^{m-1}(\mathbb{R})$ associated with Q:

$$F_L = Q \times_{G'} G'/H'_L = Q/H'_L.$$

Consequently a mapping $\pi_L : Q \to F_L$ being defined, we define a principal bundle $Q(F_L, H'_L, \pi_L)$.

Summarizing them, we have the following diagram:

$$\begin{tikzcd}
G' \arrow[r] & Q \arrow[d, "\pi_P"] \arrow[ddd, "\pi_M"'] \arrow[ddr, "\pi_L"] & H'_L \arrow[l] & \\
G_0 \arrow[r] & P \arrow[dd, "\pi"] \arrow[dr, "p_L"] & H_{0L} \arrow[l] \arrow[d] & \\
 & & F_L \arrow[dl, "\varpi_L"] & G_0/H_{0L} \cong G'/H'_L \cong P^{m-1}(\mathbb{R}). \arrow[l] \\
 & M & &
\end{tikzcd}$$

From 2.3, there exists a unique $\mathfrak{g} = \mathfrak{sl}(m+n,\mathbb{R})$-valued Cartan connection ω of type G/G' on Q. For $v \in Q$, the linear isomorphism

$$\omega : T_vQ \longrightarrow \mathfrak{g}$$

is defined. The Lie algebra $\mathfrak{g} = \mathfrak{sl}(m+n,\mathbb{R})$ of $G = SL(m+n,\mathbb{R})$ is a graded Lie algebra of first kind:

$$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1.$$

We identify $\mathbb{R}^{mn} = \mathbb{R}^n \otimes \mathbb{R}^m$ with $\mathfrak{g}_{-1}$ as follows:

$$^t(0,\dots,0,\overset{i}{1},0,\dots,0) \otimes (0,\dots,0,\overset{j}{1},0,\dots,0) \longleftrightarrow \mathbf{e}_{ij} = \begin{pmatrix} O_m & 0 \\ E_{ij} & O_n \end{pmatrix},$$

where $E_{ij} \in \mathrm{Mat}(n \times m, \mathbb{R})$ is a matrix unit, i.e., (i,j)-component is 1 and otherwise 0.

A subspace

$$\mathfrak{n}_L = \mathrm{span}(\mathbf{e}_{1m}, \mathbf{e}_{2m}, \dots, \mathbf{e}_{nm})$$

spanned by $\mathbf{e}_{1m}, \mathbf{e}_{2m}, \ldots, \mathbf{e}_{nm}$ is a null n-plane. The Lie algebra $\mathfrak{h}_L$ of H'_L is a subalgebra of $\mathfrak{g}' = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ and has the following form:

$$\mathfrak{h}_L = \left\{ \left(\begin{array}{ccccc|c} a_{11} & a_{12} & \cdots & a_{1m-1} & a_{1m} & \\ a_{21} & a_{22} & \cdots & a_{2m-1} & a_{2m} & \\ \vdots & \vdots & \ddots & \vdots & \vdots & D \\ a_{m-11} & a_{m-12} & \cdots & a_{m-1m-1} & a_{m-1m} & \\ 0 & 0 & \cdots & 0 & a_{mm} & \\ \hline & & O & & & C \end{array} \right) \right\} \subset \mathfrak{g}',$$

where $a_{ij} \in \mathbb{R}$, $C \in \mathrm{Mat}(n, \mathbb{R})$, $D \in \mathrm{Mat}(m \times n, \mathbb{R})$, and $\sum_{i=1}^m a_{ii} + \operatorname{trace} C = 0$.

For the vector subspace $\mathfrak{n}_L + \mathfrak{h}_L$ of $\mathfrak{g}$, we have the following:

Lemma 5.1. *The space $\mathfrak{n}_L + \mathfrak{h}_L$ is invariant under the adjoint actions of H'_L and $\mathfrak{h}_L$.*

Remark that the space $\mathfrak{n}_L$ is invariant under the adjoint action of H_{0L}.

Let $v \in Q$. Let $x = \pi_M(v) \in M$. Let $\lambda_x = \pi_P(v)$ and $\Pi_{L_x} = \pi_L(v)$. Then $p_L(\lambda_x) = \Pi_{L_x}$ holds. An element $\lambda_x = (e_1 \otimes f_1, \ldots, e_n \otimes f_1, \ldots, e_1 \otimes f_m, \ldots, e_n \otimes f_m) \in P$ is regarded as an isomorphism $\lambda_x : \mathbb{R}^{mn} (= \mathbb{R}^n \otimes \mathbb{R}^m) = \mathfrak{g}_{-1} \to T_x M$:

$$\mathbf{e}_{ij} \longmapsto e_i \otimes f_j.$$

Then

$$\mathfrak{n}_L \longmapsto \Pi_{L_x}$$

holds.

By using the normal Cartan connection ω, vectors $\omega^{-1}(\mathbf{e}_{ij}) \in T_v Q$ are the horizontal lift of vectors $e_i \otimes f_j \in T_x M$.

Next, putting

$$\mathcal{D}_{L_v} = \omega^{-1}(\mathfrak{n}_L + \mathfrak{h}_L),$$

we can define a distribution $\mathcal{D}_L$ on Q. From the lemma above, we have the following:

Lemma 5.2. *The distribution $\mathcal{D}_L$ on Q is invariant under the right action of H'_L.*

Therefore an n-dimensional distribution D_L is defined on the null n-plane bundle $F_L = Q/H'_L$:

$$D_L = \mathcal{D}_L \bmod H'_L.$$

This is a tautological distribution of null n-planes.

Investigating the complete integrability of the distribution D_L on F_L is equivalent to investigating the complete integrability of the distribution $\mathcal{D}_L$ on Q modulo H'_L.

Lemma 5.3. *We have*

$$\begin{aligned}&[D_L, D_L] \subset D_L \text{ on } F_L\\ &\quad \Longleftrightarrow [\mathcal{D}_L \bmod H'_L, \mathcal{D}_L \bmod H'_L] \subset \mathcal{D}_L \bmod H'_L \text{ on } Q.\end{aligned}$$

For $v \in Q$, elements in T_vQ

$$\tilde{\mathbf{e}}_{im|v} = \omega^{-1}(\mathbf{e}_{im}), \quad 1 \le i \le n,$$

are defined. Put

$$\tilde{\mathfrak{n}}_L = \omega^{-1}(\mathfrak{n}_L).$$

We will investigate the condition modulo H'_L satisfying

$$[\tilde{\mathbf{e}}_{im}, \tilde{\mathbf{e}}_{jm}] \in \tilde{\mathfrak{n}}_L,$$

for vector fields $\tilde{\mathbf{e}}_{im}$ $(1 \le i \le n)$ on Q.

Before investigating the complete integrability of the distribution, we prepare several lemmas.

Corresponding to the decomposition

$$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1,$$

the normal Cartan connection form ω and the curvature form Ω are decomposed respectively as follows:

$$\begin{aligned}\omega &= \omega_{-1} \oplus \omega_0 \oplus \omega_1,\\ \Omega &= \Omega_{-1} \oplus \Omega_0 \oplus \Omega_1.\end{aligned}$$

Put

$$\Omega' = \Omega_0 \oplus \Omega_1.$$

Recall that the curvature form Ω is defined as follows:

$$\Omega = d\omega + \frac{1}{2}[\omega, \omega].$$

Denote by hX and vX the components of the decomposition of $\omega^{-1}(\mathfrak{g}_{-1})$ and $\omega^{-1}(\mathfrak{g}_0 \oplus \mathfrak{g}_1)$ respectively (cf. a horizontal component and a vertical component). Denote by A^* the fundamental vector field on Q corresponding to $A \in \mathfrak{g}_0 \oplus \mathfrak{g}_1$.

First we remark the following lemma. Put $\mathbf{e}_i = \mathbf{e}_{im}$ $(1 \leq i \leq n)$. For an element $\tilde{\mathbf{e}}_{i|va}$ $(1 \leq i \leq n)$ in $T_{va}Q$ at $R_a v = va$ $(a \in G')$, we have the relation to an element in T_vQ:

Lemma 5.4.

$$\tilde{\mathbf{e}}_{i|va} = R_{a*}\omega_{|v}^{-1}(Ad(a)\mathbf{e}_i) \quad (1 \leq i \leq n).$$

Proof. At $v \in Q$, take $\tilde{X}$ such that $\omega(\tilde{X}) = Ad(a)\mathbf{e}_i$: $\tilde{X} = \omega^{-1}(Ad(a)\mathbf{e}_i)$. At $va \in Q$,

$$\omega(R_{a*}\tilde{X}) = R_a^*\omega(\tilde{X}) = Ad(a^{-1}) \cdot \omega(\tilde{X}) = Ad(a^{-1}) \cdot Ad(a)\mathbf{e}_i = \mathbf{e}_i.$$

Therefore

$$\tilde{\mathbf{e}}_{i|va} = \omega^{-1}(\mathbf{e}_i) = R_{a*}\tilde{X} = R_{a*}\omega^{-1}(Ad(a)\mathbf{e}_i).$$

$\square$

Next we have the relations between vX, hX and Ω', Ω_{-1} respectively.

Lemma 5.5. *We have*
(i) $v[\tilde{\mathbf{e}}_i, \tilde{\mathbf{e}}_j] = -2\Omega'(\tilde{\mathbf{e}}_i, \tilde{\mathbf{e}}_j)^*$ $(1 \leq i \leq n)$,
(ii) $v[\tilde{\mathbf{e}}_i, \tilde{\mathbf{e}}_j]_{|va} = -2(Ad(a^{-1})) \cdot \Omega'\big(\omega_{|v}^{-1}(Ad(a)\mathbf{e}_i), \omega_{|v}^{-1}(Ad(a)\mathbf{e}_j)\big)^*$.

Proof. (i): The following holds:

$$\begin{aligned}\Omega_0(\tilde{\mathbf{e}}_i, \tilde{\mathbf{e}}_j) &= d\omega_0(\tilde{\mathbf{e}}_i, \tilde{\mathbf{e}}_j) + \frac{1}{2}[\omega_{-1}(\tilde{\mathbf{e}}_i), \omega_1(\tilde{\mathbf{e}}_j)] + \frac{1}{2}[\omega_0(\tilde{\mathbf{e}}_i), \omega_0(\tilde{\mathbf{e}}_j)] \\ &= d\omega_0(\tilde{\mathbf{e}}_i, \tilde{\mathbf{e}}_j) = -\frac{1}{2}\omega_0([\tilde{\mathbf{e}}_i, \tilde{\mathbf{e}}_j]).\end{aligned}$$

Thus

$$\omega_0([\tilde{\mathbf{e}}_i, \tilde{\mathbf{e}}_j]) = -2\Omega_0(\tilde{\mathbf{e}}_i, \tilde{\mathbf{e}}_j).$$

Similarly, from $\Omega_1 = d\omega_1 + [\omega_1, \omega_0]$,

$$\omega_1([\tilde{\mathbf{e}}_i, \tilde{\mathbf{e}}_j]) = -2\Omega_1(\tilde{\mathbf{e}}_i, \tilde{\mathbf{e}}_j).$$

Hence

$$\begin{aligned} v[\tilde{\mathbf{e}}_i, \tilde{\mathbf{e}}_j] &= \omega^{-1}(\omega_0([\tilde{\mathbf{e}}_i, \tilde{\mathbf{e}}_j]) + \omega_1([\tilde{\mathbf{e}}_i, \tilde{\mathbf{e}}_j])) \\ &= \omega^{-1}(-2\Omega_0(\tilde{\mathbf{e}}_i, \tilde{e}_j) - 2\Omega_1(\tilde{\mathbf{e}}_i, \tilde{\mathbf{e}}_j)) \\ &= -2\omega^{-1}\Omega'(\tilde{\mathbf{e}}_i, \tilde{\mathbf{e}}_j) \\ &= -2\Omega'(\tilde{\mathbf{e}}_i, \tilde{\mathbf{e}}_j)^*. \end{aligned}$$

(ii): It follows that

$$\begin{aligned} v[\tilde{\mathbf{e}}_i, \tilde{\mathbf{e}}_j]_{|va} &= v[R_{a*}\omega^{-1}(Ad(a)\mathbf{e}_i), R_{a*}\omega^{-1}(Ad(a)\mathbf{e}_j)] \\ &= vR_{a*}[\omega^{-1}(Ad(a)\mathbf{e}_i), \omega^{-1}(Ad(a)\mathbf{e}_j)] \\ &= R_{a*}v[\omega^{-1}(Ad(a)\mathbf{e}_i), \omega^{-1}(Ad(a)\mathbf{e}_j)] \\ &= -2R_{a*}\Omega'(\omega^{-1}(Ad(a)\mathbf{e}_i), \omega^{-1}(Ad(a)\mathbf{e}_j))^* \\ &= -2(Ad(a^{-1}) \cdot \Omega'(\omega^{-1}(Ad(a)\mathbf{e}_i), \omega^{-1}(Ad(a)\mathbf{e}_j))^*, \end{aligned}$$

where the third equality is obtained from $vR_{a*} = R_{a*}v$ and the fifth equality from $(R_{a*}A^*_v)_{|va} = (Ad(a^{-1})A)^*_{|va}$. □

LEMMA 5.6. *We have*
(i) $h[\tilde{\mathbf{e}}_i, \tilde{\mathbf{e}}_j] = -2\omega^{-1}(\Omega_{-1}(\tilde{\mathbf{e}}_i, \tilde{\mathbf{e}}_j))$ $(1 \le i \le n)$,
(ii) $h[\tilde{\mathbf{e}}_i, \tilde{\mathbf{e}}_j]_{|va} = -2\omega^{-1}\big(a^{-1}\Omega_{-1}\omega^{-1}_{|v}(Ad(a)\mathbf{e}_i, \omega^{-1}_{|v}(Ad(a)\mathbf{e}_j))\big)$.

Proof. (i): From $\Omega_{-1} = d\omega_{-1} + [\omega_{-1}, \omega_0]$,

$$\begin{aligned} \Omega_{-1}(\tilde{\mathbf{e}}_i, \tilde{\mathbf{e}}_j) &= d\omega_{-1}(\tilde{\mathbf{e}}_i, \tilde{\mathbf{e}}_j) + [\omega_{-1}(\tilde{\mathbf{e}}_i), \omega_0(\tilde{\mathbf{e}}_j)] \\ &= d\omega_{-1}(\tilde{\mathbf{e}}_i, \tilde{\mathbf{e}}_j) \\ &= \frac{1}{2}(\tilde{\mathbf{e}}_i \cdot \omega_{-1}(\tilde{\mathbf{e}}_j) - \tilde{\mathbf{e}}_j \cdot \omega_{-1}(\tilde{\mathbf{e}}_i) - \omega_{-1}([\tilde{\mathbf{e}}_i, \tilde{\mathbf{e}}_j])) \\ &= -\frac{1}{2}\omega_{-1}([\tilde{\mathbf{e}}_i, \tilde{\mathbf{e}}_j] = -\frac{1}{2}h[\tilde{\mathbf{e}}_i, \tilde{\mathbf{e}}_j]. \end{aligned}$$

Here note that $\tilde{\mathbf{e}}_i$, $\tilde{\mathbf{e}}_j$ are the basic vector fields of $\mathbf{e}_i$, $\mathbf{e}_j$, that is to say, $e_i \otimes f_m$, $e_j \otimes f_m$. Hence

$$h[\tilde{\mathbf{e}}_i, \tilde{\mathbf{e}}_j] = -2\Omega_{-1}(\tilde{\mathbf{e}}_i, \tilde{\mathbf{e}}_j).$$

(ii): It follows that

$$\begin{aligned} h[\tilde{\mathbf{e}}_i, \tilde{\mathbf{e}}_j]_{|va} &= h[R_{a*}\omega^{-1}(Ad(a)\mathbf{e}_i), R_{a*}\omega^{-1}(Ad(a)\mathbf{e}_j)] \\ &= hR_{a*}[\omega^{-1}(Ad(a)\mathbf{e}_i), \omega^{-1}(Ad(a)\mathbf{e}_j)] \\ &= R_{a*}h[\omega^{-1}(Ad(a)\mathbf{e}_i), \omega^{-1}(Ad(a)\mathbf{e}_j)] \\ &= -2R_{a*}\omega^{-1}(\Omega_{-1}(\omega^{-1}(Ad(a)\mathbf{e}_i), \omega^{-1}(Ad(a)\mathbf{e}_j))) \\ &= -2(\omega^{-1}(a^{-1}\cdot\Omega_{-1}(\omega^{-1}(Ad(a)\mathbf{e}_i), \omega^{-1}(Ad(a)\mathbf{e}_j))), \end{aligned}$$

where in the last term $\mathfrak{g}_{-1}$ and $\mathbb{R}^{mn} = \mathbb{R}^n \otimes \mathbb{R}^m$ are identified. ∎

5.3. Complete integrability

Let M be a manifold with a Grassmannian structure of type (n, m). An n-dimensional tautological distribution D_L of null n-planes on the null n-plane bundle F_L over M and the distribution $\mathcal{D}_L$ on a frame bundle Q of second order with structure group G' on M are defined.

Now, assume that the distribution D_L on F_L, namely, the distribution $\mathcal{D}_L$ on Q is completely integrable.

Let $x \in M$. Let $\{e_1\otimes f_1, \dots, e_n\otimes f_1, \dots, e_1\otimes f_m, \dots, e_n\otimes f_m\}$ be a local basis field about x. Then $\lambda_x = (e_1\otimes f_1, \dots, e_n\otimes f_1, \dots, e_1\otimes f_m, \dots, e_n\otimes f_m)_x$ belongs to $\pi^{-1}(x) \subset P$. Let $v \in \pi_M^{-1}(x) \subset Q$ such that $\pi_P(v) = \lambda_x$. And let $\tilde{\mathbf{e}}_{im|v}, \tilde{\mathbf{e}}_{jm|v} \in \mathcal{D}_{Lv} \subset T_vQ$. Here $\tilde{\mathbf{e}}_{im} = \omega^{-1}(\mathbf{e}_{im})$ and $\pi_{M_{*|v}}(\tilde{\mathbf{e}}_{im}) = e_i \otimes f_m$ $(i = 1, \dots, n)$. We describe conditions such that $[\tilde{\mathbf{e}}_{im}, \tilde{\mathbf{e}}_{jm}]_{|v} \in \mathcal{D}_{Lv}$.

