Nihonkai Math. J.
Vol.8 (1997), 29-36

On geodesic hyperspheres in a complex projective space

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1. Introduction

Let $P_n(\mathbb{C})$ be an n-dimensional complex projective space with Fubini-Study metric of constant holomorphic sectional curvature $4c$. R. Takagi ([4] and [5]) classified all homogeneous real hypersurfaces in $P_n(\mathbb{C})$ which are orbits under analytic subgroups of the projective unitary group $PU(n+1)$ in $P_n(\mathbb{C})$. Due to his work, we see that such a homogeneous real hypersurface in $P_n(\mathbb{C})$ is locally congruent to one of the six model spaces of type A_1, A_2, B, C, D and E (for details, see Theorem A in [4]).

On the other hand, it is an open question whether a real hypersurface in $P_n(\mathbb{C})$ has a rigidity or not. More precisely, if M is a $(2n-1)$-dimensional Riemannian manifold and ι, $\hat{\iota}$ are two isometric immersions of M into $P_n(\mathbb{C})$, then are ι and $\hat{\iota}$ congruent ?

To this problem, Y.-W.Choe, H.S.Kim, Y.J.Suh, R.Takagi and one of the present authors gave some partial solutions (see [1] and [3]). As a special case of the rigidity problem, we can consider the following one.

If M is a real hypersurface in $P_n(\mathbb{C})$ isometric to one of the model spaces of six types, then is M congruent to the model space ?

In this paper we shall give a partial affirmative answer to this question. The model space of type A_1 is just a geodesic hypersphere in $P_n(\mathbb{C})$ ([5]). The main purpose is to prove the following

Theorem. Let M be a $(2n-1)$-dimensional connected complete Riemannian manifold, and let $\hat{\iota}$ and ι be two isometric immersions of M into $P_n(\mathbb{C})(n \geq 3)$. If $\hat{\iota}(M)$ is a geodesic hypersphere in $P_n(\mathbb{C})$, then so is $\iota(M)$, that is, $\hat{\iota}$ and ι are rigid.

The authors would like to express their sincere gratitude to Professor Ryoichi Takagi for many valuable suggestions.

2. Preliminaries on real hypersurfaces

Supported by TGRC-KOSEF

Let ι be an isometric immersion of a $(2n-1)$-dimensional Riemannian manifold M into the complex projective space $P_n(\mathbb{C})$ with the metric of constant holomorphic sectional curvature $4c$. For a local orthonormal frame field $\{e_1, e_2, \ldots, e_{2n-1}\}$ of M, we denote its dual 1-forms by θ_i. Then the connection forms θ_{ij} and the curvature forms Θ_{ij} of M are defined by

$$(2.1) \qquad d\theta_i + \sum \theta_{ij} \wedge \theta_j = 0, \qquad \theta_{ij} + \theta_{ji} = 0,$$

$$(2.2) \qquad \Theta_{ij} = d\theta_{ij} + \sum \theta_{ik} \wedge \theta_{kj}$$

respectively, where and in the sequel the indices $i, j, k, l, \ldots$ run over the range $\{1, 2, \ldots, 2n-1\}$, unless otherwise stated.

With respect to the orthonormal frame field $\{\tilde{e}_1, \tilde{e}_2, \ldots, \tilde{e}_{2n}\}$ of $P_n(\mathbb{C})$ such that $\tilde{e}_i = \iota_* e_i$, we denote the connection forms of $P_n(\mathbb{C})$ by $\tilde{\theta}_{AB}$, where the indices $A, B, \ldots$ run over the range $\{1, 2, \ldots, 2n\}$. We put

$$(2.3) \qquad \iota^* \tilde{\theta}_{2ni} = \psi_i = \sum A_{ij} \theta_j, \quad J_{ij} \circ \iota = \phi_{ij} \quad \text{and} \quad J_{2ni} \circ \iota = \xi_i,$$

where J is the complex structure of $P_n(\mathbb{C})$ and A_{ij} are components of the shape operator or the second fundamental tensor of (M, ι). The rank of the matrix (A_{ji}) is called the *type number* of (M, ι). Then from (2.1), (2.2) and (2.3), we have the equations of Gauss and Weingarten

