Nihonkai Math. J.
Vol.7 (1996), 41-50

On some uniqueness and existence results for initial value problems of ordinary differential equations

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Introduction

Consider the initial value problem

(1) $$y' = f(x,y),\ y(x_0) = y_0$$

where the function $f(x,y)$ is at least continuous in $\bar{S}_+ : x_0 \leq x \leq x_0 + a,\ |y - y_0| \leq b$. In such a case by a solution of (1) in the interval $[x_0, x_0 + a]$, we mean a function $y(x)$ satisfying:

$$\begin{cases} y(x_0) = y_0, \\ \text{for all } x \in [x_0, x_0 + a], \text{ the points } (x, y(x)) \text{ in } \bar{S}_+, \\ y'(x) \text{ exists and continuous in } [x_0, x_0 + a] \text{ ; and} \\ y'(x) = f(x, y(x)). \end{cases}$$

J.M.Bownds [1], J.M.Bownds and F.T.Metcalf [2] used a certain factorization of the function $f(x,y)$, further T.C.Gard [3] imposed a condition on the $f(x,y)$ with a certain function $\phi(x)$ defined in $[x_0, x_0 + a)$ to obtain the uniqueness of the classical solutions[††] of the initial value problem (1). The results we shall prove are based on both these works.

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[††]If $f(x,y)$ is only defined in $\bar{S}_+$, by a classical solution of (1) in $[x_0, x_0 + a]$ we mean a $y(x)$ satisfying $y(x_0) = y_0$, $(x, y(x)) \in \bar{S}_+$ and $y(x)$ is continuous in $[x_0, x_0 + a]$, $y'(x)$ exists in $(x_0, x_0 + a)$ and $y'(x) = f(x, y(x))$.

In the next section we will show a generalization of O.Kooi [4]'s uniqueness theorem. Also H.Uryu, M.Murakami and Y.Todo [5] proved that his uniqueness result remains true without continuity of $f(x,y)$ in some sense. Further It will be shown by Corollary 3 that the conditions O.Kooi imposed in his existence and uniqueness theorem are special cases of our results.

We could fined these studies [1]~[4] in the book written by R.P.Agarwal and V.Lakshmikantham [6].

The uniqueness results

Theorem 1. Suppose that it is possible to find three functions f_1, f_2, f_3, in $\bar{S}_+$ such that $f = f_1 f_2 - f_3$ in $\bar{S}_+$, where f_1, f_2, f_3 are continuous in $\bar{S}_+$, and $\frac{\partial f_1(x,y)}{\partial x}$ exists, is nonnegative, and is continuous in $\bar{S}_+$. Further, suppose that f_1, f_2 and f_3 have the following properties

i) the function f_1 is strictly positive along the solution curves,

ii) for each fixed $x \in (x_0, x_0 + a)$ and $y_0 - b \leq y < \bar{y} \leq y_0 + b$

$$f_3(x,\bar{y}) f_1(x,y) \geq f_3(x,y) f_1(x,\bar{y}) \text{ ; and}$$

iii) let $\phi(x)$ be a differentiable function defined in $(x_0, x_0 + a)$ such that $\phi(x) > 0$ in $(x_0, x_0 + a)$ and for all $(x,y), (x,\bar{y})$ in $\tilde{S}_+ \ : \ x_0 < x < x_0 + a, \ |y - y_0| \leq b$ with $y < \bar{y} \quad f_2$ satisfies

$$f_2(x,\bar{y}) - f_2(x,y) \leq \frac{\phi'(x)}{\phi(x)}(\bar{y} - y). \tag{2}$$

