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Time-Dependent Nonlinear Perturbations of Integrated Semigroups

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0. Introduction

The present paper is concerned with nonlinear evolution operators which provide weak solutions to the time-dependent semilinear problems of the form

$$\text{(SP)} \qquad \frac{d}{dt}u(t) = (A + B(t))u(t), \ s < t < \tau; \ u(s) = v, \ 0 \leq s < \tau$$

in a real Banach space $(X, |\cdot|)$. Here A is assumed to be the generator of an integrated semigroup $\{W(t) : t \geq 0\}$ in X and $B(t)$, $0 \leq t \leq \tau$, are nonlinear operators from convex subsets $D(t)$, $0 \leq t \leq \tau$, of X into X, respectively.

The importance of semilinear problems of the type (SP) has constantly been recognized in the studies of mathematical models arising in various fields of mathematical sciences. In this paper we introduce two general classes of time-dependent nonlinear perturbations of linear integrated semigroups and discuss necessary and sufficient conditions on $A + B(t)$ for the integral solutions of (SP) to exist in a global sense. As in the case of time-independent perturbations, it seems to be most natural to deal with mild solutions to (SP). However, uniqueness of mild solutions is not necessarily valid in the case that the perturbing operator $B(t)$ depends upon t and the semilinear operators $A + B(t)$ are only quasidissipative in X in a local sense. Accordingly, we here employ the notion of integral solution in the sense of Bénilan [3] for the semilinear problem (SP) which is stronger than that of mild solution.

The notion of integrated semigroup is a natural extension of the notion of semigroup of class (C_0) and the generation theory is applied to a variety of linear evolution problems. On the other hand, there has been a substantial development in the theory of nonlinear evolution operators associated with semilinear problems of the form (SP) in which A is assumed to be the infinitesimal generator of a (C_0)-semigroup in X. We here focus our attention to a nonlinear perturbation theory for (C_0)-semigroups which was treated by Oharu and Takahashi [19]. In their treatise various types of characterizations of nonlinear semigroups providing weak solutions of autonomous semilinear problems (SP) with $B(t) \equiv B$ have been obtained in terms of the corresponding semilinear infinitesimal generators.

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Their arguments contain three features : Firstly, a lower semicontinuous functional $\varphi : X \to [0,\infty]$ is employed to define local quasidissipativity of $A+B$ and the growth of solutions to (SP) is restricted in terms of the nonnegative function $\varphi(u(\cdot))$. In case of concrete partial differential equations, the use of such functionals corresponds to a priori estimates or energy estimates which ensure the global existence of the solutions as well as their asymptotic properties. Secondly, the semilinear operators $A+B$ is assumed to be quasidissipative on φ-bounded sets. Thirdly, $A+B$ is assumed to satisfy the so-called implicit subtangential condition or explicit subtangential condition. Because of the simplicity and universality of those conditions the perturbation theory is applied to various evolution problems. Therefore, it is important from both theoretical and practical points of view to discuss semilinear problems (SP) in terms of nonlinear perturbations of linear integrated semigroups, and a time-independent nonlinear perturbation theory for integrated semigroups has been established in the recent work Matsumoto *et al.* [17].

However, it is much more useful to extend the nonlinear perturbation theory for linear integrated semigroups to the case of time-dependent nonlinear perturbations. In fact, a variety of semilinear evolution problems are treated in this framework and a broad class of quasilinear evolution equations may be studied via the theory presented in this paper. Therefore it is important from both theoretical and practical points of view to discuss time-dependent semilinear problems (SP) in terms of nonlinear perturbations of integrated semigroups. Here we consider a class of integrated semigroups treated in Kellermann and Hieber [10] and investigate nonlinear perturbations of such integrated semigroups in the same spirit of the work [19]. Locally Lipschitz perturbations of such integrated semigroups have been studied in Thieme [21]. Our results are affected by those works. We interpret the above-mentioned problem as a characterization problem of a nonlinear evolution operator which provides the integral solutions of (SP). In fact, our principal results may be described as follows: Under a subtangential condition for the semilinear operators $A+B(t)$ a nonlinear evolution operator $\mathcal{U} \equiv \{U(t,s)\}$ can be constructed in such a way that $\mathcal{U}$ provides unique integral solutions in the sense of Bénilan and hence mild solutions to (SP). Conversely, it follows from the existence of a nonlinear evolution operator $\mathcal{U}$ providing mild solutions to (SP) and an appropriate convexity condition on the domain of $\mathcal{U}$ that the semilinear operators $A+B(t)$ satisfy the subtangential condition.

This paper is organized as follows: In Section 1 a class of integrated semigroups treated in [10] by Kellermann and Hieber is introduced and the basic results are outlined so that they may be directly applied to our nonlinear problems. In Section 2, two generalized notions of solutions to (SP) and a notion of nonlinear evolution operators associated with semilinear problems (SP) are discussed. Furthermore, a natural notion of semilinear infinitesimal generator is introduced for such nonlinear evolution operators. Section 3 contains our first main result (Theorem 3.1) which gives a generation theorem of nonlinear evolution operators providing weak solutions to (SP) under the implicit subtangential conditions. In Iwamiya [9], a time-dependent nonlinear perturbation theorem is given for (C_0)-semigroups. Section 4 discusses an extension of his result to the case of integrated semigroups. This extension is applied to obtain our second main result (Theorem 4.3) which gives a characterization of nonlinear evolution operators providing weak solutions to (SP) under the explicit subtangential conditions. Finally, Section 5 is devoted to the application of the first main theorem to a certain reaction-diffusion system.

1. A Class of Integrated Semigroups

In this section we introduce a class of integrated semigroups and state some basic facts on such integrated semigroups. In what follows, $(X^*, |\cdot|)$ denotes the dual space of X. For $x \in X$ and $f \in X^*$ the value of f at x is written as $\langle x, f\rangle$. The duality mapping of X is denoted by $\mathcal{F}$. For $x, y \in X$ the symbols $\langle x, y\rangle_i$ and $\langle x, y\rangle_s$ stand for the infimum and the supremum of the set $\{\langle s, f\rangle : f \in \mathcal{F}(y)\}$, respectively. In (SP) the operator A is assumed to be a closed linear operator in X whose domain is not necessarily dense in X. We write A^* for the dual operator of A. If A is densely defined in X, then A^* is defined as a closed linear operator in X^*. If the domain $D(A)$ is not dense in X, then A^* is multi-valued and the identity $\langle Ax, f\rangle = \langle x, g\rangle$ holds for $x \in D(A)$, $f \in D(A^*)$ and $g \in A^*f$; hence the value $\langle x, g\rangle$ does not depend upon the choice of $g \in A^*f$. It follows that $\langle x, g\rangle$ does not depend upon the choice of $g \in A^*f$ provided that $x \in Y$, where Y is the closure of $D(A)$:

$$Y \equiv \overline{D(A)}.$$

Hence for $y \in Y$ and $f \in D(A^*)$ it is justified and convenient to introduce the notation

$$\langle y, A^*f\rangle \equiv \langle y, g\rangle, \quad g \in A^*f.$$

In concrete cases it is not straightforward in general to characterize the set-valued operator A^*. In order to overcome this difficulty, we here introduce the notion of A-total set which is an essential part of $D(A^*)$ and makes it easier to formulate the notion of weak solution to the problem (SP).

A subset D of $D(A^*)$ is said to be A-total, if for λ sufficiently small the set

$$(I - \lambda A^*)D \equiv \{f - \lambda g : f \in D, g \in A^*f\}$$

separates points in Y in the sense that $x \in Y$ equals 0 whenever $\langle x, h\rangle = 0$ for $h \in (I - \lambda A^*)D$. In the case that a complex number λ is contained in the resolvent set $\rho(A)$, an A-total set can be easily constructed : Take any subset Z of X^* which separates points in Y. Then $D = [(I - \lambda A)^{-1}]^*(Z)$ is A-total. The point of this concept consists in the fact that one can equivalently replace $D(A^*)$ by an A-total subset of $D(A^*)$ in the notion of weak solution to (SP) which is discussed later. In many applications it will be possible to choose an A-total subset of $D(A^*)$ on which A^* has a useful representation while this is not the case on the whole of $D(A^*)$.

To describe the notion of once integrated semigroup in conjunction with linear evolution equations in X, we consider the initial-value problem

$$\text{(IVP)}_\psi \qquad u'(t) = Au(t) + \psi(t), \quad t > 0; \quad u(0) = v \in X,$$

where A is a closed linear operator in X, $\psi(\cdot) \in C((0,\infty); X) \cap L^1_{\ell oc}(0,\infty; X)$ and v is an initial-value given in X. For the problem (IVP)_ψ we may employ the following three kinds of notions of generalized solution. The first notion was first employed in Da Prato and Sinestrari [6].

DEFINITION 1.1. An X-valued function $u(\cdot)$ on $[0,\infty)$ is said to be an *integrated solution* of (IVP)_ψ, if $u(\cdot) \in C((0,\infty); X) \cap L^1_{\ell oc}(0,\infty; X)$, $\int_0^t u(s)ds \in D(A)$ and

$$u(t) = v + A\int_0^t u(s)ds + \int_0^t \psi(s)ds \qquad \text{for } t \geq 0.$$

Moreover, a function $u(\cdot) \in C([0,\infty);X)$ is called a C^1-solution of $(\mathrm{IVP})_\psi$, if $u(\cdot) \in C^1((0,\infty);X)$, $u'(t) = Au(t) + \psi(t)$ for $t > 0$, and $u(0) = v$.

The following notion of solution in a generalized sense to $(\mathrm{IVP})_\psi$ is due to J. Ball [2].

DEFINITION 1.2. A function $u(\cdot) \in C([0,\infty);X)$ is said to be a *weak solution* of $(\mathrm{IVP})_\psi$, if $u(0) = v$ and for each $f \in D(A^*)$ the scalar-valued function $\langle u(\cdot), f\rangle$ is of class $C^1(0,\infty)$ and satisfies

$$\frac{d}{dt}\langle u(t), f\rangle = \langle u(t), A^*f\rangle + \langle \psi(t), f\rangle \qquad \text{for } t > 0.$$

The third notion of generalized solution is similar to that of weak solution but the last relation being restricted to f in an A-total subset D of $D(A^*)$. As seen from Theorem 1.7 below, these two notions are equivalent; hence we use the same terminology of weak solution.

If $u(\cdot)$ is an integrated solution of $(\mathrm{IVP})_\psi$ and is continuous at $t = 0$, then $u(t) = \lim_{h\downarrow 0} h^{-1}\int_t^{t+h} u(s)ds \in \overline{D(A)}$ for $t \geq 0$ and so $v = u(0) \in \overline{D(A)}$. Also, $u(\cdot)$ is an integrated solution of $(\mathrm{IVP})_\psi$ if and only if the indefinite integrated $w(t) = \int_0^t u(s)ds$ gives a C^1-solution of the initial-value problem

$$w'(t) = Aw(t) + v + \int_0^t \psi(s)ds, \quad t > 0; \quad w(0) = 0. \tag{1.1}$$

Let $\psi(t) \equiv 0$ in (1.1) and suppose that for each initial-value $v \in X$ there exists a unique integrated solution $u(\cdot; v)$ of $(\mathrm{IVP})_\psi$ with $\psi(t) \equiv 0$. Now for each $t \geq 0$ we define an operator $W(t)$ on X by

$$W(t)v = \int_0^t u(r;v)dr \qquad \text{for } v \in X.$$

Since (1.1) is an autonomous linear equation, it follows that each $W(t)$ is linear. Therefore we get a one-parameter family $\mathcal{W} \equiv \{W(t)\}$ of linear operators with the properties below:

$$W(0)v = 0 \quad \text{and} \quad W(\cdot)v \in C([0,\infty);X) \qquad \text{for } v \in X. \tag{w.1}$$

Since $w(\cdot) \equiv W(\cdot)v$ is a unique C^1-solution of (1.1) with $\psi(t) \equiv 0$, we have the relation

$$W(t)v = A\int_0^t W(r)vdr + tv \qquad \text{for } t \geq 0 \text{ and } v \in X. \tag{1.2}$$