$$(2.4) \qquad \Theta_{ij} = \psi_i \wedge \psi_j + c\theta_i \wedge \theta_j + c\sum(\phi_{ik}\phi_{jl} + \phi_{ij}\phi_{kl})\theta_k \wedge \theta_l,$$

$$(2.5) \qquad d\psi_i + \sum \psi_j \wedge \theta_{ji} = c\sum(\xi_j \phi_{ik} + \xi_i \phi_{jk})\theta_j \wedge \theta_k$$

respectively. From (2.3) we also have

$$(2.6) \qquad \sum \phi_{ik}\phi_{kj} = -\delta_{ij} + \xi_i \xi_j, \quad \sum \xi_j \phi_{ji} = 0, \quad \sum \xi_i^2 = 1,$$

that is, the tensor fields $\phi = (\phi_{ij})$ and $\xi = (\xi_i)$ form an almost contact structure on M. ξ is called a *structure vector field.*

Since we have $dJ_{AB} = \sum(J_{AC}\tilde{\theta}_{CB} - J_{BC}\tilde{\theta}_{CA})$ on $P_n(\mathbb{C})$, it follows from (2.3) that

$$(2.7) \qquad d\phi_{ij} = \sum(\phi_{ik}\theta_{kj} - \phi_{jk}\theta_{ki}) - \xi_i\psi_j + \xi_j\psi_i,$$

$$(2.8) \qquad d\xi_i = \sum(\xi_j\theta_{ji} - \phi_{ji}\psi_j).$$

For another immersion $\hat{\iota}$ of M into $P_n(\mathbb{C})$, we shall denote the differential forms and tensor fields of $(M, \hat{\iota})$ by the same symbol as ones in (M, ι) but with a hat. Since the canonical 1-forms, connection forms and curvature forms are independent of the choice of immersions, it follows from (2.4) that

$$
\begin{aligned}
&A_{ik}A_{jl} - A_{il}A_{jk} + c(\phi_{ik}\phi_{jl} - \phi_{il}\phi_{jk} + 2\phi_{ij}\phi_{kl}) \\
&= \hat{A}_{ik}\hat{A}_{jl} - \hat{A}_{il}\hat{A}_{jk} + c(\hat{\phi}_{ik}\hat{\phi}_{jl} - \hat{\phi}_{il}\hat{\phi}_{jk} + 2\hat{\phi}_{ij}\hat{\phi}_{kl}).
\end{aligned} \tag{2.9}
$$

As for the rigidity of (M, ι) and $(M, \hat{\iota})$, the following are known and will be used later.

Theorem A ([1]). *Let M be a $(2n-1)$-dimensional Riemannian manifold, and $\hat{\iota}$ and ι be two isometric immersions of M into $P_n(\mathbb{C})(n \geq 3)$. If two structure vector fields coincide up to sign on M and the type number of $(M, \hat{\iota})$ or (M, ι) is not equal to 2 at every point of M, then $\hat{\iota}$ and ι are rigid.*

Theorem B ([3]). *Let M be a $(2n-1)$-dimensional Riemannian manifold, and $\hat{\iota}$ and ι be two isometric immersions of M into $P_n(\mathbb{C})(n \geq 3)$. If there exists a principal direction in common and the type number of $(M, \hat{\iota})$ or (M, ι) is not equal to 2 at every point of M, then $\hat{\iota}$ and ι are rigid.*

3. Geodesic hyperspheres

Let $\hat{\iota}$ be an isometric immersion of a $(2n-1)$-dimensional connected complete Riemannian manifold M into the complex projective space $P_n(\mathbb{C})$, and $\hat{\iota}(M)$ be a geodesic hypersphere in $P_n(\mathbb{C})$. Then there exists a local orthonormal frame field $\{e_1 = \hat{\xi}, e_2, \ldots, e_{2n-1}\}$ on M such that

$$
\hat{A} = \begin{pmatrix} \hat{\alpha} & 0 & \ldots & 0 \\ 0 & & & \\ \vdots & & r\delta_{pq} & \\ 0 & & & \end{pmatrix} \quad \text{and} \quad \hat{\phi} = \begin{pmatrix} 0 & & \ldots & 0 \\ 0 & & & \\ \vdots & & \hat{\phi}_{pq} & \\ 0 & & & \end{pmatrix} \tag{3.1}
$$

where $2 \leq p, q \leq 2n-1$, and the principal curvatures $\hat{\alpha}$ and r of $(M, \hat{\iota})$ are given by

$$
\hat{\alpha} = 2\sqrt{c}\cot 2\theta, \quad r = \sqrt{c}\cot\theta \tag{3.2}
$$

(for instance, see [5]).