Further let $\lambda(x)$ be a nonnegative function in $(x_0, x_0 + a)$ such that $\int_{x_0}^{x} \lambda(t) dt$ exists and for all $(x,\bar{y}), (x,y)$ in $\tilde{S}_+$ with $y < \bar{y} \quad f_2$ satisfies

$$f_2(x,\bar{y}) - f_2(x,y) \leq \lambda(x) g(\bar{y} - y) \tag{3}$$

where $g(z) > 0$ is the continuous and nondecreasing function for $z > 0$ and it satisfies

$$G(y) = \int_0^y \frac{dz}{g(z)} < \infty, \qquad \lim_{x \to x_0^+} \frac{G^{-1}\left(M \int_{x_0}^{x} \lambda(t) dt\right)}{\phi^M(x)} = 0$$

where $M = \max_{\bar{S}_+} |f_1(x,y)|$.

Then, the initial value problem (1) has at most one solution in $[x_0, x_0 + a]$.

Proof. Suppose there are two solutions $y(x)$ and $\bar{y}(x)$ of (1) in the interval $[x_0, x_0 + a]$. Then, there exists a point $\bar{x} \in (x_0, x_0 + a)$ such that $\bar{y}(\bar{x}) - y(\bar{x}) > 0$. Let $\xi_0 = \sup\{x : x_0 \le x < \bar{x} \text{ and } \bar{y}(x) = y(x)\}$. Then, $\bar{y}(\xi_0) = y(\xi_0)$ and $\bar{y}(x) > y(x)$ in $(\xi_0, \bar{x})$. Now $f_1(\xi_0, y(\xi_0)) > 0$ and since f_1 is continuous there exists a ξ_1 such that for $\xi_0 \le x \le \xi_1$ and $y(x) \le y \le \bar{y}(x)$ it is true that $f_1(x,y)$ is bounded away from zero by a positive number, say, m. Define a function $H : [\xi_0, \xi_1] \to R$ as follows

$$(4) \qquad H(x) \equiv \int_{y(x)}^{\bar{y}(x)} \left[\frac{dt}{f_1(x,t)}\right] + \int_{\xi_0}^{x} \left\{\frac{f_3(t,\bar{y}(t))}{f_1(t,\bar{y}(t))} - \frac{f_3(t,y(t))}{f_1(t,y(t))}\right\} dt.$$

Clearly, $H(\xi_0) = 0$ and in view of the condition i) and ii) it follows that

$$(5) \qquad H(x) > 0 \quad \text{in } (\xi_0, \xi_1).$$

Also, by the hypotheses on f_1, $H(x)$ may be differentiated in (ξ_0, ξ_1), to obtain

$$\begin{aligned}
(6) \qquad H'(x) &= \int_{y(x)}^{\bar{y}(x)} \frac{\partial}{\partial x}\left[\frac{1}{f_1(x,t)}\right] dt + \frac{\bar{y}'(x)}{f_1(x,\bar{y}(x))} - \frac{y'(x)}{f_1(x,y(x))} \\
&\quad + \frac{f_3(x,\bar{y}(x))}{f_1(x,\bar{y}(x))} - \frac{f_3(x,y(x))}{f_1(x,y(x))} \\
&\le \frac{\bar{y}'(x) + f_3(x,\bar{y}(x))}{f_1(x,\bar{y}(x))} - \frac{y'(x) + f_3(x,y(x))}{f_1(x,y(x))} \\
&= \frac{f(x,\bar{y}(x)) + f_3(x,\bar{y}(x))}{f_1(x,\bar{y}(x))} - \frac{f(x,y(x)) + f_3(x,y(x))}{f_1(x,y(x))}.
\end{aligned}$$

In the inequality (6), on using the factorization of f and the condition (2), we obtain

$$\begin{aligned}
(7) \qquad H'(x) &\le f_2(x,\bar{y}(x)) - f_2(x,y(x)) \\
&\le \frac{\phi'(x)}{\phi(x)}(\bar{y}(x) - y(x)), \qquad x \in (\xi_0, \xi_1).
\end{aligned}$$