Let $t \geq 0$ and $v \in X$. Then by assumptions on the existence and uniqueness of integrated solutions of $(\mathrm{IVP})_\psi$ with $\psi(t) \equiv 0$ the function $W(\cdot)W(t)v$ is a unique solution of (1.1) with $\psi(t) \equiv 0$ and v replaced by $W(t)v$. We next consider the function

$$z(s) = \int_0^s [W(r+t)v - W(r)v]dr$$

defined on all of $[0, \infty)$. Then $z(0) = 0$ and (1.2) implies

$$\begin{aligned} z'(s) &= W(s+t)v - W(s)v \\ &= A\int_t^{t+s} W(r)vdr + A\int_0^t W(r)vdr + tv - A\int_0^s W(r)vdr \\ &= A\int_0^s [W(r+t)v - W(r)v]dr + W(t)v = Az(s) + W(t)v \end{aligned}$$

for $s > 0$. This shows that $z(\cdot)$ is a C^1-solution of (1.1) with $\psi(t) \equiv 0$ and v replaced by $W(t)v$. From this we see that $\mathcal{W}$ has the following property:

$$(w.2) \qquad W(s)W(t)v = \int_0^s [W(r+t)v - W(r)v]dr \qquad \text{for } s,\ t \geq 0 \text{ and } v \in X.$$

The above observation leads us to the following

DEFINITION 1.3. A one-parameter family $\mathcal{W} \equiv \{W(t) : t \geq 0\}$ of bounded linear operators is said to be a *once integrated semigroup* on X if it has properties $(w.1)$ and $(w.2)$ mentioned above. We say that $\mathcal{W}$ is *non-degenerate* if $W(t)v \equiv 0$ for all $t > 0$ implies $v = 0$. If there exist constants $M > 1$ and $\omega \geq 0$ such that $|W(t)| \leq Me^{\omega t}$ for $t \geq 0$, the once integrated semigroup $\mathcal{W}$ is said to be *exponentially bounded.*

In what follows, by an integrated semigroup is meant a non-degenerate exponentially bounded once integrated semigroup, unless otherwise stated.

As shown in Thieme [20, Section 3], given a once integrated semigroup $\mathcal{W} \equiv \{W(t)\}$ there exists a closed linear operator A in X such that $v \in D(A)$ and $w = Av$ are characterized by the property that the function $W(t)v$ is continuously differentiable and

$$\frac{d}{dt}W(t)v = v + W(t)w \qquad \text{for } t > 0.$$

The operator A is called the *generator* of $\mathcal{W}$ and an integrated semigroup is uniquely determined by its generator. It is also shown in [20, Section 3] that (1.2) holds for integrated semigroups and their generators. These facts may be summarized as follows:

PROPOSITION 1.4. *Let $\mathcal{W}$ be an integrated semigroup and A the generator of $\mathcal{W}$. Then $\mathcal{W}$ gives integrated solutions to $(IVP)_\psi$ with $\psi(t) \equiv 0$ in the sense that*

(a) $W(t)v \in D(A)$ *and* $AW(t)v = W(t)Av$ *for* $v \in D(A)$ *and* $t \geq 0$,

(b) $\int_0^t W(t)vdt \in D(A)$ *and* $W(t)v = A\int_0^t W(r)vdr + tv$ *for* $v \in X$ *and* $t \geq 0$.

As shown in [20, Section 3], it is seen that if $\mathcal{W}$ is exponentially bounded then the integral

$$(1.3) \qquad R(\xi)v = \xi\int_0^\infty e^{-\xi t}W(t)vdt$$

exists for $\xi > \omega$ and $v \in X$, and that $\xi R(\xi) = (I - \lambda A)^{-1}$ for $\xi = 1/\lambda > \omega$.

In the subsequent discussions, we are mainly concerned with closed linear operator A in X satisfying the following basic condition:

(H1) There is a constant $\omega \in \boldsymbol{R}$ such that for $\lambda > 0$ with $\lambda\omega < 1$ the resolvent $(I-\lambda A)^{-1}$ exists and satisfies

$$|(I-\lambda A)^{-1}| \leq (1-\lambda\omega)^{-1}.$$

The class of integrated semigroups treated in this paper is that of once integrated semigroups whose generators satisfy the above-mentioned condition (H1). From condition (H1) we see that the part A_Y of A in the smaller Banach space $Y \equiv \overline{D(A)}$ has a dense domain in Y and generates a semigroup $\mathcal{T}_Y \equiv \{T_Y(t) : t \geq 0\}$ of class (C_0) on Y by the Hille-Yosida theorem. Also, it has been shown in [11] that a closed linear operator A satisfying (H1) is the generator of an integrated semigroup $\mathcal{W} \equiv \{W(t) : t \geq 0\}$ on X. The following result is a consequence of the results established in Arendt [1], [10] and [21] and gives a characterization of the class of integrated semigroups under consideration.

THEOREM 1.5. *A closed linear operator A in X is the generator of a once integrated semigroup $\mathcal{W}$ on X such that*

$$|W(t+h) - W(t)| \leq \int_t^{t+h} e^{\omega s} ds \qquad \text{for } t,\ h \geq 0 \tag{1.4}$$

if and only if it satisfies condition (H1).

At the beginning of this section we observed that an integrated semigroup was derived from indefinite integrals of integrated solutions of (IVP)$_\psi$ with $\psi(t) \equiv 0$. One obtains the following structure theorem for integrated semigroups which illustrates this observation and eventually implies the above characterization theorem.

THEOREM 1.6. (Structure Theorem) *Let A be a closed linear operator in X satisfying* (H1) *and $\mathcal{T}_Y$ the semigroup of class (C_0) on $Y \equiv \overline{D(A)}$ generated by the part A_Y of A in Y. Then the integrated semigroup $\mathcal{W}$ generated by A is represented as*

$$W(t)v = \lim_{\lambda \downarrow 0} \int_0^t T_Y(s)(I-\lambda A)^{-1} v ds \quad \text{for } t \geq 0 \text{ and } v \in X. \tag{1.5}$$

PROOF. Let A be a closed linear operator in X. First assume that (H1) holds for A. Then, by [1, Theorem 4.1], A generates a once integrated semigroup $\mathcal{W}$ such that

$$\limsup_{h \downarrow 0} h^{-1}|W(t+h) - W(t)| \leq e^{\omega t} \qquad \text{for } t \geq 0.$$

Moreover, it is seen from [1, Proposition 3.3] that $\mathcal{W}$ satisfies (1.2). Let $\mathcal{T}_Y$ be the semigroup of class (C_0) on Y generated by A_Y. Taking any $\lambda > 0$ with $\lambda\omega < 1$ and applying the resolvent $(I-\lambda A)^{-1}$ to both sides of (1.2), we have

$$(I-\lambda A)^{-1}W(t)v = A_Y(I-\lambda A)^{-1}\int_0^t W(s)vds + t(I-\lambda A)^{-1}v$$

for $t > 0$ and $v \in X$. Differentiating both sides with respect to t, we have

$$\frac{d}{dt}(I - \lambda A)^{-1}W(t)v = A_Y(I - \lambda A)^{-1}W(t)v + (I - \lambda A)^{-1}v.$$

Let $v \in X$. Since $W(0)v = 0$, this implies that the function $W(\cdot)v$ satisfies the variation of constants formula

$$(I - \lambda A)^{-1}W(t)v = \int_0^t T_Y(s)(I - \lambda A)^{-1}vds \quad \text{for } t \geq 0.$$

Since $W(t)v \in Y$ for $t \geq 0$ by Proposition 1.4 (b), we obtain Formula (1.5) by letting $\lambda \downarrow 0$ in the above identity. Therefore the assertion of the theorem is obtained. ∎

The above structure theorem is contained in Thieme [21] as a characteristic property of locally Lipschitz once integrated semigroups. We also refer to [17]. For other representations see Lumer [14]. In [15, Theorem 3.5], Lumer emphasizes the significant role of locally Lipschitz once integrated semigroups among (multiple) integrated semigroups. The following simple consequence suggests the terminology of *integrated* semigroup.

We now return to the inhomogeneous initial-value problem $(\mathrm{IVP})_\psi$ which is the starting point of our considerations. That $(\mathrm{IVP})_\psi$ has a unique integrated solution was shown by Da Prato and Sinestrari [6] and Bénilan *et al.* [4]. There are explicit formulas for the weak solutions and several alternative characterizations. The following results illustrate the main points of these discussions.

THEOREM 1.7. *Let $\tau \in [0, \infty]$. Assume that $\psi : [0, \tau) \to X$ and $u : [0, \tau) \to Y$ be continuous. Let $v \in Y$. Then the following statements are equivalent:*

(i) *The function u is a weak solution of $(\mathrm{IVP})_\psi$ in the sense that $u(0) = v$, $\langle u(t), f\rangle$ is continuously differentiable on $[0, \tau)$ and*

$$\frac{d}{dt}\langle u(t), f\rangle = \langle u(t), A^*f\rangle + \langle \psi(t), f\rangle \quad \textit{for all } 0 \leq t < \tau.$$

(ii) *The function u is a weak solution of $(\mathrm{IVP})_\psi$ in the same sense as in (i), but with f being restricted to elements in an A-total subset of $D(A^*)$.*

(iii) *The function u is an integrated solution of $(\mathrm{IVP})_\psi$ in the sense that $\int_0^t u(s)ds \in D(A)$ and*

$$u(t) = v + A\int_0^t u(s)ds + \int_0^t \psi(s)ds, \quad 0 \leq t < \tau.$$

(iv) $u(0) = v$ *and*

$$\lim_{h \downarrow 0} h^{-1}\big(u(t+h) - T_Y(t)u(t) - W(h)\psi(t)\big) = 0, \quad 0 < t < \tau.$$

(v)
$$u(t) = T_Y(t)v + \int_0^t W(dr)\psi(t - r), \quad 0 \leq t < r.$$

(vi) $$u(t) = T_Y(t)v + \lim_{\lambda \downarrow 0} \int_0^t T_Y(r)(I - \lambda A)^{-1}\psi(t-r)dr, \quad 0 \le t < \tau.$$

(vii) $$u(t) = T_Y(t)v + \frac{d}{dt}\int_0^t W(r)\psi(t-r)dr, \quad 0 \le t < \tau$$

(viii) $$u(t) = T_Y(t)v + A\int_0^t W(r)\psi(t-r)dr + \int_0^t \psi(r)dr,, \quad 0 \le t < \tau$$

Moreover the expressions in formulas (v) *through* (viii) *are well-defined and, simultaneously, provide unique solutions as mentioned in* (i) *through* (iv).

For the proof of this theorem we refer to [17, Theorem 1.9].

2. Nonlinear Evolution Operators Associated with (SP)

In this section we first introduce a class of semilinear operators in X and the associated semilinear problems of the form (SP). We then discuss generalized solutions to the semilinear problem (SP) and consider a general class of nonlinear evolution operators providing such generalized solutions to (SP). Let A be a closed linear operator in X and write $Y = \overline{D(A)}$. Let $B(t)$, $0 \le t \le \tau$, be possibly nonlinear operators in X which are defined on convex subsets $D(t)$, $0 \le t \le \tau$, of the closed linear subspace Y, respectively. If $D(A) \cap D(t) \neq \phi$, then the sum $A + B(t)$ defines a semilinear operator in X with domain $D(A + B(t)) = D(A) \cap D(t)$. Throughout this paper we call it a semilinear operator in X determined by A and $B(t)$.

To restrict the time dependence of the family of nonlinear operators $B(t)$, we introduce the following family of nonnegative functions defined on all of $[0,\tau]^2$. By $\mathcal{F}$ is meant the set of all $\theta \in C([0,\tau]^2)$ such that for $0 \le s \le t \le \tau$

(2.1) $$\theta(s,t) = \theta(t,s) \quad \text{and} \quad \theta(s,s) = 0.$$

Moreover in order to restrict the continuity and quasi-dissipativity in a local sense of $B(t)$, we employ a lower semicontinuous convex functional $\varphi : X \to [0,\infty]$ such that $D(t) \subset D(\varphi) \equiv \{v \in X : \varphi(v) < \infty\}$ for $0 \le t \le \tau$.