The geodesic hyperspheres in $P_n(\mathbb{C})$ are characterized by

Lemma 3.1. *$\hat{\iota}(M)$ is a geodesic hypersphere in $P_n(\mathbb{C})$ if and only if the shape operator $\hat{A}$ of $(M, \hat{\iota})$ is given by*

$$\hat{A}_{ji} = r\delta_{ji} + (\hat{\alpha} - r)\hat{\xi}_j\hat{\xi}_i \tag{3.3}$$

where $\hat{\alpha}$ and r are scalar fields, and $r \neq 0$ on M.

Proof. If $\hat{\iota}(M)$ is a geodesic hypersphere, then it is easily seen that (2.8) and (3.1) give rise to

$$\sum \hat{\phi}_{ji}\hat{\psi}_j = r\sum \hat{\phi}_{ji}\theta_j, \tag{3.4}$$

which is equivalent to (3.3).

Conversely, if (3.3) is satisfied on M, then it follows from (2.6) and (3.3) that $\sum \hat{\xi}_j\hat{A}_{ji} = \hat{\alpha}\hat{\xi}_i$, that is, $\hat{\alpha}$ is a principal curvature of M. It is well known ([2]) that $\hat{\alpha}$ is a constant on M. Moreover (3.3) is equivalent to

$$\hat{\psi}_i = r\theta_i + (\hat{\alpha} - r)\hat{\xi}_i\hat{\eta}, \tag{3.5}$$

where $\hat{\eta}$ is the associated 1-form of $\hat{\xi}$, that is, $\hat{\eta} = \sum \hat{\xi}_i\theta_i$. By applying $\hat{\phi}_{ji}$ to (3.5), it is easily seen from (2.6) that (3.5) is equivalent to (3.4).

Differentiating (3.5) and making use of (2.1), (2.5), (2.8), (3.4) and (3.5), we have

$$dr \wedge (\theta_i - \hat{\xi}_i\hat{\eta}) = [c + r(\hat{\alpha} - r)](\sum \hat{\phi}_{ji}\theta_j \wedge \hat{\eta} + \sum \hat{\xi}_i\hat{\phi}_{jk}\theta_j \wedge \theta_k).$$

Multiplying this equation by $\hat{\xi}_i$ and using (2.6), we obtain

$$[c + r(\hat{\alpha} - r)]\sum \hat{\phi}_{ji}\theta_j \wedge \theta_i = 0.$$

Since the rank of the matrix $(\hat{\phi}_{ji})$ is equal to $2n - 2$, then this equation is reduced to

$$c + r(\hat{\alpha} - r) = 0, \tag{3.6}$$

which shows that r is a non-zero constant on M. Moreover if we set $r = \sqrt{c}\cot\theta$, then (3.2) is satisfied. Therefore $\hat{\imath}(M)$ is a geodesic hypersphere in $P_n(\mathbb{C})$.

4. Proof of Theorem

Since $\hat{\imath}(M)$ is a geodesic hypersphere in $P_n(\mathbb{C})$, then the shape operator $\hat{A}$ of $(M, \hat{\imath})$ is given by (3.3) in Lemma 3.1, and the principal curvatures $\hat{\alpha}$ and $r(\neq 0)$ are all constants on M.

It follows from (2.9) and (3.3) that

$$\tag{4.1} \begin{aligned} &A_{ik}A_{jl} - A_{il}A_{jk} + c(\phi_{ik}\phi_{jl} - \phi_{il}\phi_{jk} + 2\phi_{ij}\phi_{kl}) \\ &= r^2(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) - c(\delta_{ik}\hat{\xi}_j\hat{\xi}_l - \delta_{il}\hat{\xi}_j\hat{\xi}_k + \delta_{jl}\hat{\xi}_i\hat{\xi}_k - \delta_{jk}\hat{\xi}_i\hat{\xi}_l) \\ &\quad + c(\hat{\phi}_{ik}\hat{\phi}_{jl} - \hat{\phi}_{il}\hat{\phi}_{jk} + 2\hat{\phi}_{ij}\hat{\phi}_{kl}). \end{aligned}$$

For the shape operator A of (M, ι), We shall define some scalar fields on M as

$$\tag{4.2} \alpha = \sum\xi_j\xi_i A_{ji},\ \beta = \sum\xi_j\hat{\xi}_i A_{ji},\ \gamma = \sum\hat{\xi}_j\hat{\xi}_i A_{ji} \text{ and } f = \hat{\xi}_j\xi_j.$$

Then, first of all, we see that $f^2 \leq 1$ on M.

