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Strong Convergence Theorems for Nonexpansive Nonself-mappings in Banach Spaces

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1 Introduction

Let E be a Banach space and let C be a nonempty closed convex subset of E. A mapping T from C into E is called nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. For a given $u \in C$ and each $t \in (0,1)$, we define a contraction $T_t : C \to E$ by

$$T_t x = tTx + (1-t)u \qquad \text{for all } x \in C. \tag{1}$$

If $T(C) \subset C$, then $T_t(C) \subset C$. Thus, by Banach's contraction principle , there exists a unique fixed point x_t of T_t in C, that is, we have

$$x_t = tTx_t + (1-t)u. \tag{2}$$

A question naturally arises to whether $\{x_t\}$ converges strongly as $t \to 1$ to a fixed point of T. This question has been investigated by several authors; see, for example, Browder[1], Halpern[4], Singh and Watson[8], Marino and Trombetta[6], and others. Recently, Xu and Yin[10] proved that if C is a nonempty closed convex subset of a Hilbert space H, if $T : C \to H$ is a nonexpansive nonself-mapping, and if $\{x_t\}$ is the sequence defined by (2) which is bounded, then $\{x_t\}$ converges strongly as $t \to 1$ to a fixed point of T. Next, consider a sunny nonexpansive retraction P from E onto C. Then, following Marino and Trombetta[6], for a given $u \in C$ and each $t \in (0,1)$, we define contractions S_t and U_t from

C into itself by

$$S_t x = tPTx + (1-t)u \qquad \text{for all } x \in C$$

and

$$U_t x = P(tTx + (1-t)u) \qquad \text{for all } x \in C.$$

By Banach's contraction principle, there exists a unique fixed point x_t (resp. y_t) of S_t (resp. U_t) in C, i.e.,

$$x_t = tPTx_t + (1-t)u \tag{3}$$

and

$$y_t = P(tTy_t + (1-t)u). \tag{4}$$

Xu and Yin[10] also proved that if C is a nonempty closed convex subset of a Hilbert space H, if $T : C \to H$ is a nonexpansive nonself-mapping satisfying the weak inwardness condition, and if P is the nearest projection from H onto C, then the sequence $\{x_t\}$(resp. $\{y_t\}$) defined by (3) (resp. (4)) which is bounded converges strongly as $t \to 1$ to a fixed point of T.

In this paper, we extend Xu and Yin's results[10] to Banach spaces, that is, we prove that the sequence defined by (2)(resp. (3), (4)) which is bounded in a smooth and reflexive Banach space converges strongly as $t \to 1$ to a fixed point of T.

2 Preliminaries

Throughout this paper we denote by E and E^* a Banach space and the dual space of E, respectively. The value of $x^* \in E^*$ at $x \in E$ will be denoted by $\langle x, x^* \rangle$. We also denote by $F(T)$ the set of all fixed points of T, i.e., $F(T) = \{x \in C : Tx = x\}$ and by R and R^+ the sets of all real numbers and all nonnegative real numbers, respectively. When $\{x_n\}$ is a sequence in E, then $x_n \to x$ (resp. $x_n \rightharpoonup x$, $x_n \overset{*}{\rightharpoonup} x$) will denote strong (resp. weak, *weak**) convergence of the sequence $\{x_n\}$ to x. Let C be a nonempty closed convex subset of E, let D be a subset of C and let P be a mapping of C into D. Then P is said to be sunny if

$$P(Px + t(x - Px)) = Px$$

whenever $Px + t(x - Px) \in C$ for $x \in C$ and $t \geq 0$. A mapping P of C into C is said to be a retraction if $P^2 = P$. If a mapping P of C into C is a retraction, then $Pz = z$ for every $z \in R(P)$, where $R(P)$ is the range of P. A subset D of C is said to be a sunny nonexpansive retract of C if there exists a sunny nonexpansive retraction of C onto D; for more details, see[5]. For every ϵ with $0 \leq \epsilon \leq 2$, the modulus $\delta(\epsilon)$ of convexity of E is defined by

$$\delta(\epsilon) = \inf\{1 - \|\frac{x+y}{2}\| : \|x\| \leq 1, \quad \|y\| \leq 1, \quad \|x - y\| \geq \epsilon\}.$$

E is said to be uniformly convex if $\delta(\epsilon) > 0$ for every $\epsilon > 0$. If E is uniformly convex, then E is reflexive. Let $S(E) = \{x \in E : \|x\| = 1\}$. Then the norm of E is said to be Gâteaux differentiable (and E is said to be smooth) if