For a family $\{\theta_\alpha : \alpha > 0\}$ of functions in $\mathcal{F}$, a family $\{D(t) : 0 \le t \le \tau\}$ of convex subsets of Y and a functional φ as mentioned above, we introduce a class of semilinear operators $A + B(t)$ with which we are concerned in this paper. Let

(2.2) $$\mathcal{D} = \bigcup_{0 \le s \le \tau} (\{s\} \times D(s)).$$

The set $\mathcal{D}$ forms a possibly noncylindrical domain in the product space $[0,\tau] \times X$.

DEFINITION 2.1. A one-parameter family $\{A + B(t)\}$ of semilinear operators is said to belong to the class $\mathfrak{U}(\mathcal{D},\varphi)$, if A is a closed linear operator satisfying (H1) and $B(t)$, $0 \le t \le \tau$, are possibly nonlinear operators satisfying conditions (H2) and (H3) below:

(H2) For each $\alpha > 0$ and each $t \in [0, \tau]$, the level set $D_\alpha(t) \equiv \{v \in D(t) \ : \ \varphi(v) \leq \alpha\}$ is closed in X and the operator $(t, v) \to B(t)v$ is continuous in (t, v) on $\mathcal{D}_\alpha \equiv \{(s, v) \in \mathcal{D} : \varphi(v) \leq \alpha\}$.

(H3) For each $\alpha > 0$ and each $t \in [0, \tau]$, the semilinear operator $A + B(t)$ is locally quasi-dissipative in the sense that

$$\langle (A + B(s))v - (A + B(t))w,\ v - w\rangle_i \leq \omega_\alpha |v - w|^2 + \theta_\alpha(s,t)|v - w|$$

for $v \in D(A) \cap D_\alpha(s)$, $w \in D(A) \cap D_\alpha(t)$, some constant $\omega_\alpha \in \boldsymbol{R}$ and some $\theta_\alpha \in \mathcal{F}$.

The continuity condition (H2) on the level sets $\{D_\alpha(t) : \alpha > 0\}$ is much weaker than the continuity on the whole domain $D(t)$ in general and considerably useful for the application to partial differential equations. In condition (H3) the intersections $D(A) \cap D_\alpha(t)$ may be empty; condition (H3) states that the quasi-dissipativity of $A + B(t)$ on $D_\alpha(t)$ is assumed whenever $D(A) \cap D_\alpha(t) \neq \phi$. Given a one-parameter family $\{A+B(t)\}$ of semilinear operators belonging to the class $\mathfrak{U}(\mathcal{D}, \varphi)$ we consider the initial-value problem for the semilinear evolution equation in X

$$\text{(SE)} \qquad \frac{d}{dt}u(t) = (A + B(t))u(t), \qquad s < t < \tau.$$

DEFINITION 2.2. Let $\mathcal{D}$ be the domain defined by (2.2). Let $s \in [0, \tau)$ and $v \in D(s)$. A strongly continuous function $u(\cdot) : [s, \tau) \to X$ is said to be a *$\mathcal{D}$-valued weak solution* of (SE) on $[s, \tau)$ with initial-value v, if $u(s) = v$, $u(t) \in D(t)$ for $t \in [s, \tau]$, $B(\cdot)u(\cdot) \in C([s, \tau]; X)$, and for each $f \in D(A^*)$ the scalar-valued function $\langle u(\cdot), f\rangle$ is continuously differentiable over $[s, \tau)$ and satisfies

$$\text{(2.3)} \qquad \frac{d}{dt}\langle u(t), f\rangle = \langle u(t), A^* f\rangle + \langle B(t)u(t), f\rangle \qquad \text{for } t \in (s, \tau).$$

In concrete cases the notion of weak solution to (SE) as defined above may be rather cumbersome owing to the difficulty in characterizing the set-valued operator A^*. It follows from Theorem 1.7 that the set $D(A^*)$ can be replaced by A- total subsets.

Because of the localized quasi-dissipativity condition (H3), the semilinear problem (SP) may admit only local weak solutions. Hence it is necessary to restrict the growth of the weak solutions in order to discuss the weak solutions of (SP) on $[s, \tau]$. In this paper we employ a typical growth condition in terms of the real-valued function $\varphi(u(\cdot))$, namely,

$$\text{(EG)} \qquad \varphi(u(t)) \leq e^{a(t-s)}(\varphi(v) + b(t - s)), \qquad t \in [s, \tau],$$

where a and b are constants. This type of growth condition may be called the *exponential growth condition.*

DEFINITION 2.3. Let $\mathcal{D}$ be the domain defined by (2.2). Let $s \in [0, \tau)$ and $v \in D(s)$. A strongly continuous function $u(\cdot) : [s, \tau) \to X$ satisfying (EG) is said to be an *integral solution* in the sense of Bénilan of (SE)-(EG) on $[s, \tau)$ with initial-value v, if it satisfies the following:

$$u(s) = v, \;\; u(t) \in D(t) \quad \text{for } t \in [s, \tau],$$

$$|u(t+h) - z|^2 - |u(t) - z|^2 \le 2 \int_t^{t+h} \Big[\;\langle (A + B(r))z, u(\xi) - z \rangle_s + \omega_\alpha |u(\xi) - z|^2 + \theta_\alpha(\xi, r) |u(\xi) - z| \Big] \, d\xi$$

for $s \le t \le t + h \le \tau$, $r \in [0, \tau]$ and $z \in D(A) \cap D_\alpha(r)$, where $\alpha = e^{a\tau}[\varphi(v) + b\tau]$.

REMARK 2.4. In a way similar to [19, Proposition 7.5], one can show that an integral solution in the sense of Bénilan [3] becomes a weak solution satisfying (EG). For integral solutions to (SP) the following uniqueness theorem is valid; see Kobayasi *et al.*[11]:

PROPOSITION 2.5. *Let $\{A + B(t)\}$ be a one-parameter family of semilinear operators of the class $\mathfrak{U}(\mathcal{D}, \varphi)$. Let $s, t \in [0, \tau)$, $v \in D_\alpha(s)$, $w \in D_\alpha(t)$ and let $u(\cdot; v)$, $u(\cdot; w)$ be the associated integral solutions in the sense of Bénilan. Let $\beta \ge e^{a\tau}(\alpha + b\tau)$. Then we have*

$$|u(t; v) - u(t; w)| \le \exp(\omega_\beta t)|v - w| \qquad \text{for } t \in [0, \tau], \tag{2.4}$$

where ω_β is a constant determined for β by condition (H3).

A two-parameter family $\mathcal{U} \equiv \{U(t, s) : 0 \le s \le t \le \tau\}$ of possibly nonlinear operators from $D(s)$ into D(t) is called an *evolution operator constrained in $\mathcal{D}$*, if it has the two properties below:

$$U(r, r)v = v \quad \text{and } U(t, s)U(s, r)v = U(t, r)v$$

for $0 \le r \le s \le t \le \tau$ and $v \in D(r)$.

(E2) For each $s \in [0, \tau)$ and each $v \in D(s)$, $U(\cdot, s)v \in C([s, \tau); X)$.

If in particular a nonlinear evolution operator $\mathcal{U}$ constrained in $\mathcal{D}$ provides weak solutions of (SP) in the sense that for each $s \in [0, \tau)$ and each $v \in D(s)$ the function $u(\cdot; v)$ defined by

$$u(t; v) = U(t, s)v \qquad \text{for } t \in [s, \tau] \tag{2.5}$$

is a $\mathcal{D}$-valued weak solution of (SP) on $[s, \tau]$, then we say that $\mathcal{U}$ is *associated with* the semilinear evolution equation (SE). Under conditions (H1), (H2) and (H3) a family of solution operators in the sense of (2.5) gives rise to a evolution operator $\mathcal{U}$ constrained in $\mathcal{D}$, as stated below.

PROPOSITION 2.6. *Let $\{A + B(t)\}$ be a one-parameter family of semilinear operators belonging to the class $\mathfrak{U}(\mathcal{D}, \varphi)$. Suppose that for each $s \in [0, \tau)$ and each $v \in D(s)$ there is an integral solution $u(\cdot; v)$ of* (SP)-(EG) *on $[s, \tau]$. Then there is an evolution operator $\mathcal{U} \equiv \{U(t, s) : 0 \le s \le t \le \tau\}$ on $\mathcal{D}$ associated with* (SE) *such that for $\alpha > 0$ and $\beta \ge e^{a(\tau - s)}(\alpha + b(\tau - s))$, $BU(\cdot, s) \in C([s, \tau]; X)$ and*

$$|U(t, s)v - U(t, s)w| \le \exp(\omega_\beta(t - s))|v - w| \tag{2.6}$$

for $v,\ w \in D_\alpha(s),\ t \in [s,\tau]$ *and some constant* $\omega_\beta \in \boldsymbol{R}$.

PROOF. Proposition 2.5 states that $u(\cdot;v)$ is a unique integral solution of (SP)-(EG). Also, $B(\cdot)u(\cdot;v) \in \mathcal{C}([s,\tau];X)$ by (EG) and (H2). Therefore one can define for each $s \in [0,\tau)$ and each $t \in [s,\tau]$ an operator $U(t,s)$ from $D(s)$ into D(t) by

$$U(t,s)v = u(t;v) \qquad \text{for } v \in D(s).$$

By the uniqueness of the integral solutions in the sense of Bénilan, the family $\mathcal{U} \equiv \{U(t,s) : 0 \le s \le t \le \tau\}$ of operators so defined have the properties (E1) and (E2). The local equi-Lipschitz continuity (2.6) of the evolution operator $\mathcal{U}$ follows from (2.4). ∎

In what follows, we say that an evolution operator $\mathcal{U}$ constrained in $\mathcal{D}$ is *locally equi-Lipschitz continuous* with respect to φ, if for each $\alpha > 0$ and $s \in [0,\tau)$ there is a number $\omega(\alpha,\tau)$ such that

$$|U(t,s)v - U(t,s)w| \le e^{\omega(\alpha,\tau)(t-s)}|v-w| \qquad \text{for } t \in [s,\tau] \text{ and } v,\ w \in D_\alpha(s).$$

Proposition 2.6 states that if $\{A + B(t)\}$ belong to the class $\mathfrak{U}(\mathcal{D},\varphi)$ then the evolution operator associated with (SE) is necessarily locally equi-Lipschitz continuous with respect to φ. In the remainder of this section we investigate the differentiability of the evolution operator associated with (SE) and then show that such an evolution operator can be characterized in several ways. In view of the characterization theorem below, we introduce a notion of semilinear infinitesimal generators.

THEOREM 2.7. (Differentiability Theorem) *Let A be a closed linear operator in X satisfying condition* (H1), $Y = \overline{D(A)}$, *and let $B(t)$, $0 \le t \le \tau$, be possibly nonlinear operators defined on convex subsets $D(t)$, $0 \le t \le \tau$, of Y, respectively. Let $\mathcal{U} \equiv \{U(t,s) : 0 \le s \le t \le \tau\}$ be an evolution operator constrained in $\mathcal{D}$ such that $B(\cdot)U(\cdot,s)v \in \mathcal{C}([s,\tau];X)$ for each $s \in [0,\tau)$ and each $v \in D(s)$. Then the following statements are equivalent:*

(a) *The evolution operator $\mathcal{U}$ is associated with* (SE) *in the sense that the scalar-valued function $\langle U(\cdot,s)v,f\rangle$ is continuously differentiable over $[s,\tau)$ and*

$$\frac{d}{dt}\langle U(t,s)v,f\rangle = \langle U(t,s)v,A^*f\rangle + \langle B(t)U(t,s)v,f\rangle$$

for $s \in [0,\tau)$, $t \in [s,\tau)$, $v \in D(s)$ and $f \in D(A^*)$.

(b) *The evolution operator $\mathcal{U}$ is associated with* (SE) *in the sense that for some A-total subset D of $D(A^*)$, the scalar-valued function $\langle U(\cdot,s)v,f\rangle$ is continuously differentiable over $[s,\tau)$ and*

$$\frac{d}{dt}\langle U(t,s)v,f\rangle = \langle U(t,s)v,A^*f\rangle + \langle B(t)U(t,s)v,f\rangle$$

for $s \in [0,\tau)$, $t \in [s,\tau)$, $v \in D(s)$, $f \in D$.