By Ω^R_{ab} $(1 \le a, b \le m)$ and Ω^L_{ij} $(1 \le i, j \le n)$ we denote (a, b)-component of $\mathfrak{g}^R_0$ and (i, j)-component of $\mathfrak{g}^L_0$ with respect to the decomposition $\Omega = \Omega_{-1} \oplus \Omega_0 \oplus \Omega_1 = \Omega_{-1} \oplus \Omega'$ of the curvature form Ω of the normal Cartan connection ω respectively, and by $\Omega^{\alpha,\beta}_{-1}$ $(1 \le \alpha \le n,\ 1 \le \beta \le m)$ (α, β)-component of $\mathfrak{g}_{-1}$.

From lemmas in 5.2, we have the following.

Proposition 5.1. *We have*

$$[\tilde{\mathbf{e}}_{im}, \tilde{\mathbf{e}}_{jm}]_{|v} \in \mathcal{D}_{Lv}$$
$$\Longleftrightarrow \begin{cases} \text{(i) } \Omega^R_{mb}(\tilde{\mathbf{e}}_{im}, \tilde{\mathbf{e}}_{jm}) = 0, & (1 \le b \le m-1) \\ \text{(ii) } \Omega^{\alpha\beta}_{-1}(\tilde{\mathbf{e}}_{im}, \tilde{\mathbf{e}}_{jm}) = 0, & (1 \le \alpha \le n,\ 1 \le \beta \le m-1). \end{cases}$$

In the case $n, m \ge 3$:

In particular, we have $\Omega^{n1}_{-1}(\tilde{\mathbf{e}}_{1m}, \tilde{\mathbf{e}}_{2m}) = 0$, namely, $\mathbf{e}_{n1} \otimes (\mathbf{e}^*_{1m} \wedge \mathbf{e}^*_{2m}) \in \mathfrak{g}_{-1} \otimes \Lambda^2_R$-component of K^0 is 0. From Proposition 3.3, 1 (ii) in 3.3, this component is the component of one nonzero generator as $\mathfrak{g}_0$-module in H^2. Therefore the Grassmannian structure is right-half torsion-free, i.e., $HK^0_R = 0$. Here $K^0 = HK^0 = HK^0_L + HK^0_R$ ($HK^0_L \subset \mathfrak{g}_{-1} \otimes \Lambda^2_L$, $HK^0_R \subset \mathfrak{g}_{-1} \otimes \Lambda^2_R$ in Proposition 3.3, 1 (i), (ii) respectively).

Conversely, assume that $HK^0_R = 0$. If $HK^0_R = 0$, the component of the generator $\mathbf{e}_{n1} \otimes (\mathbf{e}^*_{1m} \wedge \mathbf{e}^*_{2m}) \in \mathfrak{g}_{-1} \otimes \Lambda^2_R$ as $\mathfrak{g}_0$-module in H^2 of K^0 is 0. Thus, by lemmas in 5.2, we get (ii) in the above proposition. Further, from Proposition 5.2, 1 in 5.5 which appears later on (cf. Proposition 4.1 in 4.4) for a co-Grassmannian structure on F_L, we get (i) in the above proposition. Therefore D_L on F_L is completely integrable.

In the case $n \geq 3$, $m = 2$:

In particular, we have $\Omega^{n1}_{-1}(\tilde{\mathbf{e}}_{12}, \tilde{\mathbf{e}}_{22}) = 0$, namely, $\mathbf{e}_{n1} \otimes (\mathbf{e}^*_{12} \wedge \mathbf{e}^*_{22}) \in \mathfrak{g}_{-1} \otimes \Lambda^2_R$ component of K^0 is 0. From Proposition 3.3, 2 (i) in 3.3, this component is the component of one nonzero generator as $\mathfrak{g}_0$-module in H^2. Therefore the Grassmannian structure is torsion-free (especially right-half Grassmannian flat), i.e., $K^0 = HK^0 = 0$.

Conversely, assume that $K^0 = HK^0 = 0$. If $HK^0 = 0$, the component of the generator $\mathbf{e}_{n1} \otimes (\mathbf{e}^*_{12} \wedge \mathbf{e}^*_{22}) \in \mathfrak{g}_{-1} \otimes \Lambda^2_R$ as $\mathfrak{g}_0$-module in H^2 of K^0 is 0. Thus, by lemmas in 5.2, we get (ii) in the above proposition. Further, from Proposition 5.2, 2 in the next 5.5 (cf. Proposition 4.1 in 4.4) for a co-Grassmannian structure on F_L, we get (i) in the above proposition. Therefore D_L on F_L is completely integrable.

In the case $n = 2$, $m = 2$:

In particular, we have $\Omega^R_{21}(\tilde{\mathbf{e}}_{12}, \tilde{\mathbf{e}}_{22}) = 0$, namely, $\mathbf{g}_{21} \otimes (\mathbf{e}^*_{12} \wedge \mathbf{e}^*_{22}) \in \mathfrak{g}^R_0 \otimes \Lambda^2_R$ component of K^1 is 0. From Proposition 3.3, 3 (ii) in 3.3, this component is the component of one nonzero generator as $\mathfrak{g}_0$-module in H^2. Therefore, in consideration of Proposition 3.2 in 3.2, the Grassmannian structure is right-half Grassmannian flat, i.e., $HK^1_R = 0$. Here $K^0 = 0$, $K^1 = HK^1 = HK^1_L + HK^1_R$ ($HK^1_L \subset \mathfrak{g}^L_0 \otimes \Lambda^2_L$, $HK^1_R \subset \mathfrak{g}^R_0 \otimes \Lambda^2_R$ in Proposition 3.3, 3 (i), (ii) respectively).

Conversely, assume that $HK^1_R = 0$. If $HK^1_R = 0$, the component of the generator $\mathbf{g}_{21} \otimes (\mathbf{e}^*_{12} \wedge \mathbf{e}^*_{22}) \in \mathfrak{g}^R_0 \otimes \Lambda^2_R$ as $\mathfrak{g}_0$-module in H^2 of K^1 is 0. Thus, by lemmas in 5.2, we get (i) in the above proposition. Further, from Proposition 5.2, 3 in the next 5.5 (cf. Proposition 4.1 in 4.4) for a co-Grassmannian structure on F_L, we get (ii) in the above proposition. Therefore D_L on F_L is completely integrable.

Summarizing them, we have the following.

THEOREM 5.1. *Let M be a manifold with a Grassmannian structure of type (n, m) and equipped with the normal Cartan connection ω. Then the tautological distribution D_L on the null n-plane bundle F_L over M is completely integrable if and only if the Grassmannian structure on M is*

1. *if $n, m \geq 3$, right-half torsion-free, i.e., $HK^0_R = 0$,*
2. *if $n \geq 3$, $m = 2$, torsion-free, i.e., $K^0 = HK^0 = 0$,*
3. *if $n = 2$, $m = 2$, right-half Grassmannian flat, i.e., $HK^1_R = 0$.*

If $n = 2$, $m = 2$, $K^0 = 0$ holds. Therefore $K = K^1 = HK^1$ is nothing but the conformal Weyl tensor of a conformal structure of type $(2, 2)$ (see 1.5, 2.3. cf. [O]).

5.4. An example of type $(n, 2)$

We give an example M with the completely integrable condition of 2 in the above theorem.

We study the example described in 1.3 (2):

$$M = H^{m,m}(\mathbb{C}) \cong U(m+1, m)/U(1) \times U(m, m).$$

Since $U(m+1, m)$ acts transitively on M, we consider the origin $o = U(1) \times U(m, m)$. Moreover, since $U(1) \times U(m, m)$ acts transitively on the set of all null n-planes in the tangent space T_oM at the origin o, we consider a distinguished null n-plane.

In $T_oM \cong \mathfrak{m}$,

$$\mathfrak{n} = \left\{ \begin{pmatrix} 0 & -{}^tx & y \\ x & 0 & 0 \\ y & 0 & 0 \end{pmatrix} \right\}, \quad (x, y \in \mathbb{R}^m \subset \mathbb{C}^m)$$

is a null n-plane. We show that $\exp \mathfrak{n}$ is a null n-manifold in M, that is, each null vector in $\mathfrak{n}$ is mapped to a null vector in the tangent space at each point in $\exp \mathfrak{n}$ under the differential $\exp_*$. If so, it follows that, for any point in M and any null n-plane in tangent space at the point, there exists a null n-manifold through the point such that the tangent space at the point is the null n-plane. Then, from the above Theorem, $M = H^{m,m}(\mathbb{C})$ is right-half Grassmannian flat.

Since M is a symmetric space, according to [He], for $X \in \mathfrak{n} \subset \mathfrak{m} \subset \mathfrak{g}$,

$$\exp_{*X} = (L_{\exp})_{*e} \circ \frac{1 - e^{-adX}}{adX}$$

holds, where e is the unit element of $G = U(m+1, m)$ and $(1-e^{-A})/A = \sum_{m=0}^{\infty} \frac{1}{(m+1)!}(-A)^m$. Thus, it suffices that, for $Y \in \mathfrak{n}$,

$$\frac{1-e^{-adX(Y)}}{adX(Y)} = \sum_{m=0}^{\infty} \frac{1}{(m+1)!}(-ad^m X(Y))$$

belongs to $\mathfrak{n}$ mod $\mathfrak{h}$. Since $[\mathfrak{n}, \mathfrak{n}] \subset [\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$, if m is odd, $ad^m X(Y)$ is an element of $\mathfrak{h}$. By simple calculations, if m is even, it follows directly that $ad^m X(Y)$ is an element of $\mathfrak{n}$. Thus $\exp_{*X}(Y)$ is a null vector. Therefore $\exp \mathfrak{n}$ is a null n-manifold.

5.5. Co-Grassmannian structure of type $(n, m-1)$ and its normal Cartan connection

Let M be a manifold with a Grassmannian structure of type (n, m) and equipped with the normal Cartan connection ω. Suppose that an n-dimensional tautological distribution D_L of null n-planes on the null n-plane bundle F_L over M is completely integrable. For the natural projection $\varpi_L : F_L \to M$, put $E_L = \mathrm{Ker}(\varpi_L)_*$. Then E_L is a completely integrable $(m-1)$-dimensional distribution on F_L. In the following, by considering a differential system $\widehat{D}_L = D_L \oplus E_L$, we will see that a transversal pair (D_L, E_L) defines a co-Grassmannian structure of type $(n, m-1)$ on F_L, that is, the symbol algebra of $\widehat{D}_L = D_L \oplus E_L$ at each point of F_L is isomorphic to a graded Lie algebra of type $(n, m-1)$ CGR. Moreover we will show that the normal Cartan connection (Q, ω) induced by a Grassmannian structure of type (n, m) on M decides the normal Cartan connection $(Q, \overline{\omega})$ of the co-Grassmannian structure of type $(n, m-1)$ on F_L.

In Sections 2 and 3, we considered $\mathfrak{g} = \mathfrak{sl}(m+n, \mathbb{R}) = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$. Now we write it as $\mathfrak{k} = \mathfrak{sl}(m+n, \mathbb{R}) = \mathfrak{k}_{-1} \oplus \mathfrak{k}_0 \oplus \mathfrak{k}_1$. On the other hand, in Section 4, we have the decomposition $\mathfrak{g} = \mathfrak{sl}(m+n, \mathbb{R}) = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$ as type $(n, m-1)$ CGR. Thus we have the two decompositions $\mathfrak{g} = \mathfrak{k} = \mathfrak{sl}(m+n, \mathbb{R})$ of $G = K = SL(m+n, \mathbb{R})$, as graded Lie algebras of first kind and second kind respectively, as follows:

$$\begin{aligned} \mathfrak{k} &= \mathfrak{k}_{-1} \oplus \mathfrak{k}_0 \oplus \mathfrak{k}_1 \\ &= \mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2. \end{aligned}$$

We have

$$\mathfrak{k}_{-1} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}^L, \quad (\mathfrak{g}_{-1}^L = \mathfrak{e})$$

$$\mathfrak{k}_0 = \mathfrak{g}^R_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}^R_1, \quad (\mathfrak{g}^R_{-1} = \mathfrak{f},\, \mathfrak{g}^R_1 = (\mathfrak{g}^R_{-1})^*)$$
$$\mathfrak{g}_{-1} = \mathfrak{g}^L_{-1} \oplus \mathfrak{g}^R_{-1}, \quad \mathfrak{g}_{-2} \cong \mathfrak{g}^L_{-1} \otimes \mathfrak{g}^R_{-1}.$$

Moreover we have the following:

$$\begin{aligned} \Lambda^2\mathfrak{k}^*_{-1} &= \Lambda^2_L \oplus \Lambda^2_R \\ &= \Lambda^2\mathfrak{g}^*_{-2} \oplus (\mathfrak{g}^*_{-2} \wedge (\mathfrak{g}^L_{-1})^*) \oplus \Lambda^2(\mathfrak{g}^L_{-1})^*, \\ \Lambda^2(\mathfrak{g}^L_{-1})^* &\subset \Lambda^2_R. \end{aligned}$$

We identify $\mathbb{R}^{mn} = \mathbb{R}^n \otimes \mathbb{R}^m$ with $\mathfrak{k}_{-1}$ and $\mathbb{R}^{mn+m-1} = \mathbb{R}^n \otimes \mathbb{R}^{m-1} \oplus \mathbb{R}^n \oplus \mathbb{R}^{m-1}$ with $\mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} = \mathfrak{g}_{-2} \oplus \mathfrak{g}^L_{-1} \oplus \mathfrak{g}^R_{-1}$ respectively. The set of all isomorphisms of $\mathfrak{k}_{-1}$ to the tangent space at each point of M is the linear frame bundle of M and the structure group is the Lie group of $\mathfrak{k}_0$. The set of all isomorphisms of $\mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} = \mathfrak{g}_{-2} \oplus \mathfrak{g}^L_{-1} \oplus \mathfrak{g}^R_{-1}$ to the tangent space at each point of F_L is the linear frame bundle of F_L and the structure group is the Lie group of $\mathfrak{g}_0$.

Let K' be the Lie group of $\mathfrak{k}' = \mathfrak{k}_0 \oplus \mathfrak{k}_1$ and G' the Lie group of $\mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$. The flat model with a Grassmannian structure of type (n, m) is $G_{m,n+m} \cong K/K'$ (see 1.2) and the flat model with a co-Grassmannian structure of type $(n, m-1)$ is $F_{m-1,m} \cong G/G'$ (see 4.2).

From the diagram in 5.2, we can regard the frame bundle Q of second order with structure group K' on M as a principal bundle with structure group G' over F_L. The normal Cartan connection ω of type K/K' on Q is a $\mathfrak{g} = \mathfrak{k} = \mathfrak{sl}(m+n, \mathbb{R})$-valued 1-form and a linear isomorphism $\omega : T_vQ \to \mathfrak{k}$ for $v \in Q$. At the same time, ω defines a Cartan connection $\overline{\omega}$ of type G/G' on Q. The curvature function $\overline{K}$ by the curvature form $\overline{\Omega}$ of ω is said to be the lift of the curvature function K by the curvature form Ω of ω.

We show that $\overline{\omega}$ satisfies normality condition, that is, $\overline{K}^{-1} = 0$, $\partial^*\overline{K}^p = 0$ $(p \geq 0)$.

$\overline{K}^{-1} = 0$:
$\overline{K}^{-1} = \overline{K}^{-1,2}$ has the values in

$$\begin{aligned} C^{-1,2} &= \mathfrak{g}_{-2} \otimes \Lambda^2_{-2} = \mathfrak{g}_{-2} \otimes \Lambda^2\mathfrak{g}^*_{-1} \\ &= \mathfrak{g}_{-2} \otimes \Lambda^2(\mathfrak{g}^L_{-1})^* \oplus \left(\mathfrak{g}_{-2} \otimes (\mathfrak{g}^L_{-1})^* \wedge (\mathfrak{g}^R_{-1})^*\right) \oplus \mathfrak{g}_{-2} \otimes \Lambda^2(\mathfrak{g}^R_{-1})^*. \end{aligned}$$

Since $\overline{K}$ is the lift of K, it follows that

$$\overline{K}^{-1} = 0 \Longleftrightarrow \mathfrak{g}_{-2} \otimes \Lambda^2(\mathfrak{g}^L_{-1})^*\text{-component of } K = 0.$$

If an n-dimensional tautological distribution D_L of null n-planes on F_L is completely integrable (see 5.3), these equivalent conditions are satisfied.

$\partial^* \overline{K}^p = 0$:

Let $\{e_i\}$ be a basis of $\mathfrak{m} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$ and take $X \in \mathfrak{m}$. Then

$$\begin{aligned}(\partial^* \overline{K})(X) &= \sum_j [e_j^*, \overline{K}(e_j \wedge X)] + \frac{1}{2} \sum_j \overline{K}([e_j^*, X]_- \wedge e_j) \\ &= \sum_j [e_j^*, K(e_j \wedge X)] + \frac{1}{2} \sum_j K([e_j^*, X]_- \wedge e_j).\end{aligned}$$

Since $e_j^* \in \mathfrak{m}^* = \mathfrak{g}_{-2}^* \oplus (\mathfrak{g}_{-1}^L)^* \oplus (\mathfrak{g}_{-1}^R)^* = \mathfrak{k}_{-1}^* \oplus (\mathfrak{g}_{-1}^R)^*$ and $[\mathfrak{g}_1^L, \mathfrak{g}_{-2}] \subset \mathfrak{g}_{-1}^R$, the second term does not appear. Remark that $K(e_j \wedge X) = 0$ for $e_j \in \mathfrak{g}_{-1}^R$. Thus it follows that

$$\partial^* \overline{K} = \partial^* K.$$

From normality condition of K, we have $\partial^* K^p = 0$ $(p \geq 0)$. Therefore we get $\partial^* \overline{K}^p = 0$.