Multiplying (4.1) by $\xi_i\xi_k, \hat{\xi}_i\xi_k$ and $\hat{\xi}_i\hat{\xi}_k$, we have

$$\tag{4.3} \begin{aligned} \alpha A_{ji} - \sum\xi_k A_{kj}\xi_l A_{li} - 3c\sum\xi_k\hat{\phi}_{kj}\xi_l\hat{\phi}_{li} &= (r^2 - cf^2)\delta_{ji} - r^2\xi_j\xi_i \\ &\quad - c\hat{\xi}_j\hat{\xi}_i + cf(\hat{\xi}_j\xi_i + \xi_j\hat{\xi}_i) \end{aligned}$$

$$\tag{4.4} \beta A_{ji} - \sum\xi_k A_{kj}\hat{\xi}_l A_{li} = \hat{\alpha}r(f\delta_{ji} - \xi_j\hat{\xi}_i),$$

$$\tag{4.5} \gamma A_{ji} - \sum\hat{\xi}_k A_{kj}\hat{\xi}_l A_{li} + 3c\sum\hat{\xi}_k\phi_{kj}\hat{\xi}_l\phi_{li} = \hat{\alpha}r(\delta_{ji} - \hat{\xi}_j\hat{\xi}_i)$$

respectively, where we have used (2.6) and (3.6). If we take the symmetric parts of (4.4), we obtain

$$\sum(\hat{\xi}_k A_{kj}\xi_l A_{li} - \xi_k A_{kj}\hat{\xi}_l A_{li}) = \hat{\alpha}r(\hat{\xi}_j\xi_i - \xi_j\hat{\xi}_i).$$

Under the same consideration as the above, it is easily verified that this equation is reduced to

$$\tag{4.6} \gamma\sum\xi_j A_{ji} - \beta\sum\hat{\xi}_j A_{ji} = \hat{\alpha}r(\xi_i - f\hat{\xi}_i),$$

$$\tag{4.7} \beta\sum\xi_j A_{ji} - \alpha\sum\hat{\xi}_j A_{ji} = \hat{\alpha}r(f\xi_i - \hat{\xi}_i),$$

and

$$\alpha\gamma - \beta^2 = \hat{\alpha}r(1 - f^2). \tag{4.8}$$

Now we shall prove

Lemma 4.1. *Let $\hat{\iota}$ and ι be two isometric immersions of a $(2n-1)$-dimensional connected complete Riemannian manifold M into $P_n(\mathbb{C})$, and $\hat{\iota}(M)$ be a geodesic hypersphere. If the principal curvature $\hat{\alpha}$ is equal to zero on M, then the two structure vector fields coincide up to sign on M.*

Proof. At first we see from (3.6) and (4.8) that $r^2 = c$ and $\alpha\gamma = \beta^2$. Therefore our case can be occurred when $\theta = \tan^{-1}(\pm 1)$ as seen in (3.2).

If there is a point p of M such that $\alpha(p) = 0$, then (4.3) is reduced to

$$\sum \xi_k A_{kj} \xi_l A_{li} + 3c \sum \xi_k \hat{\phi}_{kj} \xi_l \hat{\phi}_{li} = c[(f^2-1)\delta_{ji} + \xi_j\xi_i + \hat{\xi}_j\hat{\xi}_i - f(\hat{\xi}_j\xi_i + \xi_j\hat{\xi}_i)].$$

Summing up for i and j, we have

$$\|A\xi\|^2 + 3c\|\hat{\phi}\xi\|^2 = (2n-3)c(f^2-1), \tag{4.9}$$

where $\|\ \ \|$ denotes the magnitude. Since $f^2 - 1 \leq 0$ and $n \geq 3$, then (4.9) gives rise to $\hat{\phi}\xi = 0$ and hence $\xi = \pm\hat{\xi}$ at p.