(c) *For $s \in [0,\tau)$ and $v \in D(s)$, $\int_s^t U(\xi,s)v d\xi \in D(A)$ and*

$$U(t,s)v = v + A\int_s^t U(\xi,s)vd\xi + \int_s^t B(\xi)U(\xi,s)vd\xi \qquad \text{for } t \in [s,\tau].$$

(d) *For $s \in [0,\tau)$, $v \in D(s)$, $\lim_{h\downarrow 0} h^{-1}[(U(s+h,s)v - T_Y(h)v) - W(h)B(s)v] = 0$.*

(e) *For $s \in [0,\tau)$, $v \in D(s)$ and $t \in [s,\tau]$,*

$$U(t,s)v = T_Y(t-s)v + \int_0^{t-s} W(d\xi)B(t-\xi)U(t-\xi,s)v$$

where the integral is taken in the sense of Stieltjes.

(f) *For $s \in [0,\tau)$, $v \in D(s)$ and $t \in [s,\tau]$,*

$$U(t,s)v = T_Y(t-s)v + \lim_{\lambda\downarrow 0}\int_s^t T_Y(t-\xi)(I-\lambda A)^{-1}B(\xi)U(\xi,s)vd\xi,$$

where the limit is taken with respect to the norm of X.

(g) *For $s \in [0,\tau)$, $v \in D(s)$ and $f \in D(A^*)$*

$$\lim_{h\downarrow 0}\langle h^{-1}(U(s+h,s)v - v), f\rangle = \langle v, A^*f\rangle + \langle B(s)v, f\rangle.$$

(h) *If D is an arbitrarily given A-total subset of $D(A^*)$ and $s \in [0,\tau)$, $v \in D(s)$, $f \in D$, then*

$$\lim_{h\downarrow 0}\langle h^{-1}(U(s+h,s)v - v), f\rangle = \langle v, A^*f\rangle + \langle B(s)v, f\rangle.$$

PROOF. Set $u(t) = U(t,s)v$ and $\psi(t) = B(t)U(t,s)v$. The equivalence of (a) through (f) then follows from Theorem 1.7. Clearly, (a) implies (g) and (g) implies (h). It now remains to show that (b) follows from (h). This implication can be derived from the evolution property (E1) in the following way: Suppose that (h) holds. Let $s \in [0,\tau)$, $v \in D(s)$ and $f \in D$ and let D be an A-total subset of $D(A^*)$. Using the evolution property of $\mathcal{U}$, we get

$$\frac{d^+}{dt}\langle U(t,s)v, f\rangle = \langle U(t,s)v, A^*f\rangle + \langle B(t)U(t,s)v, f\rangle \qquad \text{for } t \in (s,\tau),$$

where the left-hand side denotes the right-hand derivative of the function $\langle U(\cdot,s)v, f\rangle$. But the right-hand side of the above relation is continuous in $t \in (s,\tau)$, and so $\langle U(\cdot,s)v, f\rangle$ is of class $C^1[s,\tau]$. This shows that (b) holds. The proof is now complete. ∎

The above theorem is a straightforward extension of [19, Theorem 3.1] to the case where A need not be densely defined and $B(t)$ depends upon t. The equivalence of conditions (a), (c) and (g) is discussed in [4] in the context that A is a *multi-valued* linear operator. The formula given in condition (f) may be regarded as a straightforward extension of the so-called variation of constants formula for (SE). In fact, if A is densely defined, (f) is equivalent to the statement that for $s \in [0,\tau)$ and $v \in D(s)$, $u(t) \equiv U(t,s)v$ becomes a usual mild solution.

In the case that $D(A)$ is dense in X, $Y \equiv X$ and $\{T_X(t) : t \geq 0\}$ is a (C_0)-semigroup on X, it follows that for any $s \in [0,\tau)$ and any $v \in D(s)$ the formula

$$\lim_{h \downarrow 0} h^{-1}(U(s+h,s)v - T_X(h)v) = B(s)v \tag{2.7}$$

is valid, which means that $A + B(s)$ are the full infinitesimal generators of $\mathcal{U}$ in the sense of [19, Definition 3.1]. However, the statement (d) does not seem to be appropriate for defining semilinear infinitesimal generators of the evolution operator $\mathcal{U}$ because of the terms $h^{-1}W(h)B(s)v$, $h > 0$. We here employ the statement (g) to introduce a notion of semilinear infinitesimal generator at s of $\mathcal{U}$.

DEFINITION 2.8. Let $\mathcal{U} \equiv \{U(t,s) : 0 \leq s \leq t \leq \tau\}$ by an evolution operator constrained in $\mathcal{D}$ such that $B(\cdot)U(\cdot,s)v \in \mathcal{C}([s,\tau];X)$ for $s \in [0,\tau)$ and $v \in D(s)$. Then $A + B(s)$ are said to be the *full infinitesimal generator at* s of $\mathcal{U}$, if

$$\lim_{h \downarrow 0} \langle h^{-1}(U(s+h,s)v - v), f\rangle = \langle v, A^* f\rangle + \langle B(s)v, f\rangle \tag{2.8}$$

for $s \in [0,\tau)$, $v \in D(s)$ and $f \in D(A^*)$.

It should be noted that (2.8) holds on all of $D(s)$ and makes sense even if $D(A+B(s)) = D(A) \cap D(s) = \phi$. This fact motivates the terminology of *full* infinitesimal generators. Formula (2.8) may be interpreted as follows: The vector field generated by $A + B(s)$ is tangential in a weak sense to the continuous curve $U(\cdot,s)v$ in X for any $s \in [0,\tau)$ and $v \in D(s)$.

3. Generation under Implicit Subtangential Conditions

In this section a general sufficient condition is given for a one-parameter family of semilinear operators $\{A + B(t)\}$ in the class $\mathfrak{U}(\mathcal{D},\varphi)$ to generate an evolution operator $\mathcal{U}$ constrained in $\mathcal{D}$ associated with (SE) and satisfying the growth condition (EG). We then demonstrate that under certain conditions on the domain $\mathcal{D}$ under consideration, necessary and sufficient conditions can be obtained for a given family $\{A + B(t)\}$ of semilinear operators to be the family of semilinear infinitesimal generators of $\mathcal{U}$. Now the generation theorem is stated as follows:

THEOREM 3.1. (GENERATION THEOREM) *Let $\{A + B(t)\}$ be a one-parameter family of semilinear operators belonging to the class $\mathfrak{U}(\mathcal{D},\varphi)$. Suppose that the domain $\mathcal{D}$ has the following closure property:*

(H4) *If $w_n \in D_R(t_n)$ for $n = 1, 2, \cdots$ and some $R > 0$, $t_n \uparrow t$ in $[s,\tau]$ and $w_n \to w$ in X as $n \to \infty$, then $w \in D(t)$.*

Let a, $b \geq 0$. Then (I) *implies* (II)*:*

(I) *For each $s \in [0,\tau)$ and each $v \in D(s)$, there exists a null sequence (h_n) of positive numbers and a sequence (v_n) in $D(A) \cap D(s+h_n)$ such that*

(I.a) $$\lim_{n\to\infty} h_n^{-1}|v_n - h_n(A + B(s + h_n))v_n - v| = 0,$$

(I.b) $$\limsup_{n\to\infty} h_n^{-1}[\varphi(v_n) - \varphi(v)] \le a\varphi(v) + b,$$

(I.c) $$\lim_{n\to\infty} |v_n - v| = 0.$$

(II) *There exists an evolution operator* $\mathcal{U} \equiv \{U(t,s) : t \ge 0\}$ *constrained in* $\mathcal{D}$ *such that for* $s \in [0,\tau)$, $v \in D(s)$, $t \in [s,\tau]$ *and* $f \in D(A^*)$,

(II.a) *the* $\boldsymbol{R}$*-valued function* $\langle U(\cdot,s)v, f\rangle$ *is continuously differentiable over* $[s,\tau)$,

$$\frac{d}{dt}\langle U(t,s)v, f\rangle = \langle U(t,s)v, A^*f\rangle + \langle B(t)U(t,s)v, f\rangle,$$

(II.b) $U(\cdot,s)v$ *satisfies the growth condition*

$$\varphi(U(t,s)v) \le e^{a(t-s)}[\varphi(v) + b(t-s)].$$

According to [19, Section 5], condition (I) is called an implicit subtangential condition. In this regard we refer to for instance a recent paper by Clément *et al.* [5]. The second statement (II) means the existence of a nonlinear evolution operator $\mathcal{U}$ constrained in $\mathcal{D}$ such that for $s \in [0,\tau)$ and $v \in D(s)$ the X-valued function $u(\cdot) \equiv U(\cdot,s)v$ gives a $\mathcal{D}$-valued weak solution of (SP) satisfying the exponential growth condition (EG).

PROOF OF THEOREM 3.1. The proof is similar to [19, Theorem 5.2]. Let $s \in [0,\tau)$, $v \in D(s)$ and let $\varepsilon \in (0,1]$. Set $\alpha = e^{a\tau}[\varphi(v) + (b+1)\tau]$. We denote by ω_α and θ_α a constant and function given by (H3), respectively. For each $t \in [s,\tau]$ we write $D^s(t)$ for the set

$$D^s(t) \equiv \{w \in D(t) : \varphi(w) \le e^{a(t-s)}[\varphi(v) + (b+\varepsilon)(t-s)]\}.$$

We define a nonlinear operator $B^s(t)$ from $D^s(t)$ into X by

$$B^s(t)w = B(t)w \quad \text{for } w \in D^s(t).$$

Then, we see from (H2) and (H3) that the following statements are valid:

(a) For each $t \in [s,\tau]$, $D^s(t)$ is closed; if $w_n \in D^s(t_n)$, $t_n \uparrow t$ in $[s,\tau]$ and $w_n \to w$ in X as $n \to \infty$, then $w \in D^s(t)$.

(b) $B^s(t)w$ is continuous with respect to (t,w) in the (non-cylindrical) domain $\cup_{s\le t\le\tau}(\{t\} \times D^s(t))$ in $[0,\tau] \times X$.

(c) For each $t \in [s,\tau]$, $A + B^s(t)$ is quasi-dissipative in the sense that

$$\langle (A + B^s(r))v - (A + B^s(t))w,\ v - w\rangle_i \le \omega_\alpha|v-w|^2 + \theta_\alpha(r,t)|v-w|$$

for $v \in D(A) \cap D^s(r)$ and $w \in D(A) \cap D^s(t)$, and $r,t \in [s,\tau]$.

We next show that the subtangential condition

$$\liminf_{h\downarrow 0} h^{-1}d(w, R(I-h(A+B^s(t+h)))) = 0 \tag{3.1}$$

holds for $t \in [s,\tau]$ and $w \in D^s(t)$. Let $t \in [s,\tau]$ and $w \in D^s(t)$. By the implicit subtangential condition (I), one can choose sequences (h_n) and (w_n) satisfying (I.a) and (I.b). It follows from (I.b) that

$$\begin{aligned}\varphi(w_n) &\le \varphi(w) + h_n[a\varphi(w) + (b+\varepsilon)] \\ &\le e^{ah_n}[\varphi(w) + (b+\varepsilon)h_n]\end{aligned}$$

for n sufficiently large. Since $\varphi(w) \le e^{a(t-s)}[\varphi(v) + (b+\varepsilon)(t-s)]$, we have

$$\varphi(w_n) \le e^{a(t+h_n-s)}[\varphi(v) + (b+\varepsilon)(t+h_n-s)] \tag{3.2}$$

for n sufficiently large. Hence (3.1) follows from (3.2) and (I.a). Combining (a), (b), (c) and (3.1), we may apply the results of [11, Theorem 3.5 and Remark 3.4] to conclude that the Cauchy problem

$$\frac{d}{dt}u(t) = (A + B^s(t))u(t), \quad s < t < \tau, \quad u(s) = v$$

admits an integral solution in the sense of Bénilan $u(\cdot) \in C([s,\tau];X)$ such that $u(t) \in D^s(t)$ for $t \in [s,\tau]$. Since $u(t) \in \overline{D(A) \cap D_\alpha(t)} \subset D_\alpha(t)$ for $t \in [s,\tau]$, it follows from the uniqueness of the integral solutions that $u(\cdot)$ is independent of $\varepsilon \in (0,1]$. Noting that $\varphi(u(t)) \le e^{a(t-s)}[\varphi(v) + (b+\varepsilon)(t-s)]$ and letting ε tend to 0, we have $\varphi(u(t)) \le e^{a(t-s)}[\varphi(v) + b(t-s)]$. This together with Remark 2.4 asserts that (I) implies (II). ■

In the case that $D(t)$ varies with t and is not convex for all $t \in [0,\tau]$, it is not straightforward to obtain the converse implication (II) $\Rightarrow$ (I). In order to show that (II) implies (I) we necessitate imposing appropriate conditions on $\mathcal{D}$. We here employ the following condition:

(H5) $D(s) \subset D(t)$ and $\theta D(s) + (1-\theta)D(t) \subset D(\theta s + (1-\theta)t)$ for $s, t \in [0,\tau]$ with $s \le t$ and $\theta \in [0,1]$.