We have the following.

THEOREM 5.2. *Let M be a manifold with a Grassmannian structure of type (n, m) and equipped with the normal Cartan connection ω. Suppose that an n-dimensional tautological distribution D_L of null n-planes on the null n-plane bundle F_L over M is completely integrable. Then a pair $(D_L, E_L = \mathrm{Ker}(\varpi_L)_*)$ defines a co-Grassmannian structure of type $(n, m-1)$ on F_L. Moreover the normal Cartan connection (Q, ω) of a Grassmannian structure of type (n, m) induces the normal Cartan connection $(Q, \overline{\omega})$ of the co-Grassmannian structure of type $(n, m-1)$ on F_L.*

We have the harmonic part HK of the curvature function K of ω and the harmonic part $H\overline{K}$ of the curvature function $\overline{K}$ of $\overline{\omega}$ that is its lift. The relation of them is as follows.

PROPOSITION 5.2. *If the conditions of the theorem above are satisfied,*

1. *if $n, m \geq 3$, for $K^0 = HK^0 = HK_L^0 + HK_R^0$,*
   (a) *the vanishing of the generator of $HK_R^0 \subset \mathfrak{k}_{-1} \otimes \Lambda_R^2$ implies the normality condition of $\overline{K}$ (plus the complete integrability of D_L),*

(b) *the generator of* $HK^0_L \subset \mathfrak{k}_{-1} \otimes \Lambda^2_L$ *is lifted to the generator of* $H\overline{K}^0 \subset \mathfrak{g}_{-2} \otimes (\mathfrak{g}^*_{-2} \wedge (\mathfrak{g}^L_{-1})^*)$, (*the other generator of* $H\overline{K}^0 \subset \mathfrak{g}_{-2} \otimes (\mathfrak{g}^*_{-2} \wedge (\mathfrak{g}^R_{-1})^*)$ *vanishes*)

2. *if* $n \geq 3$, $m = 2$, *for* $K^0 = HK^0$ *and* HK^1,

   (a) *the vanishing of the generator of* $K^0 = HK^0 \subset \mathfrak{k}_{-1} \otimes \Lambda^2_R$ *implies the normality condition of* $\overline{K}$ (*plus the complete integrability of* D_L),

   (b) *the generator of* $HK^1 \subset \mathfrak{k}^L_0 \otimes \Lambda^2_L$ *is lifted to the generator of* $H\overline{K}^2 \subset \mathfrak{g}_0 \otimes (\mathfrak{g}^*_{-2} \wedge (\mathfrak{g}^L_{-1})^*)$, (*the generator of* $H\overline{K}^1 \subset \mathfrak{g}^L_{-1} \otimes (\mathfrak{g}^*_{-2} \wedge (\mathfrak{g}^R_{-1})^*)$ *vanishes*)

3. *if* $n = 2$, $m = 2$, *for* $K^1 = HK^1 = HK^1_L + HK^1_R$,

   (a) *the generator of* $HK^1_R \subset \mathfrak{k}^R_0 \otimes \Lambda^2_R$ *is lifted to the generator of* $H\overline{K}^0 \subset \mathfrak{g}^R_{-1} \otimes \Lambda^2(\mathfrak{g}^L_{-1})^*$, *the vanishing of the generator of* HK^1_R *implies the normality condition* (*plus the complete integrability of* D_L),

   (b) *the generator of* $HK^1_L \subset \mathfrak{k}^L_0 \otimes \Lambda^2_L$ *is lifted to the generator of* $H\overline{K}^2 \subset \mathfrak{g}_0 \otimes (\mathfrak{g}^*_{-2} \wedge (\mathfrak{g}^L_{-1})^*)$. (*the generator of* $H\overline{K}^1 \subset \mathfrak{g}^L_{-1} \otimes (\mathfrak{g}^*_{-2} \wedge (\mathfrak{g}^R_{-1})^*)$ *vanishes*).

Here $\mathfrak{k}^L_0 = \mathfrak{sl}_L = \mathfrak{sl}(n, \mathbb{R})$ and $\mathfrak{k}^R_0 = \mathfrak{sl}_R = \mathfrak{sl}(m, \mathbb{R})$.

Remark that, from Proposition 4.1, 2 (a), (b) in 4.4, if $n \geq 2$, $m = 3$, the relations are the same ones as the above 1 (a), (b). If $n = 2$, $m = 3$, the generator of $H\overline{K}^0 \subset \mathfrak{g}^R_{-1} \otimes \Lambda^2(\mathfrak{g}^L_{-1})^*$ vanishes besides, because of the complete integrability of D_L. (The other generator of $H\overline{K}^0 \subset \mathfrak{g}^L_{-1} \otimes \Lambda^2(\mathfrak{g}^R_{-1})^*$ vanishes.)

§6. Null m-plane bundle

6.1. Definition

Let M be a manifold with a Grassmannian structure of type (n, m). Considering a set of all the null m-planes as in 5.1 in the tangent space at each point of M, we have a fibre bundle F_R with fibre $P^{n-1}(\mathbb{R})$ over M, called *a null m-plane bundle*:

$$\begin{array}{ccc} F_R & \longleftarrow & P^{n-1}(\mathbb{R}) \\ \Big\downarrow{\scriptstyle \varpi_R} & & \\ M & & \end{array} \quad .$$

6.2. Tautological distribution

An m-dimensional distribution D on the null m-plane bundle F_R over M is called *an m-dimensional tautological distribution of null m-planes* if it satisfies the following: for the m-dimensional subspace $D_\Pi \subset T_\Pi F_R$,

$$\varpi_{R*}(D_\Pi) = \Pi \subset T_{\varpi_R(\Pi)}M.$$

Here note that $\Pi \in F_R$ is a null m-plane in $T_{\varpi_R(\Pi)}M$. As in the way that we defined an n-dimensional tautological distribution of null n-planes on the null n-plane bundle F_L over M in Section 5, by the use of the normal Cartan connection on Q we will define an m-dimensional tautological distribution D_R of null m-planes on the null m-plane bundle F_R over M.

Let M be a manifold with a Grassmannian structure of type (n, m). We identify TM with $V \otimes W$ under σ. For $x \in M$, we denote T_xM by $V_x \otimes W_x$. Take a basis $\{e_i\}$ $(1 \leq i \leq n)$ of V_x and a basis $\{f_j\}$ $(1 \leq j \leq m)$ of W_x. A set $\{e_i \otimes f_j\}$ $(1 \leq i \leq n,\ 1 \leq j \leq m)$ is a null basis of $V_x \otimes W_x$. Therefore $\lambda_x = (e_1 \otimes f_1, e_2 \otimes f_1, \ldots, e_n \otimes f_1, \ldots, e_1 \otimes f_m, e_2 \otimes f_m, \ldots, e_n \otimes f_m)$ belongs to P.

Put

$$\Pi_{R_x} = \mathrm{span}(e_1 \otimes f_1, e_1 \otimes f_2, \ldots, e_1 \otimes f_m).$$

The subspace Π_{R_x} is a null m-plane in T_xM. Namely, Π_{R_x} belongs to F_R. Then a mapping $p_R : P \to F_R$ is defined by

$$p_R : \lambda_x \longmapsto \Pi_{R_x}.$$

A subgroup of $G_0 = GR(n, m)$ which leaves the null m-plane Π_{R_x} invariant is

$$H_{0R} = S(GL(m, \mathbb{R}) \times \{C\}),$$

where

$$C = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & a_{n2} & \cdots & a_{nn} \end{pmatrix} \in GL(n, \mathbb{R}).$$

Consequently we define a principal bundle $P(F_R, H_{0R}, p_R)$. The null m-plane bundle F_R over M is the fibre bundle with fibre $G_0/H_{0R} \cong P^{n-1}(\mathbb{R})$ associated with P:

$$F_R = P \times_{G_0} G_0/H_{0R} = P/H_{0R}.$$

For a linear frame bundle P with structure group $G_0 = GR(n,m)$ over M, let us consider the frame bundle Q of second order with structure group G'. Let π_P be a canonical projection $Q \to P$. Then the null m-plane bundle F_R over M is the fibre bundle with fibre $G'/H'_R \cong P^{n-1}(\mathbb{R})$ associated with Q:

$$F_R = Q \times_{G'} G'/H'_R = Q/H'_R.$$

Consequently a mapping $\pi_R : Q \to F_R$ being defined, we define a principal bundle $Q(F_R, H'_R, \pi_R)$.

Summarizing them, we have the following diagram:

$$\begin{tikzcd}
G' \arrow[r] & Q \arrow[ddd, bend right, "\pi_M"'] \arrow[d, "\pi_P"] \arrow[ddr, "\pi_R"] & H'_R \arrow[l] & \\
G_0 \arrow[r] & P \arrow[dr, "p_R"] \arrow[dd, "\pi"] & H_{0R} \arrow[l] & \\
 & & F_R \arrow[dl, "\varpi_R"] & G_0/H_{0R} \cong G'/H'_R \cong P^{n-1}(\mathbb{R}). \arrow[l] \\
 & M & &
\end{tikzcd}$$

We use the same notations as in 5.2.

A subspace

$$\mathfrak{n}_R = \mathrm{span}(\mathbf{e}_{11}, \mathbf{e}_{12}, \ldots, \mathbf{e}_{1m})$$

spanned by $\mathbf{e}_{11}, \mathbf{e}_{12}, \ldots, \mathbf{e}_{1m}$ is a null m-plane. The Lie algebra $\mathfrak{h}_R$ of H'_R is a subalgebra of $\mathfrak{g}' = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ and has the following form:

$$\mathfrak{h}_R = \left\{ \begin{pmatrix} B & & D & \\ & a_{11} & a_{12} \cdots & a_{1n} \\ & 0 & a_{22} \cdots & a_{2n} \\ O & \vdots & \vdots \ \ddots & \vdots \\ & 0 & a_{n2} \cdots & a_{nn} \end{pmatrix} \right\} \subset \mathfrak{g}',$$

where $a_{ij} \in \mathbb{R}$, $B \in \mathrm{Mat}(m, \mathbb{R})$, $D \in \mathrm{Mat}(m \times n, \mathbb{R})$, and trace $B + \sum_{i=1}^{n} a_{ii} = 0$.

For the vector subspace $\mathfrak{n}_R + \mathfrak{h}_R$ of $\mathfrak{g}$, We have the following:

LEMMA 6.1. *The space $\mathfrak{n}_R + \mathfrak{h}_R$ of $\mathfrak{g}$ is invariant under the adjoint action of H'_R and $\mathfrak{h}_R$.*

Remark that the space $\mathfrak{n}_R$ is invariant under the adjoint action of H_{0R}.

Let $v \in Q$. Let $x = \pi_M(v) \in M$. Let $\lambda_x = \pi_P(v)$ and $\Pi_{R_x} = \pi_R(v)$. Then $p_R(\lambda_x) = \Pi_{R_x}$ holds. An element $\lambda_x = (e_1 \otimes f_1, \dots, e_n \otimes f_1, \dots, e_1 \otimes f_m, \dots, e_n \otimes f_m) \in P$ is regarded as an isomorphism $\lambda_x : \mathbb{R}^{mn} \, (= \mathbb{R}^n \otimes \mathbb{R}^m) = \mathfrak{g}_{-1} \to T_x M$:

$$\mathbf{e}_{ij} \longmapsto e_i \otimes f_j.$$

Then

$$\mathfrak{n}_R \longmapsto \Pi_{R_x}$$

holds.

By using the normal Cartan connection ω, vectors $\omega^{-1}(\mathbf{e}_{ij}) \in T_v Q$ are the horizontal lift of vectors $e_i \otimes f_j \in T_x M$.

Next, putting

$$\mathcal{D}_{R_v} = \omega^{-1}(\mathfrak{n}_R + \mathfrak{h}_R),$$

we can define a distribution $\mathcal{D}_R$ on Q. From the lemma above, we have the following:

LEMMA 6.2. *The distribution $\mathcal{D}_R$ on Q is invariant under the right action of H'_R.*

Therefore an m-dimensional distribution D_R is defined on the null m-plane bundle $F_R = Q/H'_R$:

$$D_R = \mathcal{D}_R \bmod H'_R.$$

This is a tautological distribution of null m-planes.

Investigating the complete integrability of the distribution D_R on F_R is equivalent to investigating the complete integrability of the distribution $\mathcal{D}_R$ on Q modulo H'_R.

LEMMA 6.3. *We have*

$$
\begin{aligned}
&[D_R, D_R] \subset D_R \text{ on } F_R \\
&\quad \Longleftrightarrow [\mathcal{D}_R \bmod H'_R, \mathcal{D}_R \bmod H'_R] \subset \mathcal{D}_R \bmod H'_R \text{ on } Q.
\end{aligned}
$$

For $v \in Q$, elements in $T_v Q$

$$\tilde{\mathbf{e}}_{1j|v} = \omega^{-1}(\mathbf{e}_{1j}), \quad 1 \leq j \leq m,$$

are defined. Put

$$\tilde{\mathfrak{n}}_R = \omega^{-1}(\mathfrak{n}_R).$$

We will investigate the condition modulo H'_R satisfying

$$[\tilde{\mathbf{e}}_{1i}, \tilde{\mathbf{e}}_{1j}] \in \tilde{\mathfrak{n}}_R,$$

for vector fields $\tilde{\mathbf{e}}_{1j}$ $(1 \leq j \leq m)$ on Q.

6.3. Complete integrability

Now, assume that the distribution D_R on F_R, namely, the distribution $\mathcal{D}_R$ on Q is completely integrable.

We describe conditions such that $[\tilde{\mathbf{e}}_{1i}, \tilde{\mathbf{e}}_{1j}]_{|v} \in \mathcal{D}_{Rv}$ for $\tilde{\mathbf{e}}_{1i|v}, \tilde{\mathbf{e}}_{1j|v} \in \mathcal{D}_R \subset T_v Q$.

We use the same notations as in 5.3.

From lemmas in 5.2, we have the following.

PROPOSITION 6.1. *We have*

$$
\begin{aligned}
&[\tilde{\mathbf{e}}_{1i}, \tilde{\mathbf{e}}_{1j}]_{|v} \in \mathcal{D}_{Rv} \\
&\quad \Longleftrightarrow \begin{cases} \text{(i) } \Omega^L_{k1}(\tilde{\mathbf{e}}_{1i}, \tilde{\mathbf{e}}_{1j}) = 0, & (2 \leq k \leq n) \\ \text{(ii) } \Omega^{\alpha\beta}_{-1}(\tilde{\mathbf{e}}_{1i}, \tilde{\mathbf{e}}_{1j}) = 0, & (2 \leq \alpha \leq n,\ 1 \leq \beta \leq m). \end{cases}
\end{aligned}
$$

In the case $n, m \geq 3$:

In particular, we have $\Omega^{n1}_{-1}(\tilde{\mathbf{e}}_{1m-1}, \tilde{\mathbf{e}}_{1m}) = 0$, namely, $\mathbf{e}_{n1} \otimes (\mathbf{e}^*_{1m-1} \wedge \mathbf{e}^*_{1m}) \in \mathfrak{g}_{-1} \otimes \Lambda^2_L$ component of K^0 is 0. From Proposition 3.3, 1 (i) in 3.3, this component is the component of one nonzero generator as $\mathfrak{g}_0$-module in H^2. Therefore the Grassmannian structure is left-half torsion-free, i.e., $HK^0_L = 0$. Here $K^0 = HK^0 = HK^0_L + HK^0_R$ ($HK^0_L \subset \mathfrak{g}_{-1} \otimes \Lambda^2_L$, $HK^0_R \subset \mathfrak{g}_{-1} \otimes \Lambda^2_R$ in Proposition 3.3, 1 (i), (ii) respectively).

Conversely, assume that $HK^0_L = 0$. If $HK^0_L = 0$, the component of the generator $\mathbf{e}_{n1} \otimes (\mathbf{e}^*_{1m-1} \wedge \mathbf{e}^*_{1m}) \in \mathfrak{g}_{-1} \otimes \Lambda^2_L$ as $\mathfrak{g}_0$-module in H^2 of

K^0 is 0. Thus, by lemmas in 5.2, we get (ii) in the above proposition. Further, from Proposition 6.2, 1 in 6.4 which appears in the next subsection (cf. Proposition 4.1 in 4.4) for a co-Grassmannian structure on F_R, we get (i) in the above proposition. Therefore D_R on F_R is completely integrable.

In the case $n \geq 3$, $m = 2$:

In particular, we have $\Omega_0^{n1}(\tilde{\mathbf{e}}_{11}, \tilde{\mathbf{e}}_{12}) = 0$, namely, $\mathbf{h}_{n1} \otimes (\mathbf{e}_{11}^* \wedge \mathbf{e}_{12}^*) \in \mathfrak{g}_0^L \otimes \Lambda_L^2$ component of K^1 is 0. From Proposition 3.3, 2 (ii) in 3.3, this component is the component of one nonzero generator as $\mathfrak{g}_0$-module in H^2. Therefore, in consideration of Proposition 3.1 in 3.2, the Grassmannian structure is left-half Grassmannian flat, i.e., $HK^1 = 0$.