From ii) and (4) it follows that

$$H(x) \geq \int_{y(x)}^{\bar{y}(x)} \frac{dt}{M} = \frac{\bar{y}(x) - y(x)}{M}. \tag{8}$$

Thus a combination of (7) and (8) leads to the differential inequality

$$H'(x) - \frac{\phi'(x)}{\phi(x)} M H(x) \leq 0, \qquad x \in (\xi_0, \xi_1).$$

Hence for $0 < \varepsilon < \xi_1 - \xi_0$, we have

$$\frac{d}{dx}\left[\ln H(x) - \ln \phi^M(x)\right] = \frac{d}{dx}\left(\ln \frac{H(x)}{\phi^M(x)}\right) \leq 0, \qquad x \in [\xi_0 + \varepsilon, \xi_1).$$

Therefore, it follows that

$$\ln \frac{H(x)}{\phi^M(x)} \leq \ln \frac{H(\xi_0 + \varepsilon)}{\phi^M(\xi_0 + \varepsilon)}$$

which may be written as

$$0 \leq \frac{H(x)}{\phi^M(x)} \leq \frac{H(\xi_0 + \varepsilon)}{\phi^M(\xi_0 + \varepsilon)}, \qquad x \in [\xi_0 + \varepsilon, \xi_1). \tag{9}$$

If $x_0 < \xi_0$, then (9), in view of $\lim_{\varepsilon \to 0^+} \frac{H(\xi_0+\varepsilon)}{\phi^M(\xi_0+\varepsilon)} = 0$, implies that $H(x) = 0$ in (ξ_0, ξ_1). But this contradicts (5).

If $x_0 = \xi_0$ then (9) is

$$0 \leq \frac{H(x)}{\phi^M(x)} \leq \frac{H(x_0 + \varepsilon)}{\phi^M(x_0 + \varepsilon)}, \qquad x \in [x_0 + \varepsilon, \xi_1). \tag{10}$$

From (3) and the hypothesis on g, it follows that

$$H'(x) \leq \lambda(x) g(\bar{y}(x) - y(x)) \leq \lambda(x) g(M H(x))$$

which implies that

$$G(MH(x_0 + \varepsilon)) \leq M \int_{x_0}^{x_0 + \varepsilon} \lambda(t) dt.$$

On using the monotoneity of G, we find

$$(11) \qquad MH(x_0+\varepsilon) \le G^{-1}\left(M\int_{x_0}^{x_0+\varepsilon}\lambda(t)dt\right).$$

From a combination of (11) and (10), we get

$$\begin{aligned} 0 \le \frac{H(x)}{\phi^M(x)} &\le \frac{H(x_0+\varepsilon)}{\phi^M(x_0+\varepsilon)} \\ &\le \frac{1}{M}\frac{G^{-1}\left(M\int_{x_0}^{x_0+\varepsilon}\lambda(t)dt\right)}{\phi^M(x_0+\varepsilon)}, \qquad x \in [x_0+\varepsilon, \xi_1). \end{aligned}$$

This leads to the desired contradiction. ■

Corollary 1. In Theorem 1 condition iii) can be replaced by

iii)′ the function f_2 satisfies for all $(x,y),(x,\bar{y}) \in \tilde{S}_+$ with $y \le \bar{y}$

$$f_2(x,\bar{y}) - f_2(x,y) \le \frac{k(\bar{y}-y)}{x-x_0},$$

$$f_2(x,\bar{y}) - f_2(x,y) \le \frac{c(\bar{y}-y)^\alpha}{(x-x_0)^\beta}$$

where the constants k, c, α and β, satisfy the inequalities $k>0$, $c>0$, $0<\alpha<1$, $\beta<1$, $Mk<\frac{1-\beta}{1-\alpha}$, and $M=\max_{\bar{S}_+}|f_1(x,y)|$.