Condition (H5) is a combination of incontractibility and convexity conditions. It should be noted that (H5) holds if $D(t)$ is independent of t, $D(t) \equiv D$ and D is convex in X.

THEOREM 3.2. *Let $\{A + B(t)\}$ be a one-parameter family of semilinear operators belonging to the class $\mathfrak{U}(\mathcal{D},\varphi)$. Suppose that* (H5) *holds. Then* (II) *implies* (I).

PROOF. To show that (II) implies (I), we first introduce a family $\{J_h(s)\}$ of operators which provides for each $(s,v) \in \mathcal{D}$ and a null sequence (h_n) of positive numbers, a sequence (v_h) in $D(A) \cap D(s+h_h)$ satisfying (I.a) through (I.c). For each $s \in [0,\tau)$ and each $h > 0$ we define an operator $J_h(s)$ from $D(s)$ into X by

$$J_h(s)v = (a_h(s))^{-1}\int_s^\tau e^{-t/h}U(t,s)v\,dt \qquad \text{for } v \in D(s), \tag{3.3}$$

where

$$a_h(s) = \int_s^\tau e^{-t/h}dt = h(e^{-s/h} - e^{-\tau/h}).$$

Then the operators $J_h(s)$ have the three properties below which are similar to those listed in [19, Proposition 6.1]:

(a) $J_h(s)v \in D(A) \cap D(s+h)$ and $(I - hA)J_h(s)v$ can be written as

$$v + h(a_h(s))^{-1}\int_s^\tau e^{-t/h}B(t)U(t,s)vdt - he^{-\tau/h}(a_h(s))^{-1}(U(\tau,s)v - v).$$

(b) $\lim_{h\downarrow 0} h^{-1}|(I - hA)J_h(s)v - (v + hB(s)v)| = 0$ and $\lim_{h\downarrow 0}|J_h(s)v - v| = 0$.

(c) $\lim_{h\downarrow 0} h^{-1}[\varphi(J_h(s)v) - \varphi(v)] \le a\varphi(v) + b$ and $\lim_{h\downarrow 0}\varphi(J_h(s)v) = \varphi(v)$.

It should be noted that condition (H5) is essential in the first assertion (a). Once the properties (a), (b) and (c) have been obtained, one can easily show that the implicit subtangential condition (I) holds. In fact, given $(s,v) \in \mathcal{D}$ it is sufficient to take any null sequence (h_n) satisfying $s + h_n \le \tau$ and a sequence (v_n) in $D(A) \cap D(s+h_n)$ defined by $v_n = J_{h_n}(s)v$.

Although the proofs of (a), (b) and (c) are obtained in the same way as in [19, Proposition 6.1], we give the proofs of the facts (a), (b) and (c) to make our argument self-contained.

Let $s \in [0,\tau)$, $h \in [0,\tau - s]$, $v \in D(s)$ and $w \in D(0)$. Set $R = \varphi(w) + e^{a\tau}[\varphi(v) + b\tau]$. We then define a subset of $[0,\tau] \times X$ by

$$\hat{\mathcal{D}}_R \equiv \bigcup_{0 \le r \le \tau}\left(\{r\} \times \widehat{D}_R(r)\right),$$

where

$$\widehat{D}_R(r) = \begin{cases} \bigcap_{0<\xi\le\tau} D_R(\xi) & \text{if } r = 0 \\ D_R(r) & \text{otherwise.} \end{cases}$$

We observe that $\widehat{D}_R(r)$ is closed by condition (H2). We first show that $\hat{\mathcal{D}}_R$ is closed in $[0,\tau] \times X$. Suppose that $(r_n, u_n) \in \hat{\mathcal{D}}_R$ and that $(r_n, u_n) \to (r, u)$ in $[0,\tau] \times X$ as $n \to \infty$. If $r = 0$, then we have $u \in \widehat{D}_R(0)$ by the definition of $\widehat{D}_R$. In the case that $r > 0$, we put

$$\hat{u}_n = \begin{cases} u_n & \text{if } r_n \le r, \\ \left(1 - \dfrac{r}{r_n}\right)w + \dfrac{r}{r_n}u_n & \text{if } r_n > r. \end{cases}$$

Then it follows from (H5) that $\hat{u}_n \in \widehat{D}_R(r)$. Since $\widehat{D}_R(r)$ is closed and $\hat{u}_n \to u$ as $n \to \infty$, we conclude that $u \in \widehat{D}_R(r)$. Therefore, $\hat{\mathcal{D}}_R$ is closed in $[0,\tau] \times X$.

We now show that (a) holds. It follows from the convexity of $\varphi(\cdot)$, (II.b) and (H5) that

$$\left((a_h(s))^{-1}\int_s^\tau te^{-t/h}\,dt, J_h(s)v\right) \in \hat{\mathcal{D}}_R.$$

Since

$$(a_h(s))^{-1}\int_s^\tau te^{-t/h}\,dt = (se^{-s/h}-\tau e^{-\tau/h})(e^{-s/h}-e^{-\tau/h})^{-1}+h \le s+h,$$

we see that $J_h(s)v \in D_R(s+h)$ by condition (H5). By Theorem 2.7 (c) we have that for $t \in [s,\tau]$, $\int_s^t U(\xi,s)v\,d\xi \in D(A)$ and

$$A\int_s^t U(\xi,s)v\,d\xi = U(t,s)v - v + \int_s^t B(\xi)U(\xi,s)\,d\xi.$$

Multiplying both sides of the above relation by $(a_h(s))^{-1}$ and integrating the resultant identity with respect to t over $[s,\tau]$, we obtain

$$\begin{aligned}
&(a_h(s))^{-1}\int_s^\tau \left(e^{-t/h}A\int_s^t U(\xi,s)v\,d\xi\right)dt \\
&= (a_h(s))^{-1}\int_s^\tau e^{-t/h}(U(t,s)v-v)\,dt - (a_h(s))^{-1}\int_s^\tau \left(e^{-t/h}B(\xi)U(\xi,s)v\,d\xi\right)dt \\
&= J_h(s)v - v + he^{-\tau/h}(a_h(s))^{-1}\int_s^\tau B(\xi)U(\xi,s)v\,d\xi - h(a_h(s))^{-1}\int_s^\tau e^{-t/h}B(\xi)U(\xi,s)v\,d\xi.
\end{aligned}$$

By Theorem 2.7 (c) again, the third term on the extreme right-hand side can be written as

$$he^{-\tau/h}(a_h(s))^{-1}\left[U(\tau,s)v - v - A\int_s^\tau U(\xi,s)v\,d\xi\right].$$

On the other hand, the left-hand side becomes, by the closedness of A,

$$hAJ_h(s)v - he^{-\tau/h}(a_h(s))^{-1}A\int_s^\tau U(\xi,s)v\,d\xi.$$

Hence we obtain the assertion (a). Since $e^{-\tau/h}(a_h(s))^{-1} \to 0$ and

$$(a_h(s))^{-1}\int_s^\tau e^{-t/h}B(\xi)U(\xi,s)v\,d\xi \to B(s)v \quad \text{as } h\downarrow 0,$$

it follows from (a) that

$$\lim_{h\downarrow 0} h^{-1}|(I-hA)J_h(s)v - (v+hB(s)v)| = 0. \tag{3.4}$$

Combining (3.4) with the inequality

$$|J_h(s)v-v| \le |J_h(s)v-(I-hA)^{-1}(v+hB(s)v)| + |(I-hA)^{-1}v - v| + h|(I-hA)^{-1}B(s)v|,$$

we obtain that $\lim_{h\downarrow 0}|J_h(s)v - v| = 0$, hence the assertion (b) is valid. Finally, we show that (c) holds. Since $\varphi(\cdot)$ is convex and lower semicontinuous, we see from (II.b) that

$$\varphi(J_h(s)v) \le (a_h(s))^{-1}\int_s^\tau e^{-t/h}e^{a(t-s)}[\varphi(v)+b(t-s)]\,dt$$

and

$$\begin{aligned}
h^{-1}[\varphi(J_h(s)v)-\varphi(v)] \le{}& h^{-1}(a_h(s))^{-1}\int_s^\tau e^{-t/h}(e^{a(t-s)}-1)\,dt\varphi(v) \\
&+ h^{-1}b(a_h(s))^{-1}\int_s^\tau e^{-t/h}e^{a(t-s)}(t-s)\,dt.
\end{aligned}$$

By integration by parts, the right-hand side is written as

$$(a_h(s))^{-1}\int_s^{\tau} e^{-t/h}e^{a(t-s)}\,dt(a\varphi(v)+b) - e^{-\tau/h}(a_h(s))^{-1}[(e^{a(\tau-s)}-1)\varphi(v)+be^{a(\tau-s)}(\tau-s)]$$
$$+\, ab(a_h(s))^{-1}\int_s^{\tau} e^{-t/h}e^{a(t-s)}(t-s)\,dt.$$

This implies that $\limsup_{h\downarrow 0} h^{-1}[\varphi(J_h(s)v)-\varphi(v)] \le a\varphi(v)+b$. This together with (b) gives

$$\varphi(v) \le \liminf_{h\downarrow 0}\varphi(J_h(s)v) \le \limsup_{h\downarrow 0}\varphi(J_h(s)v) \le \varphi(v).$$

The proof is thereby complete. ■

4. Characterization in Terms of Explicit Subtangential Conditions

In this section we investigate the generation problem of a nonlinear evolution operator $\mathcal{U}$ associated with (SE) by means of explicit subtangential conditions for the semilinear operators $A+B(t)$. For this purpose, we first discuss the existence theory due to Iwamiya [9] from our point of view and then give a characterization theorem for the nonlinear evolution operator $\mathcal{U}$ in terms of explicit subtangential condition.

Let $0 \le \sigma < \tau \le +\infty$ and consider the time-dependent semilinear problem

$$\text{(NSP)}\qquad \frac{d}{dt}u(t) = Au(t)+B(t)u(t),\ \ \sigma < t < \tau;\ \ u(\sigma)=v$$

in a Banach space $(X, |\cdot|)$. Here $B(t)$, $\sigma \le t < \tau$, is a nonlinear continuous operator from a subset $C(t)$, $\sigma \le t < \tau$, of X into X. Assuming that A is the infinitesimal generator of a nonexpansive evolution operator $\mathcal{T}_X$ of class (C_0) on X, Iwamiya [9] advanced a general existence theory for the problem (NSP). We here outline a modification of his existence theorem in the case where A is the generator of a once integrated semigroup $\mathcal{W}$ such that

$$\text{(4.1)}\qquad |W(t)-W(s)| \le |t-s| \qquad \text{for } s,\ t \ge 0.$$

Condition (4.1) for the integrated semigroup $\mathcal{W}$ is essential in our argument. However, the semilinear operators $A+B(t)$ are written as $(A-\omega)+(B(t)+\omega)$ and the same results are obtained if there is $\omega \in \boldsymbol{R}$ such that $A-\omega$ is the generator of a once integrated semigroup satisfying (4.1) and such that the function $v \mapsto g(t,v)+\omega v$ satisfies condition (g.2) below.

It follows from Theorem 1.5 that a closed linear operator A in X is the generator of a once integrated semigroup $\mathcal{W}$ satisfying (4.1) if and only if the following condition holds:

(H1)$'$ For $\lambda > 0$, $(I-\lambda A)^{-1}$ exists as a nonexpansive operator on X.

For the linear operator A in (NSP) we assume that the stronger condition (H1)$'$ is satisfied. As before, Y denotes the norm closure $\overline{D(A)}$ of the domain of A. We assume that $C(t) \subset Y$, $\sigma \le t < \tau$, and furthermore impose the following conditions on the operator $B(t)$:

(C1) If $v_n \in C(t_n)$, $t_n \uparrow t$ in $[\sigma, \tau)$ and $v_n \to v$ in X, then $v \in C(t)$.