Conversely, assume that $HK^1 = 0$. If $HK^1 = 0$, the component of the generator $\mathbf{h}_{n1} \otimes (\mathbf{e}_{11}^* \wedge \mathbf{e}_{12}^*) \in \mathfrak{g}_0^L \otimes \Lambda_L^2$ as $\mathfrak{g}_0$-module in H^2 of K^1 is 0. Thus, by lemmas in 5.2, we get (i) in the above proposition. Further, from Proposition 6.2, 2 in the next 6.4 (cf. Proposition 4.1 in 4.4) for a co-Grassmannian structure on F_R, we get (ii) in the above proposition. Therefore D_R on F_R is completely integrable.

In the case $n = 2$, $m = 2$:

In particular, we have $\Omega_{21}^L(\tilde{\mathbf{e}}_{11}, \tilde{\mathbf{e}}_{12}) = 0$, namely, $\mathbf{h}_{21} \otimes (\mathbf{e}_{11}^* \wedge \mathbf{e}_{12}^*) \in \mathfrak{g}_0^L \otimes \Lambda_L^2$ component of K^1 is 0. From Proposition 3.3, 3 (i) in 3.3, this component is the component of one nonzero generator as $\mathfrak{g}_0$-module in H^2. Therefore, by Proposition 3.2 in 3.2, the Grassmannian structure is left-half Grassmannian flat, i.e., $HK_L^1 = 0$. Here $K^0 = 0$, $K^1 = HK^1 = HK_L^1 + HK_R^1$ ($HK_L^1 \subset \mathfrak{g}_0^L \otimes \Lambda_L^2$, $HK_R^1 \subset \mathfrak{g}_0^R \otimes \Lambda_R^2$ in Proposition 3.3, 3 (i), (ii) respectively).

Conversely, assume that $HK_L^1 = 0$. If $HK_L^1 = 0$, the component of the generator $\mathbf{h}_{21} \otimes (\mathbf{e}_{11}^* \wedge \mathbf{e}_{12}^*) \in \mathfrak{g}_0^L \otimes \Lambda_L^2$ as $\mathfrak{g}_0$-module in H^2 of K^1 is 0. Thus, by lemmas in 5.2, we get (i) in the above proposition. Further, from Proposition 6.2, 3 in the next 6.4 (cf. Proposition 4.1 in 4.4) for a co-Grassmannian structure on F_R, we get (ii) in the above proposition. Therefore D_R on F_R is completely integrable.

Summarizing them, we have the following.

THEOREM 6.1. *Let M be a manifold with a Grassmannian structure of type (n, m) and equipped with the normal Cartan connection ω. Then the tautological distribution D_R on the null m-plane bundle F_R over M is completely integrable if and only if the Grassmannian structure on M is*

1. *if $n, m \geq 3$, left-half torsion-free, i.e., $HK_L^0 = 0$,*

2. *if* $n \geq 3$, $m = 2$, *left-half Grassmannian flat, i.e.,* $HK^1 = 0$,

3. *if* $n = 2$, $m = 2$, *left-half Grassmannian flat, i.e.,* $HK^1_L = 0$.

6.4. Co-Grassmannian structure of type $(n-1, m)$ and its normal Cartan connection

Let M be a manifold with a Grassmannian structure of type (n, m) and equipped with the normal Cartan connection ω. Suppose that an m-dimensional tautological distribution D_R of null m-planes on the null m-plane bundle F_R over M is completely integrable. For the natural projection $\varpi_R : F_R \to M$, put $E_R = \mathrm{Ker}(\varpi_R)_*$. Then E_R is a completely integrable $(n-1)$-dimensional distribution on F_R. In the following, by considering a differential system $\widehat{D}_R = E_R \oplus D_R$, we will see that a transversal pair (E_R, D_R) defines a co-Grassmannian structure of type $(n-1, m)$ on F_R, that is, the symbol algebra of $\widehat{D}_R = E_R \oplus D_R$ at each point of F_R is isomorphic to a graded Lie algebra of type $(n-1, m)$ CGR. Moreover we will show that the normal Cartan connection (Q, ω) induced by a Grassmannian structure of type (n, m) on M decides the normal Cratan connection $(Q, \overline{\varpi})$ of the co-Grassmannian structure of type $(n-1, m)$ on F_R.

As appeared in 5.5, we have the two decompositions $\mathfrak{g} = \mathfrak{k} = \mathfrak{sl}(m+n, \mathbb{R})$ of $G = K = SL(m+n, \mathbb{R})$, as graded Lie algebras of first kind and second kind respectively, as follows:

$$\begin{aligned} \mathfrak{k} &= \mathfrak{k}_{-1} \oplus \mathfrak{k}_0 \oplus \mathfrak{k}_1 \\ = \mathfrak{g} &= \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2. \end{aligned}$$

We have

$$\begin{aligned} \mathfrak{k}_{-1} &= \mathfrak{g}_{-2} \oplus \mathfrak{g}^R_{-1}, \quad (\mathfrak{g}^R_{-1} = \mathfrak{f}) \\ \mathfrak{k}_0 &= \mathfrak{g}^L_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}^L_1, \quad (\mathfrak{g}^L_{-1} = \mathfrak{e},\ \mathfrak{g}^L_1 = (\mathfrak{g}^L_{-1})^*) \\ \mathfrak{g}_{-1} &= \mathfrak{g}^L_{-1} \oplus \mathfrak{g}^R_{-1},\ \mathfrak{g}_{-2} \cong \mathfrak{g}^L_{-1} \otimes \mathfrak{g}^R_{-1}. \end{aligned}$$

Moreover we have the following:

$$\begin{aligned} \Lambda^2 \mathfrak{k}^*_{-1} &= \Lambda^2_L \oplus \Lambda^2_R \\ &= \Lambda^2 \mathfrak{g}^*_{-2} \oplus (\mathfrak{g}^*_{-2} \wedge (\mathfrak{g}^R_{-1})^*) \oplus \Lambda^2 (\mathfrak{g}^R_{-1})^*, \\ \Lambda^2 (\mathfrak{g}^R_{-1})^* &\subset \Lambda^2_L. \end{aligned}$$

We identify $\mathbb{R}^{mn} = \mathbb{R}^n \otimes \mathbb{R}^m$ with $\mathfrak{k}_{-1}$ and $\mathbb{R}^{mn+n-1} = \mathbb{R}^{n-1} \otimes \mathbb{R}^m \oplus \mathbb{R}^{n-1} \oplus \mathbb{R}^m$ with $\mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} = \mathfrak{g}_{-2} \oplus \mathfrak{g}^L_{-1} \oplus \mathfrak{g}^R_{-1}$ respectively. The set of all isomorphisms of $\mathfrak{k}_{-1}$ to the tangent space at each point of M is the linear frame bundle of M and the structure group is the Lie group of $\mathfrak{k}_0$. The set of all isomorphisms of $\mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} = \mathfrak{g}_{-2} \oplus \mathfrak{g}^L_{-1} \oplus \mathfrak{g}^R_{-1}$ to the tangent space at each point of F_R is the linear frame bundle of F_R and the structure group is the Lie group of $\mathfrak{g}_0$.

Let K' be the Lie group of $\mathfrak{k}' = \mathfrak{k}_0 \oplus \mathfrak{k}_1$ and G' the Lie group of $\mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$. The flat model with a Grassmannian structure of type (n, m) is $G_{m,n+m} \cong K/K'$ (see 1.2) and the flat model with a co-Grassmannian structure of type $(n-1, m)$ is $F_{m,m+1} \cong G/G'$ (see 4.2).

From the diagram in 6.2, we can regard the frame bundle Q of second order with structure group K' on M as a principal bundle with structure group G' over F_R. The normal Cartan connection ω of type K/K' on Q is a $\mathfrak{g} = \mathfrak{k} = \mathfrak{sl}(m+n, \mathbb{R})$-valued 1-form and a linear isomorphism $\omega : T_vQ \to \mathfrak{k}$ for $v \in Q$. At the same time, ω defines a Cartan connection $\overline{\omega}$ of type G/G' on Q. The curvature function $\overline{K}$ by the curvature form $\overline{\Omega}$ of ω is the lift of the curvature function K by the curvature form Ω of ω.

We show that $\overline{\omega}$ satisfies normality condition, that is, $\overline{K}^{-1} = 0$, $\partial^* \overline{K}^p = 0$ $(p \geq 0)$.

$\overline{K}^{-1} = 0$:
$\overline{K}^{-1} = \overline{K}^{-1,2}$ has the values in

$$\begin{aligned} C^{-1,2} &= \mathfrak{g}_{-2} \otimes \Lambda^2_{-2} = \mathfrak{g}_{-2} \otimes \Lambda^2 \mathfrak{g}^*_{-1} \\ &= \mathfrak{g}_{-2} \otimes \Lambda^2 (\mathfrak{g}^L_{-1})^* \oplus \left(\mathfrak{g}_{-2} \otimes (\mathfrak{g}^L_{-1})^* \wedge (\mathfrak{g}^R_{-1})^*\right) \oplus \mathfrak{g}_{-2} \otimes \Lambda^2 (\mathfrak{g}^R_{-1})^*. \end{aligned}$$

Since $\overline{K}$ is the lift of K, it follows that

$$\overline{K}^{-1} = 0 \Longleftrightarrow \mathfrak{g}_{-2} \otimes \Lambda^2 (\mathfrak{g}^R_{-1})^*\text{-component of } K = 0.$$

If an m-dimensional tautological distribution D_R of null m-planes on F_R is completely integrable (see 6.3), these equivalent conditions are satisfied.

$\partial^* \overline{K}^p = 0$:
Remark that $\mathfrak{m}^* = \mathfrak{g}^*_{-2} \oplus (\mathfrak{g}^L_{-1})^* \oplus (\mathfrak{g}^R_{-1})^* = \mathfrak{k}^*_{-1} \oplus (\mathfrak{g}^L_{-1})^*$ and $[\mathfrak{g}^R_1, \mathfrak{g}_{-2}] \subset \mathfrak{g}^L_{-1}$. Then in the similar way to 5.5 it follows that $\partial^* \overline{K}^p = 0$.

We have the following.

THEOREM 6.2. *Let M be a manifold with a Grassmannian structure of type (n,m) and equipped with the normal Cartan connection ω. Suppose that an m-dimensional tautological distribution D_R of null m-planes on the null m-plane bundle F_R over M is completely integrable. Then a pair $(E_R = \mathrm{Ker}(\varpi_R)_*, D_R)$ defines a co-Grassmannian structure of type $(n-1,m)$ on F_R. Moreover the normal Cartan connection (Q,ω) of a Grassmannian structure of type (n,m) induces the normal Cartan connection $(Q,\overline{\omega})$ of the co-Grassmannian structure of type $(n-1,m)$ on F_R.*

We have the harmonic part HK of the curvature function K of ω and the harmonic part $H\overline{K}$ of the curvature function $\overline{K}$ of $\overline{\omega}$ that is its lift. The relation of them is as follows.

PROPOSITION 6.2. *If the conditions of the theorem above are satisfied,*

1. *if $n,m \geq 3$, for $K^0 = HK^0 = HK^0_L + HK^0_R$,*
   (a) *the vanishing of the generator of $HK^0_L \subset \mathfrak{k}_{-1} \otimes \Lambda^2_L$ implies the normality condition of $\overline{K}$ (plus the complete integrability of D_R),*
   (b) *the generator of $HK^0_R \subset \mathfrak{k}_{-1} \otimes \Lambda^2_R$ is lifted to the generator of $H\overline{K}^0 \subset \mathfrak{g}_{-2} \otimes (\mathfrak{g}^*_{-2} \wedge (\mathfrak{g}^R_{-1})^*)$, (the other generator of $H\overline{K}^0 \subset \mathfrak{g}_{-2} \otimes (\mathfrak{g}^*_{-2} \wedge (\mathfrak{g}^L_{-1})^*)$ vanishes)*
2. *if $n \geq 3$, $m = 2$, for $K^0 = HK^0$ and HK^1,*
   (a) *the vanishing of the generator of $HK^1 \subset \mathfrak{k}^L_0 \otimes \Lambda^2_L$ is lifted to the generator of $H\overline{K}^0 \subset \mathfrak{g}^L_{-1} \otimes \Lambda^2(\mathfrak{g}^R_{-1})^*$, the vanishing of the generator of HK^1 implies the normality condition of $\overline{K}$ (plus the complete integrability of D_R),*
   (b) *the generator of $K^0 = HK^0 \subset \mathfrak{k}_{-1} \otimes \Lambda^2_R$ is lifted to the generator of $H\overline{K}^0 \subset \mathfrak{g}_{-2} \otimes (\mathfrak{g}^*_{-2} \wedge (\mathfrak{g}^R_{-1})^*)$, (the other generator of $H\overline{K}^0 \subset \mathfrak{g}_{-2} \otimes (\mathfrak{g}^*_{-2} \wedge (\mathfrak{g}^L_{-1})^*)$ vanishes)*
3. *if $n = 2$, $m = 2$, for $K^1 = HK^1 = HK^1_L + HK^1_R$,*
   (a) *the generator of $HK^1_L \subset \mathfrak{k}^L_0 \otimes \Lambda^2_L$ is lifted to the generator of $H\overline{K}^0 \subset \mathfrak{g}^L_{-1} \otimes \Lambda^2(\mathfrak{g}^R_{-1})^*$, the vanishing of the generator of HK^1_L implies the normality condition of $\overline{K}$ (plus the complete integrability of D_R),*

(b) *the generator of* $HK^1_R \subset \mathfrak{k}^R_0 \otimes \Lambda^2_R$ *is lifted to the generator of* $H\overline{K}^2 \subset \mathfrak{g}_0 \otimes (\mathfrak{g}^*_{-2} \wedge (\mathfrak{g}^R_{-1})^*)$ (*the generator of* $H\overline{K}^1 \subset \mathfrak{g}^R_{-1} \otimes (\mathfrak{g}^*_{-2} \wedge (\mathfrak{g}^L_{-1})^*)$ *vanishes*).

Remark that, from Proposition 4.1, 2 (b) in 4.4, if $n = 2$, $m = 3$, the relations are the same ones as the above 2 (a), (b). (The other generator of $H\overline{K}^0 \subset \mathfrak{g}^R_{-1} \otimes \Lambda^2(\mathfrak{g}^L_{-1})^*$ vanishes.)

Remark that the above (3) corresponds to the argument of the case $k = 1$, $l = 2$ in 4.4. Although we do not describe it in Proposition 4.1 in 4.4, compare with the proposition 3 (b).

6.5. Projective structure

Let M be an l-dimensional real manifold. A Grassmannian structure of type $(n, 1)$ on M is defined by an isomorphism σ from the tangent bundle TM of M to the tensor product $V \otimes W$ of two vector bundles V and W with rank n ($n \geq 2$) and 1 over M respectively (cf. 1.1). The flat model like that in 1.2 is the Grassmann manifold $G_{1,n+1} \cong G/G'$ ($G = SL(n+1, \mathbb{R})$) consisting of all 1-dimensional subspaces in the $(n+1)$-dimensional real vector space $\mathbb{R}^{n+1}$. This is nothing but the n-dimensional projective space $P^n(\mathbb{R})$. Therefore Grassmannian structure of type $(n, 1)$ implies n-dimensional projective structure.

As is well known ([Ko], [O]), there exists a normal Cartan (or projective) connection ω of type G/G' on the principal bundle Q with structure group G' over M. The Lie algebra $\mathfrak{g} = \mathfrak{sl}(n+1, \mathbb{R})$ of $G = SL(n+1, \mathbb{R})$ has the structure of a graded Lie algebra of first kind like that in 2.2. We described the component of nonzero generator as $\mathfrak{g}_0$-module in H^2 in a remark of Proposition 3.3 in 3.3.

For each point x of M, a null n-plane in the tangent space T_xM is an only form $V_x \otimes w$ ($w \in W_x \cong \mathbb{R}$). Thus the null n-plane bundle F_L over M is the tangent bundle TM itself. A null 1-plane (line) in T_xM is a form $v \otimes W_x$ ($v \in V_x$) and is a line through the origin. Thus the null line bundle F_R over M is the projective tangent bundle $P(TM)$ with fibre $P^{n-1}(\mathbb{R})$.

A 1-dimensional tautological distribution D_R of null lines on $F_R = P(TM)$ over M is completely integrable because of dimension 1. For the natural projection $\varpi_R : F_R \to M$, put $E_R = \mathrm{Ker}(\varpi_R)_*$. Then E_R is transversal to D_R and the fibre of E_R at $x \in M$ is the projective space $P^{n-1}(\mathbb{R})$ of T_xM. A pair (E_R, D_R) defines a co-Grassmannian structure of type $(n-1, 1)$ on F_R. In the same manner as mentioned Theorem 6.2 in 6.4, the normal

Cartan connection (Q, ω) of an n-dimensional projective structure induces the normal Cartan connection $(Q, \overline{\omega})$ of the co-Grassmannian structure of type $(n-1, 1)$ on F_R.

PROPOSITION 6.3. *Under the above condition, for $K^0 = 0$, $K^1 = HK^1$, the generator of $HK^1 \subset \mathfrak{g}_0 \otimes \Lambda^2$ is lifted to the generator of $H\overline{K}^1 \subset \mathfrak{g}^L_{-1} \otimes (\mathfrak{g}^*_{-2} \wedge (\mathfrak{g}^R_{-1})^*)$. (the generator of $H\overline{K}^2 \subset \mathfrak{g}_0 \otimes (\mathfrak{g}^*_{-2} \wedge (\mathfrak{g}^L_{-1})^*)$ vanishes.) (when $n = 3$, the generator of $H\overline{K}^0 \subset \mathfrak{g}^R_{-1} \otimes \Lambda(\mathfrak{g}^L_{-1})^*$ vanishes.)*

§7. Twistor theory of Grassmannian structures

7.1. Twistor diagrams

In the real $(n+m)$-dimensional vector space $V = \mathbb{R}^{n+m}$ $(m \geq 2)$, define

$$\begin{aligned}
\mathbf{G}_1 &= G_{m,n+m} = \{m\text{-dimensional subspace of } V\},\\
\mathbf{G}_2 &= G_{m-1,n+m} = \{(m-1)\text{-dimensional subspace of } V\},\\
\mathbf{F} &= F_{m-1,m}\\
&= \{(S_{m-1}, S_m) \mid S_i\text{: } i\text{-dimensional subspace of } V,\ S_{m-1} \subset S_m\}.
\end{aligned}$$

We have the double fibration that is considered as the twistor diagram of the flat model (cf. [W-W], [W]):

$$\begin{tikzcd}
 & \mathbf{F} \arrow[dl, "\mu"'] \arrow[dr, "\nu"] & \\
\mathbf{G}_2 & & \mathbf{G}_1,
\end{tikzcd}$$

where μ, ν are the natural projections.