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{5}$$

exists for each x and y in $S(E)$. It is also said to be uniformly Fréchet differentiable (and E is said to be uniformly smooth) if the limit (5) is attained uniformly for x, y in $S(E)$. With each $x \in E$, we associate the set

$$J_\phi(x) = \{x^* \in E^*; \langle x, x^* \rangle = \|x\|\|x^*\| \quad and \quad \|x^*\| = \phi(\|x\|)\},$$

where $\phi : R^+ \to R^+$ is a continuous and strictly increasing function with $\phi(0) = 0$ and $\phi(\infty) = \infty$. Then $J_\phi : E \to 2^{E^*}$ is said to be the duality mapping. Suppose that J_ϕ is single-valued. Then J_ϕ is said to be weakly sequentially continuous if for each $\{x_n\} \in E$ with $x_n \rightharpoonup x$, $J_\phi(x_n) \overset{*}{\rightharpoonup} J_\phi(x)$. For abbreviation, we set $J := J_\phi$. In all our proofs we assume, without loss of generality, that J is normalized. It is well known if E is smooth, then the duality mapping J is single-valued and strong-*weak** continuous. It is also known that E is uniformly smooth if and only if E^* is uniformly convex; for more details, see

Diestel[2]. A Banach space E is said to satisfy Opial's condition[7] if for any sequence $\{x_n\}$ in E, $x_n \rightharpoonup x$ implies that

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|$$

for all $y \in E$ with $y \neq x$. We know that if E admits a weakly sequentially continuous duality mapping, then E satisfies Opial's condition; see[3].

3 Strong convergence Theorems

In this section, we first prove a strong convergence theorem for nonexpansive nonself-mappings in a Banach space which generalizes Xu and Yin's result[10].

Theorem 1 *Let E be a smooth and reflexive Banach space with a weakly sequentially continuous duality mapping $J : E \to E^*$, let C be a nonempty closed convex subset of E, and let $T : C \to E$ be a nonexpansive nonself-mapping. Suppose that for some $u \in C$ and each $t \in (0,1)$, the contraction T_t defined by (1) has a (unique) fixed point $x_t \in C$. Then T has a fixed point if and only if $\{x_t\}$ remains bounded as $t \to 1$. In this case, $\{x_t\}$ converges strongly as $t \to 1$ to a fixed point of T.*

Proof. Let x be a fixed point of T. Then we have

$$\begin{aligned}\|x - x_t\| &= \|x - tTx_t - (1-t)u\| \\ &\leq t\|x - Tx_t\| + (1-t)\|x - u\| \\ &\leq t\|x - x_t\| + (1-t)\|x - u\|\end{aligned}$$

and hence $\|x - x_t\| \leq \|x - u\|$. So, $\{x_t\}$ is bounded. Conversely, suppose that $\{x_t\}$ is bounded when t is closed enough to 1. Then there exist a subsequence $\{x_{t_n}\}$ of the sequence $\{x_t\}$ and a point $y \in C$ such that $x_{t_n} \rightharpoonup y$. By (2), we have

$$\|x_{t_n} - Tx_{t_n}\| = (1-t_n)\|u - Tx_{t_n}\| \to 0 \qquad \text{as } n \to \infty.$$

So, we have

$$\begin{aligned}\limsup_{n\to\infty}\|x_{t_n}-Ty\| &\le \limsup_{n\to\infty}\{\|x_{t_n}-Tx_{t_n}\|+\|Tx_{t_n}-Ty\|\}\\ &\le \limsup_{n\to\infty}\|x_{t_n}-y\|.\end{aligned}$$

If $Ty \neq y$, from Theorem 1 in [3], we have

$$\begin{aligned}\limsup_{n\to\infty}\|x_{t_n}-y\| &< \limsup_{n\to\infty}\|x_{t_n}-Ty\|\\ &\le \limsup_{n\to\infty}\|x_{t_n}-y\|.\end{aligned}$$