(C2) $\liminf_{h \downarrow 0} h^{-1} d(T_Y(h)v + W(h)B(t)v, C(t+h)) = 0$ for $t \in [\sigma, \tau)$ and $v \in C(t)$.

(C3) For $t \in [\sigma, \tau)$, $v, w \in C(t)$ and $h > 0$ we have

$$|v - w| \leq |(v - W(h)B(t)v) - (w - W(h)B(t)w)| + hg(t, |v - w|),$$

where $g : [\sigma, \tau) \times \boldsymbol{R} \to \boldsymbol{R}$ is a given function such that

(g1) $g(t, w)$ satisfies the so-called Caratheodory's condition.

(g2) $g(t, 0) = 0$ and the function $w(t) \equiv 0$ is a maximal solution to the initial-value problem

$$w'(t) = g(t, w(t)), \quad \sigma < t < \tau; \quad w(\sigma) = 0. \tag{4.2}$$

Condition (C2) is nothing but the subtangential condition (ST) introduced by Thieme [21]. If in particular A is densely defined in X, then it is the infinitesimal generator of a nonexpansive semigroup $\mathcal{T}_X$ of class (C_0) on X and, by Corollary 1.7, condition (C2) is equivalent to the condition

(C2)′ $\liminf_{h \downarrow 0} h^{-1} d(T_X(h)v + hB(t)v, C(t+h)) = 0$ for $t \in [\sigma, \tau)$ and $v \in C(t)$.

Moreover the denseness of $D(A)$ and (C3) together imply that

(C3)′ $$|v - w| \leq |(v - hB(t)v) - (w - hB(t)w)| + hg(t, |v - w|)$$

for $t \in [\sigma, \tau)$ and $v, w \in C(t)$. Namely, for each $t \in [\sigma, \tau)$, the operator $B(t) : C(t) \to X$ is quasidissipative. The converse is not always true, even though A is densely defined. If, in particular, for each $t \in [\sigma, \tau)$ the operator $B(t) : C(t) \to X$ is Lipschitz continuous in the sense that there is a constant $\omega \geq 0$ and

$$|B(t)v - B(t)w| \leq \omega |v - w| \quad \text{for } t \in [\sigma, \tau) \text{ and } v, w \in C(t),$$

then condition (C3) holds for the function $g(t, |v - w|) = 3\omega |v - w|$.

In view of Theorem 2.7 (f), we introduce a notion of mild solution (NSP).

DEFINITION 4.1. Let $s \in [\sigma, \tau)$ and $v \in C(s)$. A continuous function $u(\cdot) : [s, \tau) \to X$ is said to be a *mild solution* of (NSP) on the interval $[s, \tau)$, if $u(t) \in C(t)$ for $t \in [s, \tau)$, the function $t \mapsto B(t)u(t)$ is continuous from $[s, \tau)$ into X, and $u(\cdot)$ satisfies the equation

$$u(t) = T_Y(t - s)v + \lim_{\lambda \downarrow 0} \int_s^t T_Y(t - \xi)(I - \lambda A)^{-1} B(\xi) u(\xi) \, d\xi \tag{4.3}$$

for $t \in [s, \tau)$, where the limit is taken in X and in the sense of the norm topology.

Under the conditions as mentioned above, the following modified version of Iwamiya's existence theorem is obtained.

THEOREM 4.2. *Suppose that conditions* (H1)′, (C1), (C2) *and* (C3) *are fulfilled. If* $C(t) \neq \phi$ *for all* $t \in [\sigma, \tau)$ *and* $\boldsymbol{C} \equiv \cup_{\sigma \leq t < \tau}(\{t\} \times C(t))$ *is a connected subset of* $[\sigma, \tau) \times X$ *such that , then for each* $s \in [\sigma, \tau)$ *and* $v \in C(s)$, *the problem* (NSP) *has a unique mild solution* $u(\cdot)$ *on* $[s, \tau)$ *satisfying* $u(s) = v$.

In Iwamiya's argument most of the estimates are very precise and the denseness of $D(A)$ is used in many parts of his proof, since he uses the property that $\lim_{t \downarrow 0} T(t)v = v$ for all $v \in X$. Therefore it is necessary to change every part of the proof where this property is employed. Although we need much more delicate estimates, it is possible to overcome this difficulty with the aid of Theorem 1.6. From Theorem 4.2 we obtain a generation theorem for nonlinear evolution operators associated with (NSP) under the explicit subtangential condition (C2).

In the rest of this section we establish a characterization of nonlinear evolution operators providing the mild solutions of (SP) through an explicit subtangential condition (C2) for $A + B(t)$. Let A be a closed linear operator satisfying (H1)′ and $B(t)$, $0 \leq t < \tau$, nonlinear operators in X defined on convex subsets $D(t)$ satisfying (H2).

In addition to (H1)′ and (H2) we assume the following condition which is a stronger form than (C3) :

(H3)′ For each $\alpha > 0$ there exist $\omega_\alpha \in \boldsymbol{R}$ such that

$$(1 - h\omega_\alpha)|v - w| \leq |(v - W(h)B(t)v) - (w - W(h)B(t)w)|$$

for $t \in [0, \tau)$, $h > 0$ and $v, w \in D_\alpha(t)$.

Conditions (H1)′ and (H3)′ together imply (H3). Under the above conditions we obtain the following characterization theorem for nonlinear evolution operators associated with (SP):

THEOREM 4.3. *Let* $a, b \geq 0$. *Assume that* A *and* $B(t)$ *satisfy conditions* (H1)′, (H2) *and* (H3)′. *Then the following are equivalent :*

(III) *For each* $s \in [0, \tau)$ *and each* $v \in D(s)$ *there exist a null sequence* $\{h_n\}$ *of positive numbers and a sequence* $\{v_n\}$ *in* $D(s + h_n)$ *such that*

(III.a) $$\lim_{n \to \infty} h_n^{-1} |T_Y(h_n)v - W(h_n)B(s)v - v_n| = 0,$$

(III.b) $$\limsup_{n \to \infty} h_n^{-1}[\varphi(v_n) - \varphi(v)] \leq a\varphi(v) + b.$$

(IV) *There exists a nonlinear evolution operator* $\mathcal{U} \equiv \{U(t, s) : \leq s \leq t < \tau\}$ *constrained in* $\mathcal{D}$ *such that*

(IV.a) $$U(t, s)v = T_Y(t - s)v + \lim_{\lambda \downarrow 0} \int_s^t T_Y(t - \xi)(I - \lambda A)^{-1} B(\xi) U(\xi, s)v \, d\xi,$$

(IV.b) $$\varphi(U(t, s)v) \leq e^{a(t-s)}[\varphi(v) + b(t - s)],$$

for each $s \in [0, \tau)$, $t \in [s, \tau)$, *and* $v \in D(s)$.

Prior to giving the proof of this theorem we make a remark on the relation between this theorem and Theorem 3.1 and prepare a uniqueness theorem for mild solutions of (SP) satisfying (EG). It would be interesting to compare the uniqueness theorems, Proposition 2.5 and Proposition 4.5 below.

REMARK 4.4. If in particular $B(t)$ are assumed to be locally Lipschitz continuous in the sense that

(H3)″ for each $\alpha > 0$ there exists $\omega_\alpha > 0$ and $\theta_\alpha \in \mathcal{F}$ such that

$$|B(s)v - B(t)w| \le \omega_\alpha |v - w| + \theta_\alpha(s,t)$$

for $s, t \in [0, \tau]$, $v \in D_\alpha(s)$ and $w \in D_\alpha(t)$, then it follows from Theorem 2.7, Theorem 3.1 and Theorem 3.2 that the statements (I) in Theorem 3.1 implies (III), and that (III) implies (II) under additional condition (H5).

Since $|W(h)| \le h$ for $h > 0$ by (4.1), it is readily seen that (H3)″ implies (H3)′. Also, (H3)″ implies (H3) under condition (H1)′. Hence the application of Theorem 2.3 implies that condition (II) in Theorem 3.1 is equivalent to condition (IV) in Theorem 4.3. This shows that condition (I) implies (III) and that (III) implies (I) under (H5).

PROPOSITION 4.5. *Assume that A and $B(t)$ satisfy conditions* (H1)′, (H2) *and* (H3)′. *Then given $s \in [0, \tau)$ and $v \in D(s)$ there exists at most one mild solution $u(\cdot)$ of the problem* (SP) *satisfying* (EG).

PROOF. Let $\alpha > 0$, $\beta = e^{a\tau}[\alpha + b\tau]$ and let ω_β be a constant provided for β by condition (H3)′. Let $s \in [0, \tau)$, $w, z \in D_\alpha(s)$ and let $u(\cdot), v(\cdot)$ be the corresponding global mild solutions of (SP) satisfying (EG). Then $u(t), v(t) \in D_\beta(s) \subset Y$ for $t \in [s, \tau)$. By (H3)′ we have

$$(1 - h\omega_\beta)|u(t+h) - v(t+h)| \le |(I - W(h)B(t+h))u(t+h) - (I - W(h)B(t+h))v(t+h)|$$

for $t \in [s, \tau)$ and $h > 0$ with $t + h \le \tau$. But $(I - W(h)B(t+h))u(t+h)$ is written as

$$T_Y(h)u(t) + \lim_{\lambda \downarrow 0} \int_t^{t+h} T_Y(t+h-s)(I - \lambda A)^{-1}[B(s)u(s) - B(t+h)u(t+h)]\, ds$$

and $(I - W(h)B(t+h))v(t+h)$ is also written in the same form as above. Hence we have

$$(1 - h\omega_\beta)|u(t+h) - v(t+h)| \le |u(t) - v(t)| + \int_t^{t+h} |B(s)u(s) - B(t+h)u(t+h)|\, ds$$

$$+ \int_t^{t+h} |B(s)u(s) - B(t+h)u(t+h)|\, ds$$

for $t \in [s, \tau)$ and $h \in (0, \tau - t]$. From this it follows that $D^+|u(t) - v(t)| \le \omega_\beta |u(t) - v(t)|$ for $t \in [s, \tau)$, where $D^+\phi(t)$ stands for the Dini upper right derivative of an $\boldsymbol{R}$-valued function ϕ on $[s, \tau)$ at t. Solving this differential inequality, one obtains

$$|u(t) - v(t)| \le e^{\omega_\beta(t-s)}|w - z| \qquad \text{for } t \in [s, \tau) \text{ and } w, z \in D_\alpha(s).$$

This implies the desired assertion. ∎

PROOF OF THEOREM 4.3. First we show that (IV) implies (III). Let $s \in [0,\tau)$, $v \in D(s)$. Let (h_n) be a null sequence of positive numbers and put $v_n = U(s+h_n,s)v$. Then it is easily seen that (III) follows from (IV).

We next show that (III) implies (IV). In view of Proposition 4.5 it suffices to show that for any $s \in [0,\tau)$ and any $z \in D(s)$ there exists an X-valued continuous function $u(\cdot)$ on $[s,\tau)$ such that for each $t \in [s,\tau)$, $u(t) \in D(t)$,

$$u(t) = T_Y(t-s)z + \lim_{\lambda \downarrow 0} \int_s^t T_Y(t-\xi)(I-\lambda A)^{-1}B(\xi)u(\xi)\,d\xi$$

and

$$\varphi(u(t)) \le e^{a(t-s)}[\varphi(z) + b(t-s)].$$

Let $s \in [0,\tau)$, $\varepsilon \in (0,1]$ and $z \in D(s)$. Set $\alpha = e^{a\tau}[\varphi(z) + (b+\varepsilon)\tau]$ and let ω_α and θ_α denote the constant and function given by (H3)$'$. Also, for each $t \in [s,\tau)$, we write $D^s(t)$ for the set $\{v \in D(t) : \varphi(v) \le e^{a(t-s)}[\varphi(z) + (b+\varepsilon)(t-s)]\}$ and define an operator $B^s(t)$ from $D^s(t)$ into X by $B^s(t)v = B(t)v$ for $v \in D^s(t)$. Then the following statements are valid for the operators $B^s(t)$, $t \in [s,\tau)$:

(i) Each of $D^s(t)$ is closed and has the property that if $w_n \in D^s(t_n)$, $t_n \uparrow t$ in $[s,\tau)$ and $w_n \to w$ in X as $n \to \infty$, then $w \in D^s(t)$.

