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OSCILLATION OF SOLUTIONS OF PARTIAL DIFFERENTIAL EQUATIONS WITH FUNCTIONAL ARGUMENTS

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Abstract. Sufficient conditions are established for the oscillations of partial differential equation with functional arguments of the form

$$\frac{\partial}{\partial t}(p(t)\frac{\partial}{\partial t}u(x,t)) = a(t)\Delta u(x,t) + \sum_{k=1}^{s} a_k(t)\Delta u(x,\rho_k(t)) - q(x,t)u(x,t)$$

$$-\sum_{j=1}^{m} q_j(x,t)u(x,\sigma_j(t)),\ (x,t)\in\Omega\times[0,\infty)\equiv G,$$

where Ω is a bounded domain in R^n with a piecewise smooth boundary $\partial\Omega$ and Δ is the Laplacian in Euclidean n-space R^n.

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1 Introduction

Recently, the oscillation problem for the partial functional differential equation has been studied by many authors. We refer the reader to [1,2,3] for parabolic equations and to [4,5,6,7] for hyperbolic equations.

In this paper, we study the oscillation of solutions of partial differential equations with functional arguments of the form

$$(1)\quad \frac{\partial}{\partial t}(p(t)\frac{\partial}{\partial t}u(x,t)) = a(t)\Delta u(x,t) + \sum_{k=1}^{s} a_k(t)\Delta u(x,\rho_k(t)) - q(x,t)u(x,t)$$

$$-\sum_{j=1}^{m} q_j(x,t)u(x,\sigma_j(t)),\ (x,t)\in\Omega\times[0,\infty)\equiv G,$$

where Ω is a bounded domain in R^n with a piecewise smooth boundary $\partial\Omega$, and $\Delta u(x,t)=\sum_{r=1}^{n}\frac{\partial^2 u(x,t)}{\partial x_r^2}$.

Suppose that the following conditions hold:

(A1) $p\in C^1([0,\infty);(0,\infty)), \lim_{t\to\infty}\int_{t_0}^{t}\frac{1}{p(s)}ds=+\infty, t_0>0$;

(A2) $q,q_j\in C(\overline{G};[0,\infty)), q(t)=\inf_{x\in\overline{\Omega}}q(x,t)$, and $q_j(t)=\inf_{x\in\overline{\Omega}}q_j(x,t), j\in I_m=\{1,2,\ldots,m\}$;

(A3) $a,a_k\in C([0,\infty);[0,\infty)), k\in I_s=\{1,2,\ldots,s\}$;

(A4) $\sigma_j,\rho_k\in C([0,\infty);R), \sigma_j(t)\le t, \rho_k(t)\le t, \sigma_j,\rho_k$ are nondecreasing functions and $\lim_{t\to\infty}\sigma_j(t)=\lim_{t\to\infty}\rho_k(t)=\infty, j\in I_m, k\in I_s$.

We consider two kinds of boundary conditions:

(2) $\quad \frac{\partial u(x,t)}{\partial N}+g(x,t)u(x,t)=0,(x,t)\in\partial\Omega\times[0,\infty),$

where N is the unit exterior normal vector to $\partial\Omega$ and $g(x,t)$ is a nonnegative continuous function on $\partial\Omega\times[0,\infty)$, and

(3) $\quad u(x,t)=0,(x,t)\in\partial\Omega\times[0,\infty).$

Definition 1.1. *The solution $u\in C^2(\overline{G};R)$ of the problem (1),(2) (or (1),(3)) is said to be oscillatory in the domain $G=\Omega\times[0,\infty)$ if for any positive number μ there exists a point $(x_0,t_0)\in\Omega\times[\mu,\infty)$ such that $u(x_0,t_0)=0$ holds.*

In the following two sections sufficient conditions are obtained for the oscillation of the solutions of the problem (1),(2) and (1),(3) in the domain G. We note that conditions for the oscillation of the solutions for $p(t)=1$ has been obtained in the works of Lalli, Yu and Cui [4], and for $p(t)=1,\sigma_j(t)=t-\mu_j,\mu_j=$ const. ≥ 0 has been obtained in the works of Cui, Yu and Lin [5].