Let $\alpha \neq 0$ on an open neighborhood U in M. If $\beta = 0$ at some points of U, then $\gamma = 0$ at that points by (4.8). We assume that there is a point q of U such that $\beta(q) = 0$. Then it follows from (4.7) that $\sum \hat{\xi}_j A_{ji} = 0$ at q. Comparing this relation with (4.5), we have $\phi\hat{\xi} = 0$ and hence $\xi = \pm\hat{\xi}$ at q. If $\beta \neq 0$ on an open subset V of U, then it follows from (4.7) that

$$\sum \hat{\xi}_j A_{ji} = (\beta/\alpha) \sum \xi_j A_{ji}.$$

Substituting this relation into (4.4), we obtain

$$\alpha A_{ji} = \sum \xi_k A_{kj} \xi_l A_{li}.$$

If we compare this equation with (4.3), then we have

$$3\sum \xi_k \hat{\phi}_{kj} \xi_l \hat{\phi}_{li} = (f^2-1)\delta_{ji} + \xi_j\xi_i + \hat{\xi}_j\hat{\xi}_i - f(\hat{\xi}_j\xi_i + \xi_j\hat{\xi}_i),$$

which implies that $3\|\hat{\phi}\xi\|^2 = (2n-3)(f^2-1)$ and hence $\xi = \pm\hat{\xi}$ on V. This completes the proof.

Lemma 4.2. *If the principal curvature $\hat{\alpha}$ is not equal to zero on M under the assumptions as in Lemma 4.1, then we have $f^2 = 1$, that is, the two structure vector fields coincide up to sign on M.*

Proof. Assume that there is an open neighborhood U of M such that $f^2 \neq 1$ on U. Then it follows from (4.6), (4.7) and (4.8) that

$$\sum \hat{\xi}_j A_{ji} = u\xi_i + v\hat{\xi}_i, \tag{4.10}$$

where we have put $u = (\beta - \gamma f)/(1 - f^2)$ and $v = (\gamma - \beta f)/(1 - f^2)$. Substituting (4.10) into (4.5), we have

$$\begin{aligned} \gamma A_{ji} + 3c\sum \hat{\xi}_k \phi_{kj} \hat{\xi}_l \phi_{li} = \hat{\alpha} r \delta_{ji} + u^2 \xi_j \xi_i + \\ (v^2 - \hat{\alpha} r)\hat{\xi}_j \hat{\xi}_i + uv(\hat{\xi}_j \xi_i + \xi_j \hat{\xi}_i) \end{aligned} \tag{4.11}$$

on U. Since $f^2 \neq 1$, then $\phi\hat{\xi}$ is a non-zero vector field on U. Multiplying (4.11) by $\sum \hat{\xi}_h \phi_{hi}$ and using (2.6), we obtain

$$\gamma \sum A_{ji} \hat{\xi}_k \phi_{ki} = [\hat{\alpha} r - 3c(1 - f^2)] \sum \hat{\xi}_k \phi_{kj} \tag{4.12}$$

on U.

Assume that there is a point p of U such that $\gamma(p) = 0$. Then we see from (4.8) that $-\beta^2(p) = \hat{\alpha} r(1 - f^2(p))$ and so $\hat{\alpha} r \leq 0$. We also see from (4.12) that $\hat{\alpha} r = 3c(1 - f^2(p)) > 0$ and it is contrary. Therefore $\gamma \neq 0$ on U. The equation (4.12) shows that $\phi\hat{\xi}$ is a principal direction of (U, ι). Since we see from (3.3) that $\phi\hat{\xi}$ is a principal direction of $(M, \hat{\iota})$, then $\phi\hat{\xi}$ is the principal direction in common on U.

Since the type number of $(M, \hat{\iota})$ is equal to $2n - 2$ or $2n - 1$ by Lemma 3.1, then we see that the structure vector fields ξ and $\hat{\xi}$ coincide up to sign on U by Theorems A and B. This contradicts to $f^2 \neq 1$ on U, and completes the proof.

Proof of Theorem. By Lemmas 4.1 and 4.2, the two structure vector fields coincide up to sign on M and the type number of $(M, \hat{\iota})$ is not equal to 2 at every point of M. Therefore $\hat{\iota}$ and ι are rigid by Theorem A and this completes the proof.

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Received August 21, 1996