(ii) The mapping $(t,v) \mapsto B^s(t)v$ is continuous from $\mathcal{D}^s \equiv \bigcup_{t\in[s,\tau)}(\{t\} \times D^s(t))$ into X.

(iii) For each $t \in [s,\tau)$, $h > 0$, $v, w \in D^s(t)$, we have

$$(1-h\omega_\alpha)|v-w| \le |(I - W(h)B(t))v - (I - W(h)B(t))w|$$

Let $t \in [s,\tau)$ and $v \in D^s(t)$. Then by (III) there exist sequences (h_n) and (v_n) satisfying (III.a) and (III.b). Hence, by (III.b), we have

$$\varphi(v_n) \le \varphi(v) + h_n[a\varphi(v) + b + \varepsilon/2] \le e^{ah_n}[\varphi(v) + (b+\varepsilon)h_n],$$

for n sufficiently large. Since $\varphi(v) \le e^{a(t-s)}[\varphi(z) + (b+\varepsilon)(t-s)]$, it follows that

$$\begin{aligned}\varphi(v_n) &\le e^{a(t+h_n-s)}[\varphi(z) + (b+\varepsilon)(t-s)] + e^{ah_n}(b+\varepsilon)h_n \\ &\le e^{a(t+h_n-s)}[\varphi(z) + (b+\varepsilon)(t+h_n-s)]\end{aligned}$$

for n sufficiently large. From this and (III.a) we infer that for $t \in [s,\tau)$ and $v \in D^s(t)$

(iv) $$\liminf_{h \downarrow 0} h^{-1} d(T_Y(h)v + W(h)B(t)v, D(t+h)) = 0.$$

By virtue of the facts (i)-(iv) mentioned above, we apply Theorem 4.2 to conclude that there exists a function $u(\cdot) \in \mathcal{C}([s,\tau);X)$ such that $u(t) \in D^s(t)$ and (IV.a) holds for $t \in [s,\tau)$. Since $u(t) \in D_\alpha(t)$ for $t \in [s,\tau)$, it follows from the uniqueness of the mild solutions that $u(\cdot)$ is independent of ε. The fact that $u(t) \in D^s(t)$ for $t \in [s,\tau)$ means that

$$\varphi(u(t)) \le e^{a(t-s)}[\varphi(z) + (b+\varepsilon)(t-s)] \qquad \text{for } t \in [s,\tau) \text{ and } \varepsilon \in (0,1].$$

Taking the limit as $\varepsilon \downarrow 0$, we have $\varphi(u(t)) \le e^{a(t-s)}[\varphi(z) + b(t-s)]$ for $t \in [s,\tau)$. This completes the proof. ∎

REMARK 4.6. Let C be a fixed closed convex subset of Y. In Thieme [21] it is assumed that $C(t) \equiv C$ for $t \in [\sigma, \tau)$, and that the nonlinear operator $B(t) : C(t) \to X$ in (NSP) is locally Lipschitz continuous and of linear growth in the following sense :

(LL) For any $t \in [\sigma, \tau)$ and $v \in C$ there exist positive numbers $\delta > 0$, $\Gamma > 0$ such that

$$|B(s)w - B(s)z| \leq \Gamma |w - z|$$

for $t \leq s \leq t + \delta, w, z \in C$ with $|w - v| \leq \delta$ and $|z - v| \leq \delta$.

(LG) For any $\tau > 0$ there exists a positive number $\kappa \equiv \kappa(\tau) > 0$ such that

$$|B(t)v| \leq \kappa(1 + |v|) \quad \text{for } 0 \leq t < \tau \text{ and } v \in C.$$

We infer from (4.3) that under the linear growth condition (LG) the following *a priori* estimates are obtained for mild solutions $u(\cdot)$ of (NSP) :

$$|u(t)| \leq e^{\kappa(t-\sigma)}[|v| + \kappa(t - \sigma)] \quad \text{for } t \in [\sigma, \tau),$$

where $v \in C(\sigma)$ stands for the initial-value given in (NSP).

5. Application to Nonlinear Reaction-Diffusion Equations

In this section we make an attempt to apply Theorem 3.1 to a class of semilinear reaction-diffusion systems. Let Ω be a bounded domain in $\boldsymbol{R}^n$ with smooth boundary and let $\tau > 0$. We then consider a system of nonlinear degenerate reaction-diffusion system of the form

$$\text{(RDS)} \quad \begin{aligned}
\partial_t u_1(t,x) &= \Delta u_1(t,x) - d_1(t,x)u_1(t,x)u_4(t,x) - d_2(t,x)u_1(t,x)u_3(t,x) \\
\partial_t u_2(t,x) &= \Delta u_2(t,x) - d_3(t,x)u_2(t,x)u_4(t,x) + d_2(t,x)u_1(t,x)u_3(t,x) \\
\partial_t u_3(t,x) &= \qquad\qquad\quad d_3(t,x)u_2(t,x)u_4(t,x) - d_2(t,x)u_1(t,x)u_3(t,x) \\
\partial_t u_4(t,x) &= \qquad\qquad - d_1(t,x)u_1(t,x)u_4(t,x) - d_3(t,x)u_2(t,x)u_4(t,x).
\end{aligned}$$

in the cylindrical domain $(0, \tau) \times \Omega$ under the boundary condition of Neumann type

$$\text{(BC)} \qquad \frac{\partial u_i}{\partial \nu}(t,x) = 0 \quad \text{for } (t,x) \in (0,\tau) \times \partial\Omega \text{ and } i = 1, 2$$

and the initial condition

$$\text{(IC)} \qquad u_i(0,x) = w_i(x) \quad \text{for } x \in \Omega \text{ and } i = 1,2,3,4,$$

where ν denotes the outward unit normal to $\partial\Omega$. We here employ a boundary condition of Neumann type, although various types of boundary conditions may be imposed.

This problem was formulated as a mathematical model describing evolution of chemical reaction of four kinds of molecules. For details and other related problems we refer to Fife [8], Mimura and Nakaoka [16], Konishi [13] and Kobayasi and Oharu [12]. It should be noted here that $u_3(\cdot)$ and $u_4(\cdot)$ are expected to be only $L^\infty(\Omega)$-valued functions because of the degeneracy in (RDS)

We here treat the semilinear problem for (RDS) under the boundary and initial conditions (BC)-(IC) in the Banach space

$$X = (L^\infty(\Omega))^4$$

equipped with the norm defined by $\|v\| = \sum_{i=1}^4 |v_i|$ for $v \in X$, where $|\cdot|$ denotes the usual $L^\infty(\Omega)$ norm.

In what follows, the coefficients $d_i(\cdot)$ are assumed to satisfy the following condition:

(C) $\quad d_i(\cdot) \in C([0,\tau]; L^\infty(\Omega)) \quad$ and $\quad d_i(t,x) \geq 0$

for $i = 1,2,3$, $t \in [0,\tau]$, and for almost all $x \in \Omega$.

This means that the chemical reaction takes place in a comparatively active way. In order to describe the diffusion terms, we define a closed linear operator A in X by

$$D(A) = \Big\{ v = (v_i) : v_i \in L^\infty(\Omega),\ i = 3,4,\ v_j \in W^{2,p}(\Omega) \quad \text{for } p > n,\ \Delta v_j \in L^\infty(\Omega),$$
$$\frac{\partial v_j}{\partial \nu} = 0 \quad \text{on } \partial\Omega,\ j = 1,2 \Big\},$$

$$Av = [\Delta v_1, \Delta v_2, 0, 0].$$

Since Ω is bounded and $W^{2,p}(\Omega)$ is continuously embedded in $C^1(\overline{\Omega})$ for $p > n$, the boundary condition makes sense, $\partial u_i/\partial \nu \in C(\partial\Omega)$, $i = 1,2$, and the Laplace operator Δ generates an $L^p(\Omega)$-contraction semigroup for $p > n$. Hence it follows that the linear operator A satisfies (H1) for $\omega = 0$, that is, for $\lambda > 0$ the resolvent $(I - \lambda A)^{-1}$ exists and satisfies

$$\|(I - \lambda A)^{-1}\| \leq 1.$$

It should also be noted that the resolvent $(I - \lambda A)^{-1}$ is order-preserving for $\lambda > 0$.

We then define nonlinear operators $B(t)$, $t \in [0,\tau]$, from X into X by

$$B(t)v = (B_1(t)v, B_2(t)v, B_3(t)v, B_4(t)v),$$

$$[B_1(t)v](x) = -d_1(t,x)v_1(t,x)v_4(t,x) - d_2(t,x)v_1(t,x)v_3(t,x)$$

$$[B_2(t)v](x) = -d_3(t,x)v_2(t,x)v_4(t,x) + d_2(t,x)v_1(t,x)v_3(t,x)$$

$$[B_3(t)v](x) = \ d_3(t,x)v_2(t,x)v_4(t,x) - d_2(t,x)v_1(t,x)v_3(t,x)$$

$$[B_4(t)v](x) = -d_1(t,x)v_1(t,x)v_4(t,x) - d_3(t,x)v_2(t,x)v_4(t,x),$$

$$D(t) = D(B(t)) = D \equiv C(\overline{\Omega})^+ \times C(\overline{\Omega})^+ \times L^\infty(\Omega)^+ \times L^\infty(\Omega)^+$$

for $t \in [0,\tau]$, where $C(\overline{\Omega})^+$ and $L^\infty(\Omega)^+$ denote the positive cone of $C(\overline{\Omega})$ and $L^\infty(\Omega)$, respectively. Then the reaction-diffusion system (RDS)-(BC)-(IC) is written as an abstract semilinear problem in X of the form

$$\text{(SP)} \qquad \begin{aligned} &\frac{d}{dt}\boldsymbol{v}(t) = A\boldsymbol{v}(t) + B(t)\boldsymbol{v}(t), \quad t \in (0,\tau) \\ &\boldsymbol{v}(0) = \boldsymbol{w}. \end{aligned}$$

Since $D(t)$ is independent of t and defined as the product of positive cones of $C(\overline{\Omega})$ and $L^\infty(\Omega)$, we seek the solution to (SP) in the cylindrical domain $\mathcal{D} = [0,\tau] \times D$ which satisfies condition (H5).

Let $\varphi(\boldsymbol{v}) = |v_1| + |v_4| + |v_1 + v_2| + |v_3 + v_4|$. Then $\varphi(\cdot)$ is a continuous convex functional on X. Let $D_\alpha = \{\boldsymbol{v} \in D : \varphi(\boldsymbol{v}) \leq \alpha\}$. The next lemma shows that $B(t)\boldsymbol{v}$ is Lipschitz continuous with respect to $\boldsymbol{v}$ and continuous in t in a local sense.