2 Oscillation of the problem (1),(2)

Theorem 2.1. *If there exists some $j_0\in I_m$ such that $\sigma'_{j_0}(t)\ge 0$, $q_{j_0}(t)>0, t\ge t_0>0$, and*

(4) $$\int_{t_0}^{\infty} q_{j_0}(t)dt = \infty.$$

Then every solution $u(x,t)$ of problem (1), (2) is oscillatory in G.

Proof. Suppose to the contrary that there is a nonoscillatory solution $u(x,t)$ of the problem (1),(2) which has no zero in $\Omega \times [t_0, \infty)$ for some $t_0 > 0$. Without loss of generality we may assume that $u(x,t) > 0, u(x, \rho_k(t)) > 0$, and $u(x, \sigma_j(t)) > 0$ in $\Omega \times [t_1, \infty), t_1 \geq t_0, k \in I_s, j \in I_m$.

Integrating (1) with respect to x over the domain Ω, we have

(5) $$\frac{d}{dt}(p(t)\frac{d}{dt}\int_\Omega u(x,t)dx) = a(t)\int_\Omega \Delta u(x,t)dx + \sum_{k=1}^{s} a_k(t)\int_\Omega \Delta u(x,\rho_k(t))dx$$
$$-\int_\Omega q(x,t)u(x,t)dx - \sum_{j=1}^{m}\int_\Omega q_j(x,t)u(x,\sigma_j(t))dx,\ t \geq t_1.$$

From Green's formula and boundary condition (2), it follows that

(6) $$\int_\Omega \Delta u(x,t)dx = \int_{\partial\Omega} \frac{\partial u(x,t)}{\partial N}dS = -\int_{\partial\Omega} g(x,t)u(x,t)dS \leq 0,$$

and

(7) $$\int_\Omega \Delta u(x,\rho_k(t))dx = \int_{\partial\Omega} \frac{\partial u(x,\rho_k(t))}{\partial N}dS$$
$$= -\int_{\partial\Omega} g(x,\rho_k(t))u(x,\rho_k(t))dS \leq 0, t \geq t_1, k \in I_s,$$

where dS is the surface element on $\partial\Omega$.

Noting that condition (A2), combining (5)–(7), we get

(8) $$\frac{d}{dt}(p(t)\frac{d}{dt}\int_\Omega u(x,t)dx) \leq -\int_\Omega q(t)u(x,t)dx - \sum_{j=1}^{m}\int_\Omega q_j(t)u(x,\sigma_j(t))dx, t \geq t_1.$$

Set $V(t) = \int_\Omega u(x,t)dx, t \geq t_1$, from (8) we have

(9) $$[p(t)V'(t)]' + q(t)V(t) + \sum_{j=1}^{m} q_j(t)V(\sigma_j(t)) \leq 0, t \geq t_1.$$

The inequality (9) shows that $[p(t)V'(t)]' < 0$ for $t \geq t_1$. Hence $p(t)V'(t)$ is a decreasing function in the interval $[t_1, \infty)$. We can claim that $V'(t) > 0$ for $t \geq t_1$.

In fact, if $V'(t) \leq 0$ for $t \geq t_1$, then there exists a $T > t_1$ such that $p(T)V'(T) < 0$. This implies that

$$V'(t) \leq \frac{p(T)V'(T)}{p(t)} \text{ for } t \geq T.$$

Hence

$$V(t) - V(T) \leq p(T)V'(T) \int_T^t \frac{ds}{p(s)}, \; t \geq T.$$

Therefore,

$$\lim_{t \to \infty} V(t) = -\infty,$$

which contradicts the fact that $V(t) = \int_\Omega u(x,t)dx > 0$.