Each space $\mathbf{G}_1$, $\mathbf{G}_2$ and $\mathbf{F}$ has a natural G structure (geometric structure).

The space $\mathbf{G}_1$ has the Grassmannian structure of type (n, m), as is explained in 1.2. Each null n-submanifold of $\mathbf{G}_1$ is diffeomorphic to $P^n(\mathbb{R})$. Then $\mathbf{G}_2$ can be regarded as the space of all null n-submanifolds. Let m_1 be a point in $\mathbf{G}_1$. The set of all null n-submanifolds through m_1 is diffeomorphic to $P^{m-1}(\mathbb{R})$ in $\mathbf{G}_1$. Remark that there is the other space of all null m-submanifolds of $\mathbf{G}_1$.

The space $\mathbf{G}_2$ has the Grassmannian structure of type $(n+1, m-1)$. Each null $(m-1)$-submanifold of $\mathbf{G}_2$ is diffeomorphic to $P^{m-1}(\mathbb{R})$. Then $\mathbf{G}_1$ can be regarded as the space of all null $(m-1)$-submanifolds. Let m_2 be a point in $\mathbf{G}_2$. The set of all null $(m-1)$-submanifolds through m_2 is diffeomorphic to $P^n(\mathbb{R})$ in $\mathbf{G}_1$. Remark that there is the other space of all null $(n+1)$-submanifolds of $\mathbf{G}_2$.

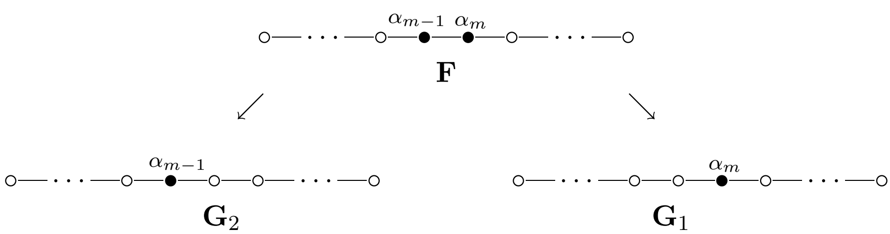
The space $\mathbf{F}$ has the co-Grassmannian structure of type $(n, m-1)$, as is explained in 4.2. There are two transversal n-dimensional and $(m-1)$-dimensional foliations. Each leaf is diffeomorphic to $P^n(\mathbb{R})$ and $P^{m-1}(\mathbb{R})$ respectively. The former leaf space is identified with $\mathbf{G}_2$ and the latter leaf space $\mathbf{G}_1$.

The three spaces are regarded as homogeneous spaces of $G = SL(m+n, \mathbb{R})$:

$$\mathbf{G}_1 = G/H_1, \quad \mathbf{G}_2 = G/H_2, \quad \mathbf{F} = G/H_{12}.$$

In complexified flag manifolds, we study them. Choose the Cartan subalgebra consisting of all the diagonal matrices in the Lie algebra A_l $(l = m+n-1) = \mathfrak{g}_{\mathbb{C}} = \mathfrak{sl}(m+n, \mathbb{C})$ of $G_{\mathbb{C}} = SL(m+n, \mathbb{C})$. Denote the simple root system by $\Delta = \{\alpha_1, \ldots, \alpha_{m+n-1}\}$.

The manifolds $\mathbf{G}_1$, $\mathbf{G}_2$ and $\mathbf{F}$ are flag manifolds corresponding to the parabolic subalgebras defined by $\Delta_1 = \{\alpha_m\}$, $\{\alpha_{m-1}\}$, $\{\alpha_{m-1}, \alpha_m\}$ respectively. We indicate them by the double fibration in terms of the Dynkin diagrams:



The simple graded Lie algebras of $\mathfrak{g} = \mathfrak{sl}(m+n, \mathbb{R})$ associated with $\mathbf{G}_1$, $\mathbf{G}_2$, $\mathbf{F}$ are of first kind, first kind (see §2, §3), second kind (see §4) respectively:

$$\begin{aligned}
\mathbf{G}_1 &: \mathfrak{g} = \mathfrak{g}^1_{-1} \oplus \mathfrak{g}^1_0 \oplus \mathfrak{g}^1_1, \quad \mathfrak{m}_1 = \mathfrak{g}^1_{-1}, \quad \dim \mathfrak{m}_1 = mn, \\
\mathbf{G}_2 &: \mathfrak{g} = \mathfrak{g}^2_{-1} \oplus \mathfrak{g}^2_0 \oplus \mathfrak{g}^2_1, \quad \mathfrak{m}_2 = \mathfrak{g}^2_{-1}, \quad \dim \mathfrak{m}_2 = mn + m - n - 1, \\
\mathbf{F} &: \mathfrak{g} = \mathfrak{g}^{12}_{-2} \oplus \mathfrak{g}^{12}_{-1} \oplus \mathfrak{g}^{12}_0 \oplus \mathfrak{g}^{12}_1 \oplus \mathfrak{g}^{12}_2, \quad \mathfrak{m}_{12} = \mathfrak{g}^{12}_{-2} \oplus \mathfrak{g}^{12}_{-1}, \\
&\qquad \dim \mathfrak{m}_{12} = mn + m - 1, \quad \dim \mathfrak{g}^{12}_{-2} = n(m-1), \\
&\qquad \dim \mathfrak{g}^{12}_{-1} = n + m - 1.
\end{aligned}$$

Nonzero generators in H^2 are as follows. See 3.3 and 4.4 (cf. [Y]) in detail.

For A_l; $m-1 \geq 3$, i.e., $m \geq 4$,

$$\begin{aligned}
H^2(\mathfrak{m}_1, \mathfrak{g}) &: x_{m,m-1} \in H^{0,2} \subset \mathfrak{g}^1_{-1} \otimes \Lambda^2_{-2}, \\
& x_{m,m+1} \in H^{0,2} \subset \mathfrak{g}^1_{-1} \otimes \Lambda^2_{-2}, \\
H^2(\mathfrak{m}_2, \mathfrak{g}) &: x_{m-1,m-2} \in H^{0,2} \subset \mathfrak{g}^2_{-1} \otimes \Lambda^2_{-2}, \\
& x_{m-1,m} \in H^{0,2} \subset \mathfrak{g}^2_{-1} \otimes \Lambda^2_{-2}, \\
H^2(\mathfrak{m}_{12}, \mathfrak{g}) &: x_{m-1,m} \in H^{0,2} \subset \mathfrak{g}^{12}_{-2} \otimes \Lambda^2_{-3}, \\
& x_{m,m-1} \in H^{0,2} \subset \mathfrak{g}^{12}_{-2} \otimes \Lambda^2_{-3}.
\end{aligned}$$

$$\circ - \cdots - \circ - \bullet - \bullet - \circ - \cdots - \circ$$

$$x_{m-1,m},\ x_{m,m-1}$$

$$\swarrow \qquad\qquad \searrow$$

$$\underset{x_{m-1,m-2},\ x_{m-1,m}}{\circ - \cdots \circ - \bullet - \circ - \circ \cdots - \circ} \qquad \underset{x_{m,m-1},\ x_{m,m+1}}{\circ - \cdots \circ - \circ - \bullet - \circ \cdots - \circ}.$$

For A_l $(l \geq 5)$, $m+n \geq 6$; $m-1 = 2$, i.e., $m = 3$,

$$\begin{aligned}
H^2(\mathfrak{m}_1, \mathfrak{g}) &: x_{32} \in H^{0,2} \subset \mathfrak{g}^1_{-1} \otimes \Lambda^2_{-2}, \\
& x_{34} \in H^{0,2} \subset \mathfrak{g}^1_{-1} \otimes \Lambda^2_{-2}, \\
H^2(\mathfrak{m}_2, \mathfrak{g}) &: x_{21} \in H^{1,2} \subset \mathfrak{g}^2_{-1} \otimes \Lambda^2_{-2}, \\
& x_{23} \in H^{0,2} \subset \mathfrak{g}^2_{-1} \otimes \Lambda^2_{-2}, \\
H^2(\mathfrak{m}_{12}, \mathfrak{g}) &: x_{21} \in H^{0,2} \subset \mathfrak{g}^{12}_{-1} \otimes \Lambda^2_{-2}, \\
& x_{23} \in H^{0,2} \subset \mathfrak{g}^{12}_{-2} \otimes \Lambda^2_{-3}, \\
& x_{32} \in H^{0,2} \subset \mathfrak{g}^{12}_{-2} \otimes \Lambda^2_{-3}.
\end{aligned}$$

$$\circ - \bullet - \bullet - \circ - \cdots - \circ$$

$$x_{21},\ x_{23},\ x_{32}$$

$$\swarrow \qquad\qquad \searrow$$

$$\underset{x_{21},\ x_{23}}{\circ - \bullet - \circ - \circ - \cdots - \circ} \qquad \underset{x_{32},\ x_{34}}{\circ - \circ - \bullet - \cdots - \circ}.$$

For A_4, $m+n = 5$; $m-1 = 2$, i.e., $m = 3$,

$$\begin{aligned}
H^2(\mathfrak{m}_1, \mathfrak{g}) &: x_{32} \in H^{0,2} \subset \mathfrak{g}^1_{-1} \otimes \Lambda^2_{-2}, \\
& x_{34} \in H^{0,2} \subset \mathfrak{g}^1_{-1} \otimes \Lambda^2_{-2}, \\
H^2(\mathfrak{m}_2, \mathfrak{g}) &: x_{21} \in H^{1,2} \subset \mathfrak{g}^2_{-1} \otimes \Lambda^2_{-2}, \\
& x_{23} \in H^{1,2} \subset \mathfrak{g}^2_{-1} \otimes \Lambda^2_{-2},
\end{aligned}$$

$$\begin{aligned} H^2(\mathfrak{m}_{12},\mathfrak{g}) : x_{21} &\in H^{0,2} \subset \mathfrak{g}^{12}_{-1} \otimes \Lambda^2_{-2}, \\ x_{23} &\in H^{0,2} \subset \mathfrak{g}^{12}_{-2} \otimes \Lambda^2_{-3}, \\ x_{32} &\in H^{0,2} \subset \mathfrak{g}^{12}_{-2} \otimes \Lambda^2_{-3}, \\ x_{34} &\in H^{0,2} \subset \mathfrak{g}^{12}_{-1} \otimes \Lambda^2_{-2}. \end{aligned}$$

$$\begin{array}{ccc} & \circ\!-\!\bullet\!-\!\bullet\!-\!\circ & \\ & x_{21},\ x_{23},\ x_{32},\ x_{34} & \\ \swarrow & & \searrow \\ \circ\!-\!\bullet\!-\!\circ\!-\!\circ & & \circ\!-\!\circ\!-\!\bullet\!-\!\circ. \\ x_{21},\ x_{23} & & x_{32},\ x_{34} \end{array}$$

We consider the case $m = 2$ especially.
In the real $(n+2)$-dimensional vector space $V = \mathbb{R}^{n+2}$, define

$$\begin{aligned} \mathbf{G} &= G_{2,n+2} = \{2\text{-dimensional subspace of } V\}, \\ \mathbf{P} &= G_{1,n+2} = P^{n+1}(\mathbb{R}) = \{1\text{-dimensional subspace of } V\}, \\ \mathbf{F} &= F_{12} = \{(S_1, S_2) \mid S_i\text{: } i\text{-dimensional subspace of } V,\ S_1 \subset S_2\}. \end{aligned}$$

We have the double fibration that is considered as the twistor diagram of the flat model (cf. [W-W], [W]):

$$\begin{tikzcd} & \mathbf{F} \arrow[dl, "\mu"'] \arrow[dr, "\nu"] & \\ \mathbf{P} & & \mathbf{G}, \end{tikzcd}$$

where μ, ν are the natural projections.

The space $\mathbf{G}$ has the Grassmannian structure of type $(n, 2)$. Each null n-submanifold of $\mathbf{G}$ is diffeomorphic to $P^n(\mathbb{R})$. Then $\mathbf{P}$ can be regarded as the space of all null n-submanifolds. Let m_1 be a point in $\mathbf{G}$. The set of all null n-submanifolds through m_1 is diffeomorphic to $P^1(\mathbb{R}) \cong S^1$ in $\mathbf{P}$. Remark that there is the other space of all null 2-submanifolds of $\mathbf{G}$.

The space $\mathbf{P}$ has the projective structure. Each projective line of $\mathbf{P}$ is diffeomorphic to $P^1(\mathbb{R})$. Then $\mathbf{G}$ can be regarded as the space of all projective lines. Let m_2 be a point in $\mathbf{P}$. The set of all projective lines through m_2 is diffeomorphic to $P^n(\mathbb{R})$ in $\mathbf{G}$.

The space $\mathbf{F}$ has the co-Grassmannian structure of type $(n, 1)$. There are two transversal n-dimensional and 1-dimensional foliations. Each leaf is diffeomorphic to $P^n(\mathbb{R})$ and $P^1(\mathbb{R})$ respectively. The former leaf space is identified with $\mathbf{P}$ and the latter leaf space $\mathbf{G}$.

The manifolds **G**, **P** and **F** are flag manifolds corresponding to the parabolic subalgebras defined by $\Delta_1 = \{\alpha_2\}$, $\{\alpha_1\}$, $\{\alpha_1, \alpha_2\}$ respectively. We indicate them by the double fibration in terms of the Dynkin diagrams:

$$\begin{array}{ccc} & \underset{\mathbf{F}}{\bullet\!-\!\!-\bullet\!-\!\!-\circ\!-\!\!-\cdots\!-\!\!-\circ} & \\ \swarrow & & \searrow \\ \underset{\mathbf{P}}{\bullet\!-\!\!-\circ\!-\!\!-\circ\!-\!\!-\cdots\!-\!\!-\circ} & & \underset{\mathbf{G}}{\circ\!-\!\!-\bullet\!-\!\!-\circ\!-\!\!-\cdots\!-\!\!-\circ}. \end{array}$$

Nonzero generators in H^2 are as follows.
For A_l $(l \geq 4)$, $n \geq 4$,

$$\begin{aligned} H^2(\mathfrak{m}_1, \mathfrak{g}) : x_{21} &\in H^{1,2} \subset \mathfrak{g}_0^1 \otimes \Lambda^2_{-2}, \\ x_{23} &\in H^{0,2} \subset \mathfrak{g}_{-1}^1 \otimes \Lambda^2_{-2}, \\ H^2(\mathfrak{m}_2, \mathfrak{g}) : x_{12} &\in H^{1,2} \subset \mathfrak{g}_0^2 \otimes \Lambda^2_{-2}, \\ H^2(\mathfrak{m}_{12}, \mathfrak{g}) : x_{12} &\in H^{1,2} \subset \mathfrak{g}_{-1}^{12} \otimes \Lambda^2_{-3}, \\ x_{21} &\in H^{2,2} \subset \mathfrak{g}_0^{12} \otimes \Lambda^2_{-3}. \end{aligned}$$

$$\begin{array}{ccc} & \underset{x_{12},\ x_{21}}{\bullet\!-\!\!-\bullet\!-\!\!-\circ\!-\!\!-\cdots\!-\!\!-\circ} & \\ \swarrow & & \searrow \\ x_{12}\ \underset{x_{12}}{\bullet\!-\!\!-\circ\!-\!\!-\circ\!-\!\!-\cdots\!-\!\!-\circ} & & \underset{x_{21},\ x_{23}}{\circ\!-\!\!-\bullet\!-\!\!-\circ\!-\!\!-\cdots\!-\!\!-\circ}. \end{array}$$

For A_3, $n = 3$,

$$\begin{aligned} H^2(\mathfrak{m}_1, \mathfrak{g}) : x_{21} &\in H^{1,2} \subset \mathfrak{g}_0^1 \otimes \Lambda^2_{-2}, \\ x_{23} &\in H^{1,2} \subset \mathfrak{g}_0^1 \otimes \Lambda^2_{-2}, \\ H^2(\mathfrak{m}_2, \mathfrak{g}) : x_{12} &\in H^{1,2} \subset \mathfrak{g}_0^2 \otimes \Lambda^2_{-2}, \\ H^2(\mathfrak{m}_{12}, \mathfrak{g}) : x_{12} &\in H^{1,2} \subset \mathfrak{g}_{-1}^{12} \otimes \Lambda^2_{-3}, \\ x_{21} &\in H^{2,2} \subset \mathfrak{g}_0^{12} \otimes \Lambda^2_{-3}, \\ x_{23} &\in H^{0,2} \subset \mathfrak{g}_{-1}^{12} \otimes \Lambda^2_{-2}. \end{aligned}$$

$$\begin{array}{ccc} & \underset{x_{12},\ x_{21},\ x_{23}}{\bullet\!-\!\!-\bullet\!-\!\!-\circ} & \\ \swarrow & & \searrow \\ \underset{x_{12}}{\bullet\!-\!\!-\circ\!-\!\!-\circ} & & \underset{x_{21},\ x_{23}}{\circ\!-\!\!-\bullet\!-\!\!-\circ}. \end{array}$$