LEMMA 5.1. *Let $d_0 = \max_{1\leq i\leq 3} \max_{t\in[0,\tau]} |d_i(t,\cdot)|$. Then we have :*

(a) *For $\alpha > 0$, $t \in [0,\tau]$ and $\boldsymbol{v}$, $\boldsymbol{w} \in D_\alpha$*

$$\|B(t)\boldsymbol{v} - B(t)\boldsymbol{w}\| \leq 3d_0\alpha\|\boldsymbol{v} - \boldsymbol{w}\|.$$

(b) *For $\alpha > 0$, $s,t \in [0,\tau]$ and $\boldsymbol{v} \in D_\alpha$*

$$\|B(s)\boldsymbol{v} - B(t)\boldsymbol{v}\| \leq 3\sum_{i=1}^{3} |d_i(s) - d_i(t)|\alpha^2.$$

(c) *For $h \in (0, (d_0\alpha)^{-1})$, $t \in [0,\tau]$ and $\boldsymbol{v} \in D_\alpha$*

$$[(I + hB(t))\boldsymbol{v}](x) \geq 0 \quad \textit{for almost all } x \in \Omega.$$

PROOF. We first show that (a) is valid. Let $\alpha > 0$, $t \in [0,\tau]$ and let $\boldsymbol{v} = (v_i)$, $\boldsymbol{w} = (w_i) \in D_\alpha$. In view of the fact that $v_i(x), w_i(x) \geq 0$ for $i = 1,2,3,4$ and $x \in \Omega$, we have

$$\begin{aligned} |B_1(t)\boldsymbol{v} - B_1(t)\boldsymbol{w}| &= |-d_1(t,\cdot)v_1v_4 - d_2(t,\cdot)v_1v_3 + d_1(t,\cdot)w_1w_4 + d_2(t,\cdot)w_1w_3| \\ &\leq d_0(|w_3 + w_4||v_1 - w_1| + |v_1||v_3 - w_3| + |v_1||v_4 - w_4|). \end{aligned}$$

Similarly, we have

$$\begin{aligned} &|B_2(t)\boldsymbol{v} - B_2(t)\boldsymbol{w}| \leq d_0(|w_3||v_1 - w_1| + |w_4||v_2 - w_2| + |v_1||v_3 - w_3| + |v_2||v_4 - w_4|), \\ &|B_3(t)\boldsymbol{v} - B_3(t)\boldsymbol{w}| \leq d_0(|w_3||v_1 - w_1| + |w_4||v_2 - w_2| + |v_1||v_3 - w_3| + |v_2||v_4 - w_4|), \\ &|B_4(t)\boldsymbol{v} - B_4(t)\boldsymbol{w}| \leq d_0(|w_4||v_1 - w_1| + |w_4||v_2 - w_2| + |v_1 + v_2||v_4 - w_4|). \end{aligned}$$

Combining these estimates, we see that (a) is valid. Let $\alpha > 0$, $t \in [0,\tau]$ and let $\boldsymbol{v} = (v_i) \in D_\alpha$. Noting that $|v_1v_4|, |v_1v_3|, |v_2v_4| \leq \alpha^2$, we have

$$\|B(s)\boldsymbol{v} - B(t)\boldsymbol{v}\| \leq 3\sum_{i=1}^{3} |d_i(s) - d_i(t)|\alpha^2.$$

This shows that (b) holds. Finally, we show that (c) holds. Let $\alpha > 0$, $t \in [0, \tau]$ and $\boldsymbol{v} \in D_\alpha$. Then for each $h \in (0, (d_0\alpha)^{-1})$ we have

$$\begin{aligned} v_1(x) + h[B_1(t)\boldsymbol{v}](x) &= v_1(x)(1 - h(d_1(t,x)v_4(x) + d_2(t,x)v_3(x))) \\ &\geq v_1(x)(1 - hd_0|v_3 + v_4|) \geq 0 \end{aligned}$$

for almost all $x \in \Omega$. In a similar way we can show that $v_j(x) + h[B_j(t)\boldsymbol{v}](x) \geq 0$ for almost all $x \in \Omega$ and $j = 2, 3, 4$. Assertion (c) follows from these inequalities. Thus, the proof is complete. ∎

The next lemma shows that the family $\{A + B(t)\}$ of semilinear operators satisfies an implicit subtangential condition.

LEMMA 5.2. *Let $s \in [0, \tau)$ and $\boldsymbol{v} = (v_i) \in D$. Put*

$$\boldsymbol{v}_n = (I - n^{-1}A)^{-1}(\boldsymbol{v} + n^{-1}B(s + n^{-1})\boldsymbol{v}) \tag{5.1}$$

for n such that $n^{-1} \in (0, \tau - s)$. Then $\boldsymbol{v}_n \in D$ for $n > \max\{d_0\varphi(v), (\tau - s)^{-1}\}$ and

$$\lim_{n\to\infty} n\|\boldsymbol{v}_n - n^{-1}(A + B(s + n^{-1}))\boldsymbol{v}_n - \boldsymbol{v}\| = 0, \tag{5.2}$$

$$\varphi(\boldsymbol{v}_n) \leq \varphi(\boldsymbol{v}), \qquad \lim_{n\to\infty} \|\boldsymbol{v}_n - \boldsymbol{v}\| = 0. \tag{5.3}$$

PROOF. Let $s \in [0, \tau)$ and $\boldsymbol{v} = (v_i) \in D$. Put $\alpha = \varphi(\boldsymbol{v})$. Since the resolvent $(I - n^{-1}A)^{-1}$ is order-preserving, it follows from Lemma 5.1 (c) that $\boldsymbol{v}_n \in D$ for $n > \max\{d_0\alpha, (\tau - s)^{-1}\}$. We then show that (5.3) holds. By (5.1) we have

$$\|\boldsymbol{v}_n - \boldsymbol{v}\| \leq \|(I - n^{-1}A)^{-1}\boldsymbol{v} - \boldsymbol{v}\| + n^{-1}\|B(s + n^{-1})\boldsymbol{v}\|.$$

This together with Lemma 5.1 (b) implies that $\lim_{n\to\infty} \|\boldsymbol{v}_n - \boldsymbol{v}\| = 0$. Since $v_1(x) \geq 0$ for $x \in \Omega$ and $d_j(t, x) \geq 0$ for $j = 1, 2, 3$ and for almost all $x \in \Omega$, we have

$$0 \leq v_1(x) + n^{-1}[B_1(s + n^{-1})\boldsymbol{v}](x) \leq v_1(x)$$

for almost all $x \in \Omega$. This implies that $|v_1 + n^{-1}B_1(s + n^{-1})\boldsymbol{v}| \leq |v_1|$. Hence we have $|(I - n^{-1}\Delta)^{-1}(v_1 + n^{-1}B_1(s + n^{-1})\boldsymbol{v}| \leq |v_1|$. In a similar way it follows that

$$|(I - n^{-1}\Delta)^{-1}(v_1 + v_2 + n^{-1}B_1(s + n^{-1})\boldsymbol{v} + n^{-1}B_2(s + n^{-1})\boldsymbol{v})| \leq |v_1 + v_2|$$

$$|v_3 + v_4 + n^{-1}B_3(s + n^{-1})\boldsymbol{v} + n^{-1}B_4(s + n^{-1})\boldsymbol{v}| \leq |v_3 + v_4|$$

$$|v_4 + n^{-1}B_4(s + n^{-1})\boldsymbol{v}| \leq |v_4|.$$

Combining these estimates, we obtain that $\varphi(\boldsymbol{v}_n) \leq \varphi(\boldsymbol{v})$. Finally, we demonstrate that (5.2) holds. Since (5.3) implies that $\boldsymbol{v}_n \in D_\alpha$ for $n > \max\{d_0\alpha, (\tau - s)^{-1}\}$, it follows from (5.1) that

$$\lim_{n\to\infty} n\|\boldsymbol{v}_n - n^{-1}(A + B(s + n^{-1}))\boldsymbol{v}_n - \boldsymbol{v}\| = \lim_{n\to\infty} \|B(s + n^{-1})\boldsymbol{v} - B(s + n^{-1})\boldsymbol{v}_n\| = 0.$$

This completes the proof. ∎

Thus, it follows from Lemma 5.1, condition (C) and Lemma 5.2 that conditions (H1), (H2),(H3) stated in Section 2 and condition (I) in Theorem 3.1 are all satisfied. Theorem 3.1 and Theorem 2.7 can now be applied to obtain the following theorem:

THEOREM 5.3. *Suppose that* (C) *holds. Then there exists a nonlinear evolution operator* $\mathcal{U} \equiv \{U(t,s) : 0 \le s \le t \le \tau\}$ *constrained in* $\mathcal{D}$ *such that*

$$U(t,0)\boldsymbol{v} = \boldsymbol{v} + A\int_0^t U(s,0)\boldsymbol{v}\,ds + \int_0^t B(s)U(s,0)\boldsymbol{v}\,ds,$$
$$\varphi(U(t,0)\boldsymbol{v}) \le \varphi(\boldsymbol{v}), \qquad \textit{for each } \boldsymbol{v} \in D \textit{ and } t \in [0,\tau].$$

REMARK 5.4. If the coefficients $d_i(\cdot)$ are assumed to satisfy the stronger condition

(D) $\quad d_i(\cdot) \in C^1([0,\tau]; L^\infty(\Omega))$ and $d_i(t,x) \ge 0$

for $i = 1,2,3$, $t \in [0,\tau]$, and for almost all $x \in \Omega$,

then the operator $(t,\boldsymbol{v}) \mapsto B(t)\boldsymbol{v}$ from $[0,\tau] \times D$ into X is continuously Fréchet differentiable in the sense discussed in [5]. In this case it is possible to show that if $\boldsymbol{v} \in D(A) \cap D$ such that $(A + B(0))\boldsymbol{v} \in Y \equiv \overline{D(A)}$, then $U(\cdot,0)\boldsymbol{v}$ gives a C^1-solution of (SP).

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References

[1] W. Arendt, Vector valued Laplace transforms and Cauchy problems, Israel J. Math., 59 (1987), 327-352.

[2] J. Ball, Strongly continuous semigroups, weak solutions, and the variation of constants formula, Proc. Amer. Math. Soc., 63 (1977), 370-373.

[3] Ph. Bénilan, Équations d'évolution dans un espace de Banach quelconque et applications, Thése, Orsay (1972).

[4] Ph. Bénilan, M. G. Crandall and A. Pazy, "Bonnes solutions" d'un problème d'evolution semi-linéaire, C. R. Acad. Sci., 306 (1988), 527-530.

[5] Ph. Clément, K. Hashimoto, S. Oharu and B. de Pagter, Nonlinear perturbations of dual semigroups, preprint.

[6] G. Da Prato and E. Sinestrari, Differential operators with non-dense domains, Ann. Sc. Norm. Pisa, 14 (1987), 285-344.

[7] G. Da Prato and E. Sinestrari, Non autonomous evolution operators of hyperbolic type, Semigroup Forum 45 (1992), 302-321.

[8] P. Fife, Mathematical aspects of reacting and diffusion systems, Lecture Notes in Biomath., 28, Springer-Verlag, Berlin, 1979.

[9] T. Iwamiya, Global existence of mild solutions to semilinear differential equations in Banach spaces, Hiroshima Math. J., 16 (1986), 499-530.

[10] H. Kellermann and M. Hieber, Integrated semigroups, J. Func. Anal., 84 (1989), 160-180.

[11] K. Kobayasi, Y. Kobayashi and S. Oharu, Nonlinear evolution operators in Banach spaces, Osaka J. Math., 21 (1984), 281-310.

[12] K. Kobayasi and S. Oharu, On nonlinear evolution operators associated with certain nonlinear equations of evolution, Mathematical Analysis on Structures in Nonlinear Phenomena, Lecture Notes in Num. Appl. Anal., 2 (1980), 139-210.

[13] Y. Konishi, Sur un systèm dégenéré dés équations paraboliques semilinéairs avec les conditions aux limites nonlinéaires, J. Fac. Sci. Univ. Tokyo, 19 (1972), 353-361.

[14] G. Lumer, Solutions généralisées et semi-groupes intégrés, C. R. Acad. Sci. Paris, 310, Série I (1990), 577-582.

[15] G. Lumer, Examples and results concerning the behavior of generalized solutions, integrated semigroups, and dissipative evolution problems, Semigroup Theory and Evolution Equations (Ph. Clément, E. Mitidieri, B. de Pagter eds.), Lecture Notes in Pure and Applied Mathematics 135, Marcel Dekker, New York, 1991, 347-356.

[16] M. Mimura and A. Nakaoka, On some degenerate diffusion system related with a certain reaction system, J. Math. Kyoto Univ. (JMKYAZ), 12-1, (1972), 95-121.

[17] T. Matsumoto, S. Oharu and H. R. Thieme, Nonlinear perturbations of a class of integrated semigroups, to appear in Hiroshima Math. J., 26 (1996).

[18] F. Neubrander, Integrated semigroups and their applications to the abstract Cauchy problem, Pacific J., 135 (1988), 111-155.

[19] S. Oharu and T. Takahashi, Characterization of nonlinear semigroups associated with semilinear evolution equations, Trans. Amer. Math. Soc., 311 (1989), 593-619.

[20] H. R. Thieme, "Integrated semigroups" and integrated solutions to abstract Cauchy problems, J. Math. Anal. Appl., 152 (1990), 416-447.

[21] H. R. Thieme, Semiflows generated by Lipschitz perturbations of non-densely defined operators, Differential and Integral Equations, 3 (1990), 1035-1066.

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