Proof. It suffices to note that the function $\phi(x)=(x-x_0)^k$, $\lambda(x)=\frac{c}{(x-x_0)^\beta}$ and $g(z)=z^\alpha$ are admissible in Theorem 1. ■

If we take $f_1 \equiv 1, f_2 \equiv f$ and $f_3 \equiv 0$ in Corollary 1 then we obtain the generalization of Kooi's uniqueness theorem (see the proof of corollary 3).

Also even if we consider (1) as an n-dimensional system, i.e., $f=(f_1,\cdots,f_n)$ and $y=(y_1,\cdots,y_n)$, the idea of the proof in Theorem 1 can be used.

For $x,y\in R^n$ we shall use the following notations.

$|y|=\sum_{i=1}^n|y_i|$, $\quad ||y||=\sqrt{\sum_{i=1}^n y_i^2}$, $\quad x\cdot y=\sum_{i=1}^n x_iy_i$

Theorem 2. Let $\phi(x) > 0$ be a differentiable function defined in (x_0, x_0+a). Further, let the function $f(x,y)$ be continuous in $\hat{S}_+$; $x_0 \leq x < x_0 + a$, $|y - y_0| \leq b$ and for all $(x,y), (x,\bar{y}) \in \tilde{S}_+$; $x_0 < x < x_0 + a$, $|y - y_0| \leq b$ with $y \neq \bar{y}$ the following inequalities hold

i) $(f(x,y) - f(x,\bar{y})) \cdot (y - \bar{y}) \leq \frac{\phi'(x)}{\phi(x)} ||y - \bar{y}||^2$; and

ii) $(f(x,y) - f(x,\bar{y})) \cdot (y - \bar{y}) \leq \lambda(x) g(||y - \bar{y}||) ||y - \bar{y}||$

where $\lambda(x) \geq 0$ is defined in $(x_0, x_0 + a)$ such that $\int_{x_0}^{x} \lambda(t)dt$ exists, $g(z) > 0$ is continuous for $z > 0$ and these satisfy

$$G(y) = \int_0^y \frac{dz}{g(z)} < \infty, \qquad \lim_{x \to x_0^+} \frac{G^{-1}\left(\int_{x_0}^{x} \lambda(t)dt\right)}{\phi(x)} = 0.$$

Then, the initial value problem (1) has at most one n-solution[‡] in $[x_0, x_0+a)$.

Proof. Let $y(x)$ and $\bar{y}(x)$ be two n-solutions of (1). Then we define a function $H(x) \equiv ||y(x) - \bar{y}(x)||$. As in Theorem 1 we obtain

$$0 \leq \frac{H(x)}{\phi(x)} \leq \frac{H(\xi_0 + \varepsilon)}{\phi(\xi_0 + \varepsilon)} \quad \text{in } [\xi_0 + \varepsilon, \xi_1)$$

which leads the contradiction. ■

The successive approximations

It is well known that a uniqueness theorem doesn't imply the convergence of sequences of functions obtained by Picard's method of successive approximations. Therefore it may be of some interest to investigate the convergence in our case.

[‡]If $f(x,y)$ is continuous in $\hat{S}_+$, by a n-solution of (1) in $[x_0, x_0 + a)$ we mean a $y(x) = (y_1(x), \cdots, y_n(x))$ satisfying $y(x_0) = y_0$, $(x, y(x)) \in \hat{S}_+$ in $[x_0, x_0 + a)$, $y'(x)$ exists and continuous in $[x_0, x_0 + a)$; and $y_i'(x) = f_i(x, y(x))$, $i = 1, \cdots, n$.