From (9) we obtain that there exists some $j_0 \in I_m$ such that

$$(10) \quad [p(t)V'(t)]' + q_{j_0}(t)V(\sigma_{j_0}(t)) \leq 0, t \geq t_1.$$

Integrating the inequality (10), we have

$$(11) \quad p(t)V'(t) - p(t_1)V'(t_1) + \int_{t_1}^t q_{j_0}(s)V(\sigma_{j_0}(s))ds \leq 0, t \geq t_1.$$

Then we obtain

$$(12) \quad \int_{t_1}^t q_{j_0}(s)V(\sigma_{j_0}(s))ds \leq -p(t)V'(t) + p(t_1)V'(t_1), t \geq t_1.$$

Hence

$$(13) \quad \int_{t_1}^t q_{j_0}(s)ds \leq \frac{1}{V(\sigma_{j_0}(t_1))}[-p(t)V'(t) + p(t_1)V'(t_1)] \leq \frac{p(t_1)V'(t_1)}{V(\sigma_{j_0}(t_1))}, t \geq t_1,$$

which contradicts the condition (4).

If $u(x,t) < 0$ for $(x,t) \in \Omega \times [t_1, \infty)$, then $-u(x,t) > 0$ is a positive solution of the problem (1), (2). This completes the proof of the Theorem 2.1.

Theorem 2.2. *If* $q(t) > 0$ *and*

$$(14) \quad \int^\infty q(t)dt = \infty,$$

then every solution $u(x,t)$ *of the problem (1), (2) oscillates in* G.

Proof. As in the proof of Theorem 2.1, we obtain (9). Therefore,

$$(15) \quad [p(t)V'(t)]' + q(t)V(t) \leq 0, t \geq t_1.$$

The remainder of the proof is similar to that of Theorem 2.1 and we omit it.

Corollary 2.1. *If the inequality (9) has no eventually positive solution, then every solution $u(x,t)$ of the problem (1),(2) is oscillatory in G.*

3 Oscillation of the problem (1),(3)

In the domain Ω we consider the following Dirichlet problem

$$(*)\begin{cases} \Delta\omega(x) + \alpha\omega(x) = 0 \text{ in } \Omega, \\ \omega(x) = 0 \text{ on } \partial\Omega, \end{cases}$$

where α is a constant. It is well known that the least eigenvalue α_0 of the problem $(*)$ is positive and the corresponding eigenfunction $\varphi(x)$ is positive in Ω.

Theorem 3.1. *If there exists some $k_0 \in I_s$ such that $\rho'_{k_0}(t) \geq 0$, $a_{k_0}(t) > 0$, and*

$$(16) \quad \int^{\infty} a_{k_0}(t)dt = \infty,$$

then every solution $u(x,t)$ of the problem (1),(3) oscillates in G.

Proof. Suppose to the contrary that there is a nonoscillatory solution $u(x,t)$ of the problem (1),(3) which has no zero in $\Omega \times [t_0, \infty)$ for some $t_0 > 0$. Without loss of generality we may assume that $u(x,t) > 0, u(x, \rho_k(t)) > 0$, and $u(x, \sigma_j(t)) > 0$ in $\Omega \times [t_1, \infty), t_1 \geq t_0, k \in I_s, j \in I_m$.

Multiplying both sides of (1) by $\varphi(x) > 0$ and integrating with respect to x over the domain Ω, we have

$$(17) \quad \frac{d}{dt}\Big(p(t)\frac{d}{dt}\int_\Omega u(x,t)\varphi(x)dx\Big)$$
$$= a(t)\int_\Omega \Delta u(x,t)\varphi(x)dx + \sum_{k=1}^{s} a_k(t)\int_\Omega \Delta u(x,\rho_k(t))\varphi(x)dx$$
$$-\int_\Omega q(x,t)u(x,t)\varphi(x)dx - \sum_{j=1}^{m}\int_\Omega q_j(x,t)u(x,\sigma_j(t))\varphi(x)dx, \quad t \geq t_1.$$

Green's formula and boundary (3) yield

$$(18) \quad \int_\Omega \Delta u(x,t)\varphi(x)dx = \int_\Omega u(x,t)\Delta\varphi(x)dx = -\alpha_0\int_\Omega u(x,t)\varphi(x)dx, t \geq t_1,$$

and

(19) $$\int_\Omega \Delta u(x,\rho_k(t))\varphi(x)dx = \int_\Omega u(x,\rho_k(t))\Delta\varphi(x)dx = -\alpha_0\int_\Omega u(x,\rho_k(t))\varphi(x)dx,$$

$$t \geq t_1, k \in I_s.$$

Then, we have

(20) $$\frac{d}{dt}(p(t)\frac{d}{dt}\int_\Omega u(x,t)\varphi(x)dx)$$
$$\leq -\alpha_0 a(t)\int_\Omega u(x,t)\varphi(x)dx - \alpha_0\sum_{k=1}^{s} a_k(t)\int_\Omega u(x,\rho_k(t))\varphi(x)dx$$
$$-q(t)\int_\Omega u(x,t)\varphi(x)dx - \sum_{j=1}^{m} q_j(t)\int_\Omega u(x,\sigma_j(t))\varphi(x)dx,\ t \geq t_1.$$