Summarizing them, we have a twistor diagram chain of Grassmannian structures. Let V be a real $(n+m)$-dimensional vector space. We consider $G = SL(m+n, \mathbb{R})$. We write $G(m,n)$ for $G_{m,n+m}$. We simply write $F(i_1, i_2, \ldots, i_k)$ for $F_{i_1,i_2,\ldots,i_k} = \{(S_{i_1}, S_{i_2}, \ldots, S_{i_k}) \mid S_{i_1} \subset S_{i_2} \subset \cdots \subset S_{i_k},\ S_{i_l}\colon i_l\text{-dimensional subspace of } V\}$. Note that $G(m,n) = F(m)$, $G(1,n) = P^n(\mathbb{R})$. Then, for $m+n \geq 6$, we have the following chain. We write nonzero generators in H^2 (Diagram 1).

$$\begin{aligned}
&\{\text{point in } G(m-1,n+1)\}\\
&\quad \longleftrightarrow \{n\text{-dimensional null surface in } G(m,n)\},\\
&\{(m-1)\text{-dimensional null surface in } G(m-1,n+1)\}\\
&\quad \longleftrightarrow \{\text{point in } G(m,n)\},\\
&\{m\text{-dimensional null surface in } G(m,n)\}\\
&\quad \longleftrightarrow \{\text{point in } G(m+1,n-1)\},\\
&\{\text{point in } G(m,n)\}\\
&\quad \longleftrightarrow \{(n-1)\text{-dimensional null surface in } G(m+1,n-1)\}.
\end{aligned}$$

For $m+n=5$, let V be a 5-dimensional real vector space and $G = SL(5,\mathbb{R})$. We have the following.

$$\begin{array}{ccccccc}
 & \bullet\!-\!\bullet\!-\!\circ\!-\!\circ & & \circ\!-\!\bullet\!-\!\bullet\!-\!\circ & & \circ\!-\!\circ\!-\!\bullet\!-\!\bullet & \\
 & F_{12} & & F_{23} & & F_{34} & \\
 & x_{12},\ x_{21} & & x_{21},\ x_{23},\ x_{32},\ x_{34} & & x_{34},\ x_{43} & \\
\swarrow & & \searrow\ \swarrow & & \searrow\ \swarrow & & \searrow \\
\circ\!-\!\bullet\!-\!\bullet\!-\!\circ & & \circ\!-\!\circ\!-\!\bullet\!-\!\bullet & & \bullet\!-\!\circ\!-\!\circ\!-\!\circ & & \circ\!-\!\circ\!-\!\circ\!-\!\bullet \\
P^4(\mathbb{R}) & & G_{2,5}(\mathbb{R}) & & G_{3,5}(\mathbb{R}) & & (P^4(\mathbb{R}))^*. \\
x_{12} & & x_{21},\ x_{23} & & x_{32},\ x_{34} & & x_{43}
\end{array}$$

For $m+n=4$, let V be a 4-dimensional real vector space and $G = SL(4,\mathbb{R})$. We have the following.

$$\begin{array}{ccccc}
 & \bullet\!-\!\bullet\!-\!\circ & & \circ\!-\!\bullet\!-\!\bullet & \\
 & F_{12} & & F_{23} & \\
 & x_{12},\ x_{21},\ x_{23} & & x_{21},\ x_{23},\ x_{32} & \\
\swarrow & & \searrow\ \swarrow & & \searrow \\
\bullet\!-\!\circ\!-\!\circ & & \circ\!-\!\bullet\!-\!\bullet & & \circ\!-\!\circ\!-\!\bullet \\
P^3(\mathbb{R}) & & G_{2,4}(\mathbb{R}) & & (P^3(\mathbb{R}))^*. \\
x_{12} & & x_{21},\ x_{23} & & x_{32}
\end{array}$$

$$
\begin{tikzcd}[column sep=small]
 & \begin{array}{c} F(1,2) \\ x_{12}, x_{21} \end{array} \arrow[dl] \arrow[dr] & & \begin{array}{c} F(2,3) \\ x_{23}, x_{32} \end{array} \arrow[dl] \arrow[dr] & \\
\begin{array}{c} F(1) \\ = G(1, n+m-1) \\ = P^{n+m-1} \\ x_{12} \end{array} & & \begin{array}{c} F(2) \\ = G(2, n+m-2) \\ \\ x_{21},\ x_{23} \end{array} & & \cdots\cdots
\end{tikzcd}
$$

$$
\begin{tikzcd}[column sep=small]
 & & \begin{array}{c} F(m-1,m) \\ x_{m-1,m},\ x_{m,m-1} \end{array} \arrow[dl] \arrow[dr] & & \begin{array}{c} F(m,m+1) \\ x_{m,m+1},\ x_{m+1,m} \end{array} \arrow[dl] \arrow[dr] & & \\
\cdots & \begin{array}{c} F(m-1) \\ = G(m-1, n+1) \\ x_{m-1,m-2},\ x_{m-1,m} \end{array} & & \begin{array}{c} F(m) \\ = G(m,n) \\ x_{m,m-1},\ x_{m,m+1} \end{array} & & \begin{array}{c} F(m+1) \\ = G(m+1, n-1) \\ x_{m+1,m},\ x_{m+1,m+2} \end{array} & \cdots
\end{tikzcd}
$$

$$
\begin{tikzcd}[column sep=small]
 & \begin{array}{c} F(n+m-3, n+m-2) \\ x_{n+m-3,n+m-2}, \\ x_{n+m-2,n+m-3} \end{array} \arrow[dl] \arrow[dr] & & \begin{array}{c} F(n+m-2, n+m-1) \\ x_{n+m-2,n+m-1}, \\ x_{n+m-1,n+m-2} \end{array} \arrow[dl] \arrow[dr] & \\
\cdots\cdots & & \begin{array}{c} F(n+m-2) \\ = G(n+m-2, 2) \\ x_{n+m-2,n+m-3}, \\ x_{n+m-2,n+m-1} \end{array} & & \begin{array}{c} F(n+m-1) \\ = G(n+m-1, 1) \\ = (P^{n+m-1})^{*} \\ x_{n+m-1,n+m-2} \end{array}
\end{tikzcd}
$$

We see the contents in detail.

$$
\begin{tikzcd}[column sep=small]
 & \begin{array}{c} F(m-1,m) \\ \text{co-Grassmannian structure} \\ \text{of type } (n, m-1) \end{array} \arrow[dl] \arrow[dr] & & \begin{array}{c} F(m,m+1) \\ \text{co-Grassmannian structure} \\ \text{of type } (n-1, m) \end{array} \arrow[dl] \arrow[dr] & \\
\begin{array}{c} G(m-1, n+1) \\ \text{Grassmannian structure} \\ \text{of type } (n+1, m-1) \end{array} & & \begin{array}{c} G(m,n) \\ \text{Grassmannian structure} \\ \text{of type } (n,m) \end{array} & & \begin{array}{c} G(m+1, n-1) \\ \text{Grassmannian structure} \\ \text{of type } (n-1, m+1). \end{array}
\end{tikzcd}
$$

Diagram 1.

7.2. Reduction theorem

Let F be a manifold with a co-Grassmannian structure of type (k,l) by a pair (D_2, D_1) and equipped with the normal Cartan connection (Q,ω). The dimension of F is $k+l+kl$. By definition, D_2 is a subbundle of TF with rank k and is completely integrable, and D_1 is a subbundle of TF with rank l and is completely integrable.

For F, leaf spaces

$$M_1 = F/D_1, \quad M_2 = F/D_2$$

are defined and they are (locally) manifolds with Grassmannian structures of type $(k,l+1)$ and of type $(k+1,l)$ respectively. Let

$$\nu : F \longrightarrow M_1, \quad \mu : F \longrightarrow M_2$$

be canonical projections. Then

$$F \longrightarrow M_2 \times M_1;\ x \longmapsto (\mu(x), \nu(x))$$

is an embedding locally. We have the following double fibration:

$$\begin{array}{ccc} & F & \\ {}^{\mu}\swarrow & & \searrow^{\nu} \\ M_2 & & M_1. \end{array}$$

We consider the harmonic part HK of the curvature function K of the normal Cartan connection (Q,ω) over F. The places of nonzero generators in H^2 in 7.1 (3.3, 4.4) show when (Q,ω) is reduced to that over M_1 or M_2. We have the following reduction theorem.

Theorem 7.1. *We have*

1. *if* $l \geq 3$, *we have* $K^0 = HK^0 = HK = (HK^0)_1 + (HK^0)_2$, *and*
   (i) $(HK^0)_1$ $(\subset \mathfrak{g}_{-2} \otimes (\mathfrak{g}_{-2}^* \wedge (\mathfrak{g}_{-1}^R)^*)) = 0$, *if and only if* (Q,ω) *is reduced to the normal Cartan connection* ω_1 *of type* G/H_1 *on* Q_1 *over* M_1 *and* $(Q,\omega) = (Q_1,\omega_1)_{|F_L}$, $\omega = \nu^*\omega_1$,
   (ii) $(HK^0)_2$ $(\subset \mathfrak{g}_{-2} \otimes (\mathfrak{g}_{-2}^* \wedge (\mathfrak{g}_{-1}^L)^*)) = 0$, *if and only if* (Q,ω) *is reduced to the normal Cartan connection* ω_2 *of type* G/H_2 *on* Q_2 *over* M_2 *and* $(Q,\omega) = (Q_2,\omega_2)_{|F_R}$, $\omega = \mu^*\omega_2$,
2. *if* $l = 2$,

(a) *for* $k+l+1 \geq 6$, *i.e.,* $k \geq 3$, *we have* $K^0 = HK^0 = HK = (HK^0)_1 + (HK^0)_2 + (HK^0)_3$, *and*

(i) $(HK^0)_1$ $(\subset \mathfrak{g}^L_{-1} \otimes \Lambda^2(\mathfrak{g}^R_{-1})^*) = 0$, $(HK^0)_2$ $(\subset \mathfrak{g}_{-2} \otimes (\mathfrak{g}^*_{-2} \wedge (\mathfrak{g}^R_{-1})*)) = 0$, *if and only if* (Q,ω) *is reduced to* (Q_1,ω_1) *on* M_1 (*as in the above* (1) (i)),

(ii) $(HK^0)_3$ $(\subset \mathfrak{g}_{-2} \otimes (\mathfrak{g}^*_{-2} \wedge (\mathfrak{g}^L_{-1})^*)) = 0$, *if and only if* (Q,ω) *is reduced to* (Q_2,ω_2) *on* M_2 (*as in the above* (1) (ii)),

(b) *for* $k+l+1 = 5$, *i.e.,* $k = 2$, *we have* $K^0 = HK^0 = HK = (HK^0)_1 + (HK^0)_2 + (HK^0)_3 + (HK^0)_4$, *and*

(i) $(HK^0)_1$ $(\subset \mathfrak{g}^L_{-1} \otimes \Lambda^2(\mathfrak{g}^R_{-1})^*) = 0$, $(HK^0)_2$ $(\subset \mathfrak{g}_{-2} \otimes (\mathfrak{g}^*_{-2} \wedge (\mathfrak{g}^R_{-1})^*)) = 0$, *if and only if* (Q,ω) *is reduced to* (Q_1,ω_1) *on* M_1 (*as in the above* (1) (i)),

(ii) $(HK^0)_3$ $(\subset \mathfrak{g}_{-2} \otimes (\mathfrak{g}^*_{-2} \wedge (\mathfrak{g}^L_{-1})^*)) = 0$, $(HK^0)_4$ $(\subset \mathfrak{g}^R_{-1} \otimes \Lambda^2(\mathfrak{g}^L_{-1})^*) = 0$, *if and only if* (Q,ω) *is reduced to* (Q_2,ω_2) *on* M_2 (*as in the above* (1) (ii)),

3. *if* $l = 1$,

(a) *for* $k+l+1 \geq 5$, *i.e.,* $k \geq 3$, *we have* $HK = HK^1 + HK^2$, *and*

(i) HK^1 $(\subset \mathfrak{g}^L_{-1} \otimes (\mathfrak{g}^*_{-2} \wedge (\mathfrak{g}^R_{-1})^*)) = 0$, *if and only if* (Q,ω) *is reduced to* (Q_1,ω_1) *on* M_1 *with a Grassmannian structure of type* $(k,2)$ (*as in the above* (1) (i)),

(ii) HK^2 $(\subset \mathfrak{g}_0 \otimes (\mathfrak{g}^*_{-2} \wedge (\mathfrak{g}^L_{-1})^*)) = 0$, *if and only if* (Q,ω) *is reduced to* (Q_2,ω_2) *on* M_2 *with a* $(k+1)$*-dimensional projective structure* (*as in the above* (1) (ii),

(b) *for* $k+l+1 = 4$, *i.e.,* $k = 2$, *we have* $HK = HK^0 + HK^1 + HK^2$, *and*

(i) HK^1 $(\subset \mathfrak{g}^L_{-1} \otimes (\mathfrak{g}^*_{-2} \wedge (\mathfrak{g}^R_{-1})^*)) = 0$, *if and only if* (Q,ω) *is reduced to* (Q_1,ω_1) *on* M_1 *with a Grassmannian structure of type* $(2,2)$ (*as in the above* (1) (i)),

(ii) HK^2 $(\subset \mathfrak{g}_0 \otimes (\mathfrak{g}^*_{-2} \wedge (\mathfrak{g}^L_{-1})^*)) = 0$, HK^0 $(\subset \mathfrak{g}^R_{-1} \otimes \Lambda^2(\mathfrak{g}^L_{-1})^*) = 0$, *if and only if* (Q,ω) *is reduced to* (Q_2,ω_2) *on* M_2 *with a 3-dimensional projective structure* (*as in the above* (1) (ii)).

In the theorem above, 3 (a), (b) are Tanaka's results ([T3]).

7.3. Relation between both Grassmannian structures

Let M_1 be a manifold with a Grassmannian structure of type (n, m). Assume that $n, m \geq 4$. Suppose that the Grassmannian structure has $HK_R^0 = 0$ for the normal Cartan connection ω_1. The component of the generator $x_{m,m+1} \in H^{0,2}(\mathfrak{m}_1, \mathfrak{g})$ is 0. Then, from Theorem 5.1 in 5.3, the n-dimensional tautological distribution D_L of null n-planes on the null n-plane bundle F_L over M_1 is completely integrable. Therefore a co-Grassmannian structure of type $(n, m-1)$ is defined on F_L. Since, of course, the structure on F_L is reduced onto M_1, by the reduction theorem 7.1 $(HK^0)_1 = 0$ for $K^0 = HK^0 = HK = (HK^0)_1 + (HK^0)_2$ of the normal Cartan connection ω on F_L. The component of the generator $x_{m-1,m} \in H^{0,2}(\mathfrak{m}_{12}, \mathfrak{g})$ is 0. A condition for a Grassmannian structure of type $(n+1, m-1)$ to be defined on the (locally) $(mn + m - n - 1)$-dimensional manifold $M_2 = F_L/D_L$ is $(HK^0)_2 = 0$. The component of the generator $x_{m,m-1} \in H^{0,2}(\mathfrak{m}_{12}, \mathfrak{g})$ is 0. That is to say, if a Grassmannian structure of type $(n+1, m-1)$ is defined on M_2, a co-Grassmannian structure of type $(n, m-1)$ on F_L must be flat. Then the Grassmannian structure of type $(n+1, m-1)$ on M_2 is also flat. And the right-half torsion-free Grassmannian structure of type (n, m) on M_1 is also flat.

If we assume that $m = 3$, we have the same conclusion as in the case $n, m \geq 4$.

We discuss the converse. Let M_2 be a manifold with a Grassmannian structure of type $(n+1, m-1)$. Assume that $n, m \geq 4$. Suppose that the Grassmannian structure has $HK_L^0 = 0$ for the normal Cartan connection ω_2. The component of the generator $x_{m-1,m-2} \in H^{0,2}(\mathfrak{m}_2, \mathfrak{g})$ is 0. Then, from Theorem 6.1 in 6.3, the $(m-1)$-dimensional tautological distribution D_R of null $(m-1)$-planes on the null $(m-1)$-plane bundle F_R over M_2 is completely integrable. Therefore a co-Grassmannian structure of type $(n, m-1)$ is defined on F_R. Since, of course, the structure on F_R is reduced onto M_2, by the reduction theorem 7.1 $(HK^0)_2 = 0$ for $K^0 = HK^0 = HK = (HK^0)_1 + (HK^0)_2$ of the normal Cartan connection ω on F_R. The component of the generator $x_{m,m-1} \in H^{0,2}(\mathfrak{m}_{12}, \mathfrak{g})$ is 0. A condition for a Grassmannian structure of type (n, m) to be defined on the (locally) mn-dimensional manifold $M_1 = F_R/D_R$ is $(HK^0)_1 = 0$. The component of the generator $x_{m-1,m} \in H^{0,2}(\mathfrak{m}_{12}, \mathfrak{g})$ is 0. That is to say, if a Grassmannian structure of type (n, m) is defined on M_1, a co-Grassmannian structure of type $(n, m-1)$ on F_R must be flat. Then the Grassmannian structure of type (n, m) on M_1 is also flat. And the left-half torsion-free Grassmannian structure of type $(n+1, m-1)$ on M_2 is also flat.

If we assume that $m = 3$, we have the same conclusion as in the case $n, m \geq 4$.

Consequently, if $n, m \geq 3$, we have the following.

Theorem 7.2.

1. *Let M_1 be a manifold with a right-half torsion-free Grassmannian structure of type (n, m). Then, if the structure on M_1 induces a Grassmannian structure of type $(n+1, m-1)$ on $M_2 = F_L/D_L$, the Grassmannian structure of type (n, m) on M_1 is flat.*
2. *Let M_2 be a manifold with a left-half torsion-free Grassmannian structure of type $(n+1, m-1)$. Then, if the structure on M_2 induces a Grassmannian structure of type (n, m) on $M_1 = F_R/D_R$, the Grassmannian structure of type $(n+1, m-1)$ on M_2 is flat.*

Here, in 1 F_L denotes the null n-plane bundle on M_1, and in 2 F_R denotes the null $(m-1)$-plane bundle on M_2.