Theorem 3. Let $\phi(x)$ be a continuous non-decreasing function in $x_0 \leq x \leq x_0 + a$ and continuously differentiable in $x_0 < x \leq x_0 + a$ such that $\phi(x_0) = 0$, $\phi(x) > 0$ for $x > x_0$. Further, let the function $f(x, y)$ be continuous in $\bar{S}_+$ and for all $(x, \bar{y}), (x, y) \in \bar{S}_+$ it satisfies

$$
(12) \qquad |f(x, \bar{y}) - f(x, y)| \leq \frac{\phi'(x)}{\phi(x)} |\bar{y} - y|, \; x > x_0
$$

$$
(13) \qquad |f(x, \bar{y}) - f(x, y)| \leq \lambda(x) g(|\bar{y} - y|), \; x > x_0
$$

where $\lambda(x) \geq 0$ is a continuous function in $x_0 < x \leq x_0 + a$ and $g(z) \geq 0$ is the continuous and nondecreasing function for $z \geq 0$. Further suppose for any constant $L \geq 0$ there exist two constants $\delta > 1, C \geq 0$ and natural number n such that $T^n(L(x - x_0)) \leq C\phi^\delta(x)$ in $x_0 \leq x \leq x_0 + a$ where T^n is an operator given as $T^n(v(x)) \equiv T^1(T^{n-1}(v(x)))$, $T^1(v(x)) \equiv \int_{x_0}^x \lambda(t) g(v(t)) dt$.

Then, the initial value problem (1) has a unique solution in $x_0 \leq x \leq x_0 + h$ where $h = \min\left(a, \dfrac{b}{M}\right)$ and $M = \max_{\bar{S}_+} |f(x, y)|$.

Proof. I) The existence

We choose a function $y_0(x) = y_0$. The sequence of functions $\{y_n(x)\}$, defined for natural n by the relation

$$
(14) \qquad y_n(x) = y_0 + \int_{x_0}^x f(t, y_{n-1}(t)) dt
$$

is the well known sequences of the successive approximations. From induction it can be proved that $y_n(x)$ is continuous and $|y_n(x) - y_0| \leq b$ in $x_0 \leq x \leq x_0 + h$ for every $n \geq 1$. Now we shall prove the uniform convergence in $x_0 \leq x \leq x_0 + h$ of a sequence $\{y_n(x)\}$. Because $y_n(x) = y_0 + \sum_{j=1}^n (y_j(x) - y_{j-1}(x))$, it suffices to prove that a sequence $\left\{\sum_{j=1}^n (y_j(x) - y_{j-1}(x))\right\}$ uniformly converges in $x_0 \leq x \leq x_0 + h$.

By (13) and definition of T^n, we have the following inequalities

$$
\begin{aligned}
|y_n(x) - y_{n-1}(x)| &\leq \int_{x_0}^x \lambda(t) g(|y_{n-1}(t) - y_{n-2}(t)|) dt = T^1(|y_{n-1}(x) - y_{n-2}(x)|) \\
&\leq T^2(|y_{n-2}(x) - y_{n-3}(x)|) \\
&\leq \cdots\cdots\cdots
\end{aligned}
$$

$$
\begin{aligned}
&\le T^{n-1}(|y_1(x) - y_0(x)|) = T^{n-1}\left(\left|\int_{x_0}^{x} f(t, y_0)dt\right|\right) \\
&\le T^{n-1}(M(x - x_0)).
\end{aligned}
$$

From the hypothesis of the operator T^n, there exists a natural N such that $T^{N-1}(M(x - x_0)) \le c\phi^\delta$ in $x_0 \le x \le x_0 + h$. Hence

$$
|y_N(x) - y_{N-1}(x)| \le c\phi^\delta(x) \quad \text{in } x_0 \le x \le x_0 + h. \tag{15}
$$

Using condition (12) and inequality (15), it follows

$$
\begin{aligned}
|y_{N+1}(x) - y_N(x)| &\le \int_{x_0}^{x} \frac{\phi'(t)}{\phi(t)} |y_N(t) - y_{N-1}(t)| dt \\
&\le \int_{x_0}^{x} C\phi'(t)\phi^{\delta-1}(t)dt \\
&= \frac{C}{\delta}\phi^\delta(x).
\end{aligned}
$$

Repeating this process i times, we get

$$
|y_{N+i}(x) - y_{N+i-1}(x)| \le \frac{C}{\delta^i}\phi^\delta(x) \quad \text{in } x_0 \le x \le x_0 + h.
$$

Therefore we arrive at the desired result.