Set $V(t) = \int_\Omega u(x,t)\varphi(x)dx,\ t \geq t_1$, from (20) we have

(21) $$[p(t)V'(t)]' + [\alpha_0 a(t) + q(t)]V(t)$$
$$+\alpha_0\sum_{k=1}^{s} a_k(t)V(\rho_k(t)) + \sum_{j=1}^{m} q_j(t)V(\sigma_j(t)) \leq 0, t \geq t_1.$$

It follows that there exists some $k_0 \in \{1, 2, \ldots, s\}$ such that

(22) $$[p(t)V'(t)]' + \alpha_0 a_{k_0}(t)V(\rho_{k_0}(t)) \leq 0, t \geq t_1.$$

It is easy to see that

$$V(t) > 0, [p(t)V'(t)]' < 0, V'(t) > 0, t \geq t_1.$$

Integrating the inequality (22) we obtain

$$p(t)V'(t) - p(t_1)V'(t_1) + \alpha_0 V(\rho_{k_0}(t_1))\int_{t_1}^{t} a_{k_0}(s)ds \leq 0,$$

which contradicts the condition (16). This completes the proof of Theorem 3.1.

Corollary 3.1. *If the differential inequality (21) has no eventually positive solution, then every solution $u(x,t)$ of the problem (1),(3) oscillates in G.*

It is not difficult to prove the following theorems.

Theorem 3.2. *If all conditions of Theorem 2.1 hold, then every solution of the problem (1),(3) is oscillatory in G.*

Theorem 3.3. *If the condition of Theorem 2.2 hold, then every solution of the problem (1),(3) is oscillatory in G.*

4 Examples

Example 4.1. Consider the partial differential equation

(23) $u_{tt}(x,t) = u_{xx}(x,t) + u_{xx}(x,t-\pi) - 2u(x,t) - u(x,t-3\pi),$

$$(x,t) \in (0,\pi) \times [0,\infty),$$

with boundary condition

(24) $u_x(0,t) = u_x(\pi,t) = 0, t \geq 0.$

Here $s = 1, m = 1, p(t) = 1, a(t) = 1, a_1(t) = 1, \rho_1(t) = t - \pi, q(x,t) = 2, q_1(x,t) = 1, \sigma_1(t) = t - 3\pi$. It is easy to check that the condition of Theorem 2.2 is satisfied. Therefore, every solution of the problem (23), (24) is oscillatory in $(0,\pi) \times [0,\infty)$. In fact, $u(x,t) = \cos x \sin t$ is such a solution.

Example 4.2. Consider the partial differential equation

(25) $\frac{\partial}{\partial t}[t\frac{\partial}{\partial t}u(x,t)] = e^t u_{xx}(x,t) + 2u_{xx}(x,t-\frac{3\pi}{2}) - tu(x,t) - 3u(x,t-\frac{\pi}{2}) - e^t u(x,t-\pi),$

$$(x,t) \in (0,\pi) \times [0,\infty),$$

with boundary condition

(26) $u(0,t) = u(\pi,t) = 0, t \geq 0.$

Here $s = 1, m = 2, p(t) = t, a(t) = e^t, a_1(t) = 2, \rho_1(t) = t - \frac{3\pi}{2}, q(x,t) = t, q_1(x,t) = 3, \sigma_1(t) = t - \frac{\pi}{2}, q_2(x,t) = e^t, \sigma_2(t) = t - \pi$. It is easy to see that the condition of Theorem 3.3 is verified. Thus every solution of the problem (25), (26) oscillates in $(0,\pi) \times [0,\infty)$. In fact, $u(x,t) = \sin x \cos t$ is such a solution.

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