In the case $m = 2$ we discuss the above argument.

Let M_1 be a $2n$-dimensional manifold with a Grassmannian structure of type $(n, 2)$. Assume that $n \geq 4$. Suppose that the Grassmannian structure has $HK^0 = 0$ for the normal Cartan connection ω_1. The component of the generator $x_{23} \in H^{0,2}(\mathfrak{m}_1, \mathfrak{g})$ is 0. Then, from Theorem 5.1 in 5.3, the n-dimensional tautological distribution D_L of null n-planes on the null n-plane bundle F_L over M_1 is completely integrable. Therefore a co-Grassmannian structure of type $(n, 1)$ is defined on F_L. Since, of course, the structure on F_L is reduced onto M_1, by the reduction theorem 7.1 $HK^1 = 0$ for $HK = HK^1 + HK^2$ of the normal Cartan connection ω on F_L. The component of the generator $x_{12} \in H^{1,2}(\mathfrak{m}_{12}, \mathfrak{g})$ is 0. A condition for a projective structure to be defined on the (locally) $(n+1)$-dimensional manifold $M_2 = F_L/D_L$ is $HK^2 = 0$. The component of the generator $x_{21} \in H^{2,2}(\mathfrak{m}_{12}, \mathfrak{g})$ is 0. That is to say, if a projective structure is defined on M_2, a co-Grassmannian structure of type $(n, 1)$ on F_L must be flat. Then the projective structure on M_2 is also flat. And the right-half Grassmannian flat Grassmannian structure of type $(n, 2)$ on M_1 is also flat.

If we assume that $n = 3$, we have the same conclusion as in the case $n \geq 4$.

We discuss the converse.

Let M_2 be an $(n+1)$-dimensional manifold with a projective structure. Assume that $n \geq 4$. In the $(2n+1)$-dimensional projective tangent bundle $F_R = P(TM_2)$ of M_2, fibers $P^n(\mathbb{R})$ over M_2 define an n-dimensional distribution D_L on F_R. On the other hand, the geodesic flow vector field on F_R with respect to the normal Cartan connection of the projective structure on M_2 defines a 1-dimensional distribution D_R on F_R. Therefore a co-Grassmannian structure of type $(n, 1)$ is defined on F_R. Since, of course, the structure on F_R is reduced onto M_2, by the reduction theorem 7.1 $HK^2 = 0$ for $HK = HK^1 + HK^2$ of the normal Cartan connection F_R. The component of the generator $x_{21} \in H^{2,2}(\mathfrak{m}_{12}, \mathfrak{g})$ is 0. A condition for a Grassmannian structure of type $(n, 2)$ to be defined on the (locally) $2n$-dimensional manifold $M_1 = F_R/D_R$ is $HK^1 = 0$. The component of the generator $x_{12} \in H^{1,2}(\mathfrak{m}_{12}, \mathfrak{g})$ is 0. That is to say, if a Grassmannian structure of type $(n, 2)$ is defined on M_1, a co-Grassmannian structure of type $(n, 1)$ on F_R must be flat. Then the projective structure on M_2 is also flat. And since the n-dimensional distribution D_L on F_R is completely integrable, the Grassmannian structure on M_1 has $HK^0 = 0$ for the normal Cartan connection. The component of the generator $x_{23} \in H^{0,2}$ is 0. Therefore the Grassmannian structure of type $(n, 2)$ on M_1 is also flat.

If we assume that $n = 3$, we have the same conclusions as in the case $n \geq 4$.

If $m = 2$, we have the following.

Theorem 7.3.

1. *Let M_1 be a $2n$-dimensional manifold with a right-half Grassmannian flat Grassmannian structure of type $(n, 2)$ (if $n \geq 3$, equivalently torsion-free). Then, if the structure on M_1 induces a projective structure on M_2, the Grassmannian structure of type $(n, 2)$ on M_1 is flat.*

2. *Let M_2 be an $(n+1)$-dimensional manifold with a projective structure. Then, if the structure on M_2 induces a Grassmannian structure of type $(n, 2)$ on the orbit space M_1 of the geodesic flow, the projective structure on M_2 is flat.*

§8. Twistor theory by Weyl connections

In this section, we will investigate the twistor theory between projective structures and Grassmannian structures of type $(n, 2)$. By imposing a restriction to projective structures, we deal with the twistor theory by Weyl connections. See the introduction.

In the next three subsections, we recall the notions of Einstein-Weyl structure, Lie contact structure and geodesic flow before studying the twistor theory by Weyl connections.

8.1. Einstein-Weyl structure

Let M be an n-dimensional manifold with a conformal structure. The conformal structure is represented by a conformal class $C = [g]$ whose representative is a Riemannian metric g on M.

Let D be a torsion-free linear connection on M which preserves the conformal class C. Namely, for $g \in C$, there exists a 1-form ω_g on M such that

$$Dg = \omega_g \otimes g.$$

For another $g' = e^{\lambda} g \in C$ ($\lambda \in C^{\infty}(M)$), the 1-form $\omega_{g'}$ is given by $\omega_g + d\lambda$. The connection D is called a Weyl connection and we say that M has a Weyl structure. To give a Weyl connection on M is equivalent to give a torsion-free $CO(n)$ connection on the linear frame bundle L of M with structure group $CO(n)$.

A Weyl structure on M is called an Einstein-Weyl structure if the symmetric part Ric^s of the Ricci tensor Ric for the Weyl connection D is proportional to $g \in C$:

$$Ric^s = \Lambda g,$$

where Λ is a generally nonconstant function on M.

Let (M, g) be an Einstein manifold. Then, taking the conformal structure $C = [g]$ of g and the Levi-Civita connection ∇ of g as a Weyl connection D (then $\omega_g = 0$), we have an Einstein-Weyl structure on M. Therefore the notion of Einstein-Weyl is a generalization of the notion of Einstein. The simplest example of an Einstein-Weyl structure that is not Einstein is $S^{n-1} \times S^1$ as a Hopf manifold. On $S^{n-1} \times S^1$ with the conformal class $C = [g_0]$ defined by the standard Riemannian product metric g_0, we can give a Weyl flat connection D by 1-form $\omega_{g_0} = -2dt$, where t is the standard coordinate on S^1. The metric g_0 is not Einstein, but the Weyl connection D is Einstein-Weyl.

Many examples of Einstein-Weyl structures are known (cf. [P-T], [P-S]). There is a twistor correspondence between complex 3-dimensional Einstein-Weyl manifolds and (complex 2-dimensional) mini-twistor spaces ([Hi], [J-T]).

We can decompose the curvature tensor R of a Weyl connection D from the irreducible $CO(n)$-decomposition as follows:

$$R = P + U + Z + W.$$

Here P is the part represented by the distance curvature $\theta = -d\omega_g$ (not depended on $g \in C$), U the part represented by $K = s_g g$ (not depended on $g \in C$) of the scalar curvature s_g, Z the part represented by the symmetric traceless Ricci tensor Ric_s^0 and W the Weyl conformal curvature tensor of C not depending on D.

If D is an Einstein-Weyl connection,

$$Z = 0, \text{ i.e., } R = P + U + W.$$

There are two important subclasses in Einstein-Weyl classes.

One class is Weyl Ricci-flat. It implies that

$$P + U + Z = 0, \text{ i.e., } R = W.$$

Therefore it is decided only by C.

Another class is the set of Weyl structures with constant curvature. It implies that

$$Z + W = 0, \text{ i.e., } R = P + U.$$

This imposes no conditions on P. Therefore it includes wider classes than the constant curvature class of a Levi-Civita connection. The classification problem of Weyl structures with constant curvature is not known.

Let Π be a 2-plane in T_xM $(x \in M)$ and let X, Y be an orthonormal basis of Π with respect to $g \in C$. Then, put

$$K_g(\Pi) = g(R(X, Y)Y, X).$$

It does not depend on the choice of an orthonormal basis of Π. It follows that K_g is constant on all 2-planes if and only if the Weyl structure is of Weyl constant curvature.

If the dimension of M is equal to 3, the notion of Einstein-Weyl structure is equivalent to that of Weyl structure with constant curvature. The proof is similar to that of the equivalence between 3-dimensional Riemannian manifolds which are Einstein and manifolds of constant curvature (cf. [K-No, p. 293]). The examples and the classification of 3-dimensional Einstein-Weyl manifolds are given in [To].

8.2. Lie contact structure

Let M be a $(2n+1)$-dimensional manifold with a contact structure. By definition, there exists a distribution D of codimension 1 on M such that at each point $x \in M$ the $2n$-dimensional subspace $D(x) \subset T_xM$ is defined by

$$D(x) = \{X \in T_xM \mid \theta(X) = 0\},$$

where θ is a local 1-form such that $\theta \wedge (d\theta)^n$ is a volume element. Remark that $d\theta_x|_D$ defines a symplectic structure on D. As θ is unique up to nonzero functions, we have a conformal symplectic structure on the vector subbundle $D \subset TM$.

Put

$$\mathfrak{g}_{-2}(x) = T_xM/D(x), \quad \mathfrak{g}_{-1}(x) = D(x), \quad \mathfrak{m}(x) = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}(x).$$

Then a graded Lie algebra of contact type on $\mathfrak{m}(x)$ is naturally defined by the Lie bracket of vector fields on M. Namely, at each point $x \in M$ $\mathfrak{m}(x)$ is equivalent to the fundamental graded Lie algebra $\mathfrak{m} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$ of contact type. Here $\mathfrak{m}$ is a nilpotent graded Lie algebra and $\dim \mathfrak{g}_{-2} = 1$, $\dim \mathfrak{g}_{-1} = 2n$, $[\,\cdot\,,\,\cdot\,] : \mathfrak{g}_{-1} \times \mathfrak{g}_{-1} \to \mathfrak{g}_{-2}$ is nondegenerate.

Let C be the contact group which is a subgroup of $GL(\mathfrak{m})$ consisting of the set of a linear isomorphism $\sigma : \mathfrak{m} \to \mathfrak{m}$ such that $\sigma(\mathfrak{g}_{-1}) = \mathfrak{g}_{-1}$ and the induced graded map $\sigma : \mathfrak{m} \to \mathfrak{m}$ is a Lie algebra isomorphism. With respect to a contact basis by the decomposition $\mathfrak{m} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$, C is represented by

$$C = \left\{ \begin{pmatrix} c & 0 \\ \xi & A \end{pmatrix} \subset GL(2n+1,\mathbb{R}) \;\middle|\; \begin{array}{l} {}^tAJA = cJ,\ A \in GL(2n,\mathbb{R}), \\ c \neq 0 \in \mathbb{R},\ \xi \in \mathbb{R}^{2n} \end{array} \right\},$$

where $J = \left(\begin{smallmatrix} O_n & -I_n \\ I_n & O_n \end{smallmatrix}\right)$. Remark that A belongs to the conformal symplectic group $CSp(n,\mathbb{R})$.

We mean by a frame z at $x \in M$ a linear isomorphism $z : \mathfrak{m} \cong \mathbb{R}^{2n+1} \to T_xM$. A frame $z : \mathfrak{m} \to T_xM$ is called adapted if $z(\mathfrak{g}_{-1}) = D(x)$ and the induced graded map $z : \mathfrak{m} \to \mathfrak{m}(x)$ is a Lie algebra isomorphism. Let $L_C(M)$ be the set of all adapted frames. Then $L_C(M)$ is a subbundle of the linear frame bundle $L(M)$ with structure group C.

Define a subgroup $\tilde{G}$ of C by

$$\tilde{G} = \left\{ \begin{pmatrix} \det\alpha & 0 \\ \xi & B \otimes \alpha \end{pmatrix} \in GL(2n+1,\mathbb{R}) \;\middle|\; \begin{array}{l} B \in O(n),\ \alpha \in GL(2,\mathbb{R}), \\ \xi \in \mathbb{R}^{2n} \end{array} \right\}.$$

A Lie contact structure on M is by definition a subbundle $\tilde{P}$ of $L_C(M)$ with structure group $\tilde{G}$.

The model space is the unit tangent bundle T_1S^{n+1} of the sphere S^{n+1}. It is a homogeneous space:

$$T_1S^{n+1} \cong G/G', \quad G = PO(n+2,2).$$

The image $\rho(G')$ of the linear isotropy representation $\rho : G' \to GL(\mathfrak{m})$ of G' is $\tilde{G}$. We remark that the Lie contact structures is a G structure of finite type while the general contact structures infinite type. In detail, see [S-Y1], [S-Y2], [Ta].

Moreover, define a subgroup $\tilde{G}_1$ of $\tilde{G}$ by

$$\tilde{G}_1 = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & aB & O \\ \zeta & bB & B \end{pmatrix} \middle| \; B \in O(n),\, a \neq 0 \in \mathbb{R},\, b \in \mathbb{R},\, \zeta \in \mathbb{R}^n \right\}.$$

For an n-dimensional manifold N with a conformal structure, a Lie contact structure is induced on the tangent sphere bundle $M = S(N)$ of N. See [S-Y1]. The conformal structure on N gives rise to the canonical reduction of the structure group of $L_C(S(N))$ to the subgroup $\tilde{G}_1$. So, a subbundle $\tilde{P}_1$ of $L_C(M)$ with structure group $\tilde{G}_1$ is called a conformal contact structure on M. In detail, see [S-Y2].

8.3. Geodesic flow

Let P be an $(n+1)$-dimensional manifold. The tangent sphere bundle $S(P)$ of P is a quotient space of $\dot{T}P$ defined as follows: Let $\dot{T}P$ denote the set removed the zero section from TP, then

$$\begin{aligned} S(P) &= \dot{T}P/\mathbb{R}_+ \\ &= \{[v] \mid [v]\text{: the equivalence class of } \{tv\},\, v \in TP \ (t \in \mathbb{R}_+)\}. \end{aligned}$$

The natural projection $p : \dot{T}P \to S(P)$ defines a principal bundle with structure group $\mathbb{R}_+$ and $p^{-1}([v]) = \{tv \in \dot{T}P \mid t \in \mathbb{R}_+\}$. The transformation $a : \mathbb{R}_+ \times \dot{T}P \to \dot{T}P$ of the structure group is $a : (t, v) \mapsto tv$ and, if $t \in \mathbb{R}_+$ is fixed, it defines the dilation $a_t : v \mapsto tv$. The fundamental vector field A^* satisfies $A^*_v = sv^V$ $(v \in \dot{T}P)$ for $Lie\mathbb{R}_+ \cong \mathbb{R} \ni s$. Here v^V denotes the vertical lift of v.

Assume that P has a conformal structure C. Take a Weyl connection ∇ on P associated with the conformal structure. A Weyl connection is defined, as a linear connection, on the tangent bundle TP of P. Let $\overline{H} \subset T\dot{T}P$ be the subbundle of the set of $(n+1)$-dimensional horizontal subspaces and $\overline{V} \subset T\dot{T}P$ the subbundle of the set of $(n+1)$-dimensional vertical subspaces. The bundle $T\dot{T}P$ is decomposed as the direct sum $\overline{H}$ and $\overline{V}$: $T\dot{T}P = \overline{H} \oplus \overline{V}$.

Let $\overline{\pi} : \dot{T}P \to P$ and $\pi : S(P) \to P$ be the natural projections. Taking an arbitrary metric g in the conformal structure C, we define a subbundle $\overline{D} \subset T\dot{T}P$ with rank $2n$ as follows: for $v \in \dot{T}P$, $x = \overline{\pi}(v)$,

$$\overline{D}_v = \{(w_1^H, w_2^V) \in \overline{H} \oplus \overline{V} \mid g(v, w_i) = 0,\ w_i \in \dot{T}_x P\ (i = 1, 2)\},$$

where w^H, w^V are the horizontal lift, vertical lift of $w \in \dot{T}_x P$ to $T_v \dot{T}P$ respectively. We denote the horizontal component, vertical component of $\overline{D}_v$ by $\overline{D}_v^H$, $\overline{D}_v^V$. The subbundle $\overline{D}$ does not depend on the choice of g in C.

The geodesic flow on $S(P)$ is given as follows.

We fix an arbitrary metric g in the conformal structure C. Define a vector field $\overline{\eta}$ on $\dot{T}(P)$ by

$$\overline{\eta}_v = \left(\frac{v^i}{|v|},\ -\sum_{j,k} \Gamma^i_{jk} \frac{v^j v^k}{|v|} \right) \in \overline{H}_v \oplus \overline{V}_v,$$

where $v = (v^i) \in \dot{T}P$ and Γ^i_{jk} $(i, j, k = 1, \ldots, n+1)$ denote the Christoffel symbol of the Weyl connection. If $|v| = 1$, $\overline{\eta}$ is equal to the horizontal lift v_v^H of v at $v \in \dot{T}P$.

Since the tangential mapping a_{t_*} for the dilation a_t is written as $a_{t_*} : (X, Y) \mapsto (X, tY)$, it follows that

$$a_{t_*} \overline{\eta}_v = \overline{\eta}_{tv}.$$

Thus a vector field η is defined on $S(P)$ from the vector field $\overline{\eta}$ on $\dot{T}P$. Remark that trajectories are the same if we change the metric g in the conformal class C.

The projection of a trajectory of η to P is equal to a geodesic with respect to the Weyl connection. The parameter is not affine. We call the vector field η the geodesic flow vector field and the flow the geodesic flow ϕ_t on $S(P)$.

Put $\overline{H}_v = \overline{D}_v \oplus \langle \overline{\eta}_v \rangle$ and $\overline{D}_v = \overline{D}_v^H \oplus \overline{D}_v^V$. We have $a_{t_*}(\overline{H}_v) = \overline{H}_{tv}, a_{t_*}(\overline{\eta}_v) = \overline{\eta}_{tv}$, $a_{t_*}(\overline{D}_v) = \overline{D}_{tv}$, $a_{t_*}(\overline{D}_v^H) = \overline{D}_{tv}^H$ and $a_{t_*}(\overline{D}v^V) = \overline{D}_{tv}^V$. Thus we get corresponding subbundles H, $\langle \eta \rangle$, D, D^H and D^V of $TS(P)$. The space $T_{[v]}S(P)$ is regarded as $D_{[v]} \oplus \langle [v^H] \rangle$.