Let $y(x)$ be the limit function of a sequence $\{y_n(x)\}$. As $f(x, y)$ is continuous in the closed rectangle $\bar{S}_+$, this function is even uniformly continuous in $\bar{S}_+$. Hence $f(x, y_n(x))$ tends uniformly in x to $f(x, y(x))$. Therefore if n tends to infinity in (14), we have

$$
y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))dt \qquad \text{in } x_0 \le x \le x_0 + h
$$

which implies the existence of the solution of (1) in the interval $x_0 \le x \le x_0 + h$.

II) The uniqueness

Let $y(x)$ and $\bar{y}(x)$ be two solutions of (1) in $x_0 \le x \le x_0 + h$. Then,

$$
|y(x) - \bar{y}(x)| \le \int_{x_0}^{x} |f(t, y(t)) - f(t, \bar{y}(t))| dt \le 2M(x - x_0).
$$

As in the proof of the existence, it follows

$$\begin{aligned}|y(x)-\bar{y}(x)| &\le T^n(|y(x)-\bar{y}(x)|)\\ &\le T^n(2M(x-x_0))\\ &\le C\phi^\delta(x) \qquad\qquad \text{in } x_0 \le x \le x_0+h.\end{aligned}$$

Thus

$$|y(x)-\bar{y}(x)| \le \int_{x_0}^{x} \frac{\phi'(t)}{\phi(t)}|y(t)-\bar{y}(t)|dt \le \frac{C}{\delta}\phi^\delta \quad \text{in } x_0 \le x \le x_0+h.$$

By iteration of this process, we get

$$|y(x)-\bar{y}(x)| \le \frac{C}{\delta^i}\phi^\delta \ \rightarrow 0 \ as\ i \rightarrow \infty.$$

So $|y(x)-\bar{y}(x)| \equiv 0$ and hence $y(x) \equiv \bar{y}(x)$ in $x_0 \le x \le x_0+h$. ■

Remark. In Theorem 3 if we require only the existence result, the assumption on the operator T^n can be replaced by the following;
there exist $\delta > 1$, $C \ge 0$ and a natural number n such that

$$T^n\left(\left|\int_{x_0}^{x} f(t,y_0)dt\right|\right) \le C\phi^\delta(x) \quad \text{in } x_0 \le x \le x_0+a.$$

Corollary 3. O.Kooi's statement is the particular case of our Theorem 3.

Proof. Kooi [4] showed the existence and uniqueness theorem under the assumptions that functions $\phi(x)=(x-x_0)^k$, $\lambda(x)=\frac{c}{(x-x_0)^\beta}$ and $g(z)=z^\alpha$ with $k>0$, $c>0$, $0<\alpha<1$, $\beta<\alpha$ and $k<\frac{1-\beta}{1-\alpha}$. We can easily see that his conditions satisfy all the hypotheses of Theorem 3. Really it holds

$$T^n(L(x-x_0)) = C_n(x-x_0)^{(\alpha-\beta)\frac{1-\alpha^n}{1-\alpha}+1}$$

where C_n is nonnegative and independent of x. In view of

$$\lim_{n\to\infty}\left((\alpha-\beta)\frac{1-\alpha^n}{1-\alpha}+1\right) = \frac{1-\beta}{1-\alpha} > k.$$

There exists a natural N such that $(\alpha-\beta)\frac{1-\alpha^N}{1-\alpha}+1>k$. Hence we can fined $\delta>1$ and N such that $T^N(L(x-x_0))=C_N(x-x_0)^{k\delta}=C\phi^\delta(x)$ in $[x_0,x_0+h]$. ■

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Received August 29, 1995