This is a contradiction. Hence we have $y \in F(T)$. Since, for any $w \in F(T)$,

$$\begin{aligned}\langle \frac{1}{t_n}x_{t_n}-(\frac{1}{t_n}-1)u-w, J(w-x_{t_n})\rangle &= \langle Tx_{t_n}-Tw, J(w-x_{t_n})\rangle\\ &\ge -\|Tx_{t_n}-Tw\|\|J(w-x_{t_n})\|\\ &\ge -\|w-x_{t_n}\|^2\\ &= \langle x_{t_n}-w, J(w-x_{t_n})\rangle,\end{aligned}$$

we have $\langle(\frac{1}{t_n}-1)(x_{t_n}-u), J(w-x_{t_n})\rangle \ge 0$. So, we have

$$\langle x_{t_n}-u, J(w-x_{t_n})\rangle \ge 0. \tag{6}$$

Thus putting $w = y$,

$$\begin{aligned}\langle y-u, J(y-x_{t_n})\rangle &= \langle y-x_{t_n}, J(y-x_{t_n})\rangle + \langle x_{t_n}-u, J(y-x_{t_n})\rangle\\ &\ge \|y-x_{t_n}\|^2.\end{aligned}$$

Since $x_{t_n} \rightharpoonup y$ and J is weakly sequentially continuous, we have $x_{t_n} \to y$. By using the argument above again, we obtain a subsequence $\{x_{t_m}\}$ of $\{x_t\}$ converging weakly to some $z \in C$ such that $z = Tz$ and $x_{t_m} \to z$. From (6), we have

$$\langle y-u, J(w-y)\rangle \ge 0 \quad and \quad \langle z-u, J(w-z)\rangle \ge 0$$

for any $w \in F(T)$ and hence

$$\langle y-u, J(z-y)\rangle \ge 0 \quad and \quad \langle z-u, J(y-z)\rangle \ge 0.$$

This implies $y = z$. Therefore we have $x_t \to z$.

Next, we consider two strong convergence theorems which generalize Xu and Yin's results[10], using a sunny nonexpansive retraction P from E onto C. Let E be a Banach space and let C be a nonempty convex subset of E. Then for $x \in C$ we define the inward set $I_C(x)$ as follows:

$$I_C(x) = \{y \in E : y = x + a(z - x) \quad for \quad some \quad z \in C \quad and \quad a \geq 0\}.$$

A mapping $T : C \to E$ is said to be inward if $Tx \in I_C(x)$ for all $x \in C$. T is also said to be weakly inward if for each $x \in C$, Tx belongs to the closure of $I_C(x)$.

Theorem 2 *Let E be a smooth and reflexive Banach space with a weakly sequentially continuous duality mapping $J : E \to E^*$, let C be a nonempty closed convex subset of E, and let $T : C \to E$ be a nonexpansive nonself-mapping satisfying the weak inwardness condition. Suppose that C is a sunny nonexpansive retract of E and for some $u \in C$ and each $t \in (0, 1)$, $x_t \in C$ is a (unique) fixed point of the contraction S_t defined by (3), where P is a sunny nonexpansive retraction of E onto C. Then T has a fixed point if and only if $\{x_t\}$ remains bounded as $t \to 1$. In this case, $\{x_t\}$ converges strongly as $t \to 1$ to a fixed point of T.*

Proof. Let x be a fixed point of T. Then $\{x_t\}$ is bounded. Conversely, suppose that $\{x_t\}$ is bounded when t is closed enough to 1. Applying Theorem 1, we obtain that $\{x_t\}$ converges strongly as $t \to 1$ to a fixed point z of PT. Next, let us show $z \in F(T)$. Since $z = PTz$ and P is a sunny nonexpansive retraction of E onto C, we have

$$\langle Tz - z, J(z - v)\rangle \geq 0$$

for all $v \in C$; see[9]. On the other hand, Tz belongs to the closure of $I_C(z)$ by the weak inwardness condition. Hence there exist, for each integer $n \geq 1$, $z_n \in C$ and $a_n \geq 0$ such that the sequence

$$y_n := z + a_n(z_n - z) \to Tz.$$

Since

$$
\begin{aligned}
0 &\leq a_n\langle Tz - z, J(z - z_n)\rangle \\
&= \langle Tz - z, J(a_n(z - z_n))\rangle \\
&= \langle Tz - z, J(z - y_n)\rangle
\end{aligned}
$$

and J is weakly sequentially continuous, we have

$$0 \leq \langle Tz - z, J(z - Tz)\rangle = -\|Tz - z\|^2$$

and hence $Tz = z$.