We recall the notion and the properties of Jacobi fields.

Let c be a geodesic on P with respect to the Weyl connection ∇. By definition a Jacobi field J along c is a vector field along c which satisfies a second order differential equation

$$\nabla_{\dot{c}}\nabla_{\dot{c}}J + R(J, \dot{c})\dot{c} = 0.$$

If we put $R_{\dot{c}}J = R(J, \dot{c})\dot{c}$, the equation above becomes $\nabla_{\dot{c}}\nabla_{\dot{c}}J + R_{\dot{c}}J = 0$. A Jacobi field along c means the transversal vector field of the variation of c by means of geodesics.

An orthogonal Jacobi field $J = J(t)$ along $c = c(t)$ on P corresponds to one-to-one geodesic flow invariant vector field $Y = Y(t) = \phi_{t_*}Y(0)$ along $\dot{c} = \dot{c}(t) = \phi_t\dot{c}(0)$ on $S(P)$ as follows:

$$T_{c(t)}P \ni J(t) \longmapsto Y(t) = (J(t)^H, (\nabla_{\dot{c}}J(t))^V) \in T^H_{\dot{c}(t)}S(P) \oplus T^V_{\dot{c}(t)}S(P).$$

The tangential mapping for the geodesic flow ϕ_t is described as follows. Let v be an element of $S(P)$ and X an element of $T_vS(P)$. If $Y = Y(t)$ is a Jacobi field such that $Y(0) = X$,

$$\phi_{t_{*|v}}(X) = Y(t)$$

holds. See [B].

Hence a vector field J along c on P is a Jacobi field if and only if, for a vector field $Y = J^C = (J^H, (\nabla_{\dot{c}}J)^V)$ (called the complete lift J^C of J, see [Y-I]) along $\dot{c}$ on $S(P)$,

$$\phi_{t_*}Y = Y$$

holds.

We have another look of Jacobi fields that satisfy the second order differential equation along c on P as the first order differential equation along $\dot{c}$ on $S(P)$. We write $W = \binom{w_1}{w_2}$ for $W = w_1^H + w_2^V \in D_v \subset T_v^H S(P) \oplus T_v^V S(P) = T_vS(P)$, where $w_1, w_2 \in T_{\pi(v)}(P)$, $g(v, w_i) = 0$ $(i = 1, 2)$. We define an endomorphism K_v on $T_vS(P)$ by

$$K_v \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} O & I \\ -R_v & O \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

for $v \in T_xP$ $(x \in P)$. This defines a cross section K of the bundle $\mathrm{Hom}(TS(P), TS(P))$ over $S(P)$.

We remark that, for the geodesic flow vector field η to $\dot{c}$ on $S(P)$ corresponding to a geodesic $c = c(t)$ on P, $\eta_{|\dot{c}(t)} = \begin{pmatrix} \dot{c}(t) \\ 0 \end{pmatrix}$ holds. For a vector field $Y = Y(t) = \begin{pmatrix} Y_1(t) \\ Y_2(t) \end{pmatrix}$ along $\dot{c} = \dot{c}(t)$, put

$$\nabla^H_\eta \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} \nabla_{\dot{c}} Y_1 \\ \nabla_{\dot{c}} Y_2 \end{pmatrix}.$$

We consider the following first order differential equation along $\dot{c}$ on $S(P)$:

$$\nabla^H_\eta Y = K_{\dot{c}} Y.$$

From $\begin{pmatrix} \nabla_{\dot{c}} Y_1 \\ \nabla_{\dot{c}} Y_2 \end{pmatrix} = \begin{pmatrix} O & I \\ -R_{\dot{c}} & O \end{pmatrix} \begin{pmatrix} Y_1(t) \\ Y_2(t) \end{pmatrix} = \begin{pmatrix} Y_2(t) \\ -R_{\dot{c}} Y_1(t) \end{pmatrix}$, this equation implies that $Y_1 = Y_1(t)$ is a Jacobi field along c on P.

8.4. Twistor theory by Weyl connections

With respect to the bases on $S(P)$ of the horizontal lift and the vertical lift of the conformal bases on P, the structure group of the tangent bundle $TS(P)$ reduces to

$$\tilde{G}_1 = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & aB & O \\ \zeta & bB & B \end{pmatrix} \;\middle|\; B \in O(n),\ a \neq 0 \in \mathbb{R},\ b \in \mathbb{R},\ \zeta \in \mathbb{R}^n \right\}.$$

This defines a Lie contact structure on $S(P)$. Remark that we call the induced Lie contact structure a conformal contact structure.

According to [S-Y1], we have the following.

PROPOSITION 8.1. *Let P be an $(n+1)$-dimensional manifold with a conformal structure C. Then, a Lie contact structure on the tangent sphere bundle $S(P)$ of P is defined by the conformal structure C on P.*

Next, we have the following.

PROPOSITION 8.2. *The Lie contact structure on $S(P)$ induced by a Weyl structure on P is invariant under the geodesic flow if and only if the Weyl structure on P is of Weyl constant curvature.*

Proof. Let $c = c(t)$ be a geodesic on P. Let $e_i = e_i(t)$ $(i = 0, 1, \ldots, n)$ be parallel orthonormal vector fields along c and with $e_0 = \dot{c} = v$. Vectors e_i^H and e_i^V are regarded as the horizontal lifts and vertical lifts of them along the geodesic flow orbit $\dot{c} = \dot{c}(t)$ respectively. For $B \in O(n)$ and $\alpha = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in GL(2,\mathbb{R})$, we define an endomorphism $\sigma_{B,\alpha}$ on D_v by

$$\sigma_{B,\alpha} = \begin{pmatrix} aB & bB \\ cB & dB \end{pmatrix} \in O(n) \otimes GL(2,\mathbb{R})$$

with respect to $e_1^H, \ldots, e_n^H, e_1^V, \ldots, e_n^V$. Furthermore, define an endomorphism $\tilde{\sigma}_{B,\alpha,\xi}$ on $T_v S(P)$ at $v \in S(P)$ by $\tilde{\sigma}_{B,\alpha} = \left(\begin{smallmatrix} \det\alpha & 0 \\ \xi & \sigma_{B,\alpha} \end{smallmatrix}\right)$, $\xi \in \mathbb{R}^{2n}$.

A Lie contact structure is to assign the following subset of frames at $v \in S(P)$:

$$\tilde{P}_v = \left\{ \tilde{\sigma}_{B,\alpha,\xi} \begin{pmatrix} e_0^H(0) \\ e_i^H(0) \\ e_i^V(0) \end{pmatrix} \;\middle|\; B \in O(n),\ \alpha \in GL(2,\mathbb{R}),\ \xi \in \mathbb{R}^{2n} \right\}.$$

Then, that the Lie contact structure on $S(P)$ is invariant under the geodesic flow ϕ_t means

$$\phi_{t*}\tilde{P}_v \subset \tilde{P}_{\phi_t(v)},$$

in other words,

$$\phi_{t*} \begin{pmatrix} e_0^H(0) \\ e_i^H(0) \\ e_i^V(0) \end{pmatrix} = \tilde{\sigma}_{B',\alpha',\xi'} \begin{pmatrix} e_0^H(t) \\ e_i^H(t) \\ e_i^V(t) \end{pmatrix}$$

for some $B' \in O(n)$, $\alpha' \in GL(2,\mathbb{R})$ and $\xi' \in \mathbb{R}^{2n}$. Here $\tilde{\sigma}_{B',\alpha',\xi'}$ depends on the parameter t.

By differentiating with respect to the variable t, it follows that the Lie contact structure on $S(P)$ is invariant under the geodesic flow ϕ_t if and only if

$$K_v = \begin{pmatrix} O & I \\ -R_v & O \end{pmatrix} = \begin{pmatrix} xI + A & yI + A \\ zI + A & wI + A \end{pmatrix} \in I_n \otimes \mathfrak{gl}(2,\mathbb{R}) \oplus \mathfrak{o}(n) \otimes I_2$$

for some $A \in \mathfrak{o}(n)$ and $\left(\begin{smallmatrix} x & y \\ z & w \end{smallmatrix}\right) \in \mathfrak{gl}(2,\mathbb{R})$. Here $I_n \otimes \mathfrak{gl}(2,\mathbb{R}) \oplus \mathfrak{o}(n) \otimes I_2$ is the Lie algebra of $O(n) \otimes GL(2,\mathbb{R})$.

Therefore it follows that $x = 0$, $w = 0$, $y = 1$, $A = O$ and $R_v = -zI$. From this, we have $\sigma_{B',\alpha'} = \left(\begin{smallmatrix} I & tI \\ zI & I \end{smallmatrix}\right)$ and

$$R_v = \lambda_v I$$

for some $\lambda_v \in \mathbb{R}$. Taking $w \in D_v$ such that $g(w,w) = 1$, we obtain

$$\begin{aligned} g(R(w,v)v,w) &= g(R_v(w),w) \\ &= g(\lambda_v w, w) \\ &= \lambda_v. \end{aligned}$$

This means that the sectional curvature of any planes including v with respect to g is constant λ_v.

For v' orthogonal to v, from a similar argument to v, the sectional curvature of any planes including v' with respect to g is constant $\lambda_{v'}$. The sectional curvature of the plane spanned by v and v' is $\lambda_v = \lambda_{v'}$.

Hence, for an arbitrary $v \in T_xP$ with $g(v,v) = 1$, $\lambda = \lambda_v$ is identically constant. Therefore P has a Weyl structure with constant curvature.

Therefore this completes the proof. □

We note that, as compared with a Levi-Civita connection, a Weyl connection does not make the metric g parallel in general. Therefore R_v is not symmetric in general. We can decompose R_v into the symmetric part and the anti-symmetric part:

$$R_v = R_v^s + R_v^a.$$

The Ricci tensor *Ric* is not also symmetric in general.

We remark that, when a conformal structure is defined from a Riemannian structure and a Weyl connection from the Levi-Civita connection on P, the Lie contact structure on the unit tangent bundle T_1P is invariant under the geodesic flow if and only if P is of constant curvature.

In order to ensure that M is a manifold, we assume that P is an enough small convex domain.

From the above proposition and the proof, we have the following:

THEOREM 8.1. *Let P be an $(n+1)$-dimensional manifold with a Weyl structure with constant curvature. Then the structure on P induces a right-half Grassmannian flat Grassmannian structure of type $(n,2)$ on the orbit space M of the geodesic flow.*

We may regard the manifold $S(P)$ as a null n-plane bundle F_L of M with the Grassmannian structure of type $(n,2)$ and the fibre is $\mathbb{R}^1$ $(\subset S^1)$. Each null n-manifold is diffeomorphic to S^n.

Assume that $n = 2$.

Remark that, in 3-dimensional Weyl structures, the notion of Einstein-Weyl and that of Weyl constant curvature are equivalent.

On the 5-dimensional $S(P)$, a CR structure of type $(1,1)$ is naturally defined as follows: for $T_vS(P)$ at $v \in S(P)$, it is defined by $\pi/2$-rotation in the horizontal lift and $\pi/2$-rotation in the vertical lift of D_v respectively. The CR structure is integrable. As there is a (graded) Lie algebra isomorphism of $\mathfrak{o}(4,2) \cong \mathfrak{su}(2,2)$, a Lie contact structure on a 5-dimensional contact manifold is equivalent to a CR structure with a nondegenerate and indefinite Levi form. See [S-Y1]. Therefore a complex structure J on M is induced by the CR structure on $S(P)$.

Remark that, on M as the manifold of geodesics, a symplectic structure is not necessarily defined from $S(P)$ with the contact structure (cf. [B, p. 58]).

Since the notion of Grassmannian structures of type $(2,2)$ and that of conformal structures of type $(2,2)$ are equivalent, we have the following:

THEOREM 8.2. *Let P be a 3-dimensional manifold with an Einstein-Weyl structure. Then the structure on P induces a self-dual conformal Hermitian structure of type $(2,2)$ on the orbit space M of the geodesic flow.*

REFERENCES

[A-G1] M. A. Akivis and V. V. Goldberg, Conformal differential geometry and its generalizations, John Wiley and Sons, Inc., New York, 1996.

[A-G2] ———, *On the theory of almost Grassmann structures*, in New developments in differential geometry (J. Szenthe, ed.), Kluwer Academic Publishers, Dordrecht, Boston, London (1998), pp. 1–37.

[A-G3] ———, *Conformal and Grassmann structures*, Differential Geom. Appl., **8** (1998), 177–203.

[A-G4] ———, *Semiintegrable almost Grassmann structures*, Differential Geom. Appl., **10** (1999), 257–294.

[B] A. L. Besse, Manifolds all of whose geodesics are closed, Springer-Verlag, Berlin, Heidelberg, New York, 1978.

[B-E] T. N. Bailey and M. G. Eastwood, *Complex paraconformal manifolds — their differential geometry and twistor theory*, Forum Math., **3** (1991), 61–103.

[G] A. B. Goncharov, *Generalized conformal structures on manifolds*, Selecta Math. Sov., **6** (1987), 307–340.

[Ha] T. Hangan, *Geómétrie différentielle Grassmannienne*, Rev. Roum. Math. Pures Appl., **11** (1966), 519–531.

[He] S. Helgason, Differential geometry, Lie groups, and symmetric spaces, Academic Press, New York, London, 1978.

[Hi] N. J. Hitchin, *Complex manifolds and Einstein's equations*, in Twistor geometry and non-linear systems (H. D. Doebner, T. D. Palev, eds.), Lecture Notes in Math. 970, Springer-Verlag, Berlin, Heidelberg, New York (1982), pp. 73–99.

[I] T. Ishihara, *On tensor-product structures and Grassmannian structures*, J. Math. Tokushima Univ., **4** (1970), 1–17.

[J-T] P. E. Jones and K. P. Tod, *Minitwistor spaces and Einstein-Weyl spaces*, Class. Quantum. Grav., **2** (1985), 565–577.

[Ka] S. Kaneyuki, *On the subalgebras* $\mathfrak{g}_0$ *and* $\mathfrak{g}_{ev}$ *of semisimple graded Lie algebras*, J. Math. Soc. Japan., **45** (1993), 1–19.

[Ko] S. Kobayashi, Transformation groups in differential geometry, Springer-Verlag, Berlin, Heidelberg, New York, 1972.

[K-M] H. Kamada and Y. Machida, *Self-duality of metrics of type* $(2, 2)$ *on four-dimensional manifolds*, Tohoku Math. J., **49** (1997), 259–275.

[K-Na] S. Kobayashi and T. Nagano, *On filterd Lie algebras and geometric structures I*, J. Math. Mech., **13** (1964), 875–907.

[K-No] S. Kobayashi and K. Nomizu, Foundations of differential geometry I, Interscience Publishers, John Wiley and Sons, New York, London, 1963.

[Ma] Y. I. Manin, Gauge field theory and complex geometry, Springer-Verlag, Berlin, Heidelberg, New York, 1988.

[Mi] Y. I. Mikhailov, *On the structure of almost Grassmannian manifolds*, Soviet Maht., **22** (1978), 54–63.

[Ma-Sa] Y. Machida and H. Sato, *Twistor spaces for real four-dimensional Lorentzian manifolds*, Nagoya Math. J., **134** (1994), 107–135.

[Mi-St] J. Milnor and J.D. Stasheff, Characteristic classes, Princeton Univ., 1974.

[O] T. Ochiai, *Geometry associated with semisimple flat homogeneous spaces*, Trans. Amer. Math. Soc., **152** (1970), 159–193.

[P-S] H. Pedersen and A. Swann, *Riemannian submersions, four-manifolds and Einstein-Weyl geometry*, Proc. London Math. Soc., **66** (1993), 381–399.

[P-T] H. Pedersen and K. P. Tod, *Three-dimensional Einstein-Weyl geometry*, Adv. in Math., **97** (1993), 74–109.

[S] S. Sternberg, Lectures on differential geometry, Prentice-Hall, Inc., New Jersey, 1964.

[S-Y1] H. Sato and K. Yamaguchi, *Lie contact manifods*, in Geometry of manifolds (K. Shiohama, ed.), Academic Press, Boston (1989), pp. 191–238.

[S-Y2] ———, *Lie contact manifolds II*, Math. Ann., **297** (1993), 33–57.

[Ta] M. Takeuchi, *A remark on Lie contact structures*, Science Report Coll. Ge. Ed. Osaka Univ., **42** (1993), 29–37.

[To] K. P. Tod, *Compact* 3*-dimensional Einstein-Weyl structures*, J. London Math. Soc., **45** (1992), 341–351.

[T1] N. Tanaka, *On the equivalence problems associated with simple graded Lie algebras*, Hokkaido Math. J., **8** (1979), 23–84.

[T2] ———, *On affine symmetric spaces and the automorphism groups of product manifolds*, Hokkaido Math. J., **14** (1985), 277–351.

[T3] ———, On geometric theory of systems of ordinary differential equations, Lectures in Colloq. on Diff. Geom. at Sendai, August, 1989.

[W] R. O. Wells, Jr., Complex geometry in mathematical physics, Presses de l'Université de Montréal, Montréal, 1982.

[W-W] R. S. Ward and R. O. Wells, Jr., Twistor geometry and field theory, Cambridge Univ. Press, 1990.

[Y] K. Yamaguchi, *Differential systems associated with simple graded Lie algebras*, Advanced Studies in Pure Math., **22** (1993), 413–494.

[Y-I] K. Yano and S. Ishihara, Tangent and cotangent bundles, Marcel Dekker, New york, 1973.

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