Theorem 3 *Let E be a smooth and reflexive Banach space with a weakly sequentially continuous duality mapping $J : E \to E^*$, let C be a nonempty closed convex subset of E, and let $T : C \to E$ be a nonexpansive nonself-mapping satisfying the weak inwardness condition. Suppose that C is a sunny nonexpansive retract of E and for some $u \in C$ and each $t \in (0, 1)$, $y_t \in C$ is a (unique) fixed point of the contraction U_t defined by (4), where P is a sunny nonexpansive retraction of E onto C. Then T has a fixed point if and only if $\{y_t\}$ remains bounded as $t \to 1$. In this case, $\{y_t\}$ converges strongly as $t \to 1$ to a fixed point of T.*

Proof. Let x be a fixed point of T. Then we have

$$
\begin{aligned}
\|x - y_t\| &= \|Px - P(tTy_t + (1 - t)u)\| \\
&\leq t\|x - Ty_t\| + (1 - t)\|x - u\| \\
&\leq t\|x - y_t\| + (1 - t)\|x - u\|
\end{aligned}
$$

and hence $\|x - y_t\| \leq \|x - u\|$. So, $\{y_t\}$ is bounded. Conversely, suppose that $\{y_t\}$ is bounded when t is closed enough to 1. Then there exist a subsequence $\{y_{t_n}\}$ of the sequence $\{y_t\}$ and a point $y \in C$ such that $y_{t_n} \rightharpoonup y$. Since $\{Ty_{t_n}\}$ is bounded and

$$
\begin{aligned}
\|y_{t_n} - PTy_{t_n}\| &\leq \|t_nTy_{t_n} + (1 - t_n)u - Ty_{t_n}\| \\
&= (1 - t_n)\|u - Ty_{t_n}\|,
\end{aligned}
$$

we have $y_{t_n} - PTy_{t_n} \to 0$. So, we have

$$\begin{aligned}\limsup_{n\to\infty} \|y_{t_n} - PTy\| &\le \limsup_{n\to\infty}\{\|y_{t_n} - PTy_{t_n}\| + \|PTy_{t_n} - PTy\|\} \\ &\le \limsup_{n\to\infty} \|y_{t_n} - y\|.\end{aligned}$$

If $PTy \neq y$, from Theorem 1 in [3], we have

$$\begin{aligned}\limsup_{n\to\infty} \|y_{t_n} - y\| &< \limsup_{n\to\infty} \|y_{t_n} - PTy\| \\ &\le \limsup_{n\to\infty} \|y_{t_n} - y\|.\end{aligned}$$

This is a contradiction. Hence $y = PTy$. So, from [9],

$$\langle Ty - y, J(y - v)\rangle \ge 0$$

for all $v \in C$. On the other hand, Ty belongs to the closure of $I_C(y)$ by the weak inwardness condition. Hence there exist, for each integer $n \ge 1$, $z_n \in C$ and $a_n \ge 0$ such that the sequence

$$y_n := y + a_n(z_n - y) \to Ty.$$

As in the proof of Theorem 2, we have $Ty = y$. For any $w \in F(T)$, we have

$$t(w - u) + u = tw + (1 - t)u = P(tw + (1 - t)u)$$

and hence

$$\begin{aligned}\|(y_t - u) - t(w - u)\|^2 &= \|P(tTy_t + (1 - t)u) - u - t(w - u)\|^2 \\ &= \|P(t(Ty_t - u) + u) - u - t(w - u)\|^2 \\ &= \|P(t(Ty_t - u) + u) - u - P(t(w - u) + u) + u\|^2 \\ &\le \|t(Ty_t - u) - t(w - u)\|^2 \\ &\le t^2\|y_t - w\|^2 \\ &= t^2\|(y_t - u) - (w - u)\|^2.\end{aligned}$$

So, we have

$$\begin{aligned} 0 &\geq \|(y_t - u) - t(w - u)\|^2 - \|t(y_t - u) - t(w - u)\|^2 \\ &\geq 2\langle (1-t)(y_t - u), J(t(y_t - w))\rangle \\ &= 2(1-t)t\langle y_t - u, J(y_t - w)\rangle \end{aligned}$$

and hence

$$\langle y_t - u, J(y_t - w)\rangle \leq 0.$$

Thus putting $w = y$,

$$\begin{aligned} \langle y - u, J(y - y_{t_n})\rangle &= \langle y - y_{t_n}, J(y - y_{t_n})\rangle + \langle y_{t_n} - u, J(y - y_{t_n})\rangle \\ &\geq \|y - y_{t_n}\|^2. \end{aligned}$$

Since $y_{t_n} \rightharpoonup y$ and J is weakly sequentially continuous, we have $y_{t_n} \to y$. As in the proof of Theorem 1, we have $y_t \to z$.

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