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CHARACTERIZATIONS OF CERTAIN REAL HYPERSURFACES OF A COMPLEX SPACE FORM

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0. Introduction

Let $M_n(c)$ be an n-dimensional complex space form with constant holomorphic sectional curvature c. It is well known that a complete and simply connected complex space form consists of a complex projective space P_nC, a complex Euclidean space C_n or a complex hyperbolic space H_nC according as $c > 0, c = 0$ or $c < 0$. In this paper we consider a real hypersurface M of P_nC or H_nC. The real hypersurface M has an almost contact metric structure (ϕ, ξ, η, g) induced from the complex structure J of $M_n(c)$.

The study of real hypersurfaces of P_nC was initiated again by Takagi[14], who proved that all homogeneous real hypersurfaces of P_nC could be divided into the six model spaces (cf. the case $c > 0$ of Theorem A). Recently, Kimura and Maeda[7] characterized a geodesic hypersphere M in P_nC in terms of the derivative of the Ricci tensor S. Moreover, they investigated real hypersurfaces M in terms of curvature operator $R(X, Y)$ of M on the Ricci tensor S and the shape operator A.

On the other hand, real hypersurfaces of H_nC have also been investigated by Berndt[1], Montiel[10], Montiel and Romero[11], etc. In particular, by using the notions of the tube in Cecil and Ryan[2], Montiel[10], also classified the real hypersurfaces of H_nC with at most two distinct principal curvatures. Recently, Berndt[1] classified all real hypersurfaces with constant principal curvatures of H_nC (cf. the case $c < 0$ of Theorem A).

The main purpose of this paper is to give characterizations of real hypersurfaces of type A_0, A_1 and A_2 of H_nC, and to compare the real hypersurfaces of H_nC with those of P_nC under the same conditions. In

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the section 2, we study the real hypersurfaces of H_nC corresponding to the real hypersurfaces of type A_1 and A_2 (resp. type A_1) of P_nC in terms of the derivative of the shape operator A (resp. the Ricci tensor S). In the last section, we investigate homogeneous real hypersurfaces of $M_n(c)$ in terms of the curvature operator $R(X, Y)$ on S and A.

1. Preliminaries

Let M be a real hypersurface of a complex n-dimensional complex space form $M_n(c)$, and let N be its local unit normal vector field. Let us denote by J the almost complex structure of $M_n(c)$. For any tangent vector field X and normal vector field N on M, the transformations of X and N under J can be given by

$$JX = \phi X + \eta(X)N, \quad JN = \xi,$$

where ϕ defines a skew-symmetric transformation on the tangent bundle TM of M, while η and ξ denote a 1-form and a vector field on a neighborhood of x in M respectively. Moreover, it is seen that $g(\xi, X) = \eta(X)$, where g denotes the induced Riemannian metric on M. By properties of the almost complex structure J, the set (ϕ, ξ, η, g) of tensors satisfies

$$\phi^2 = -I + \eta \otimes \xi, \quad \phi\xi = 0, \quad \eta(\phi X) = 0, \quad \eta(\xi) = 1, \tag{1.1}$$

where I denotes the identity matrix. Furthermore, the covariant derivatives of the structure tensors are given by

$$\nabla_X\phi(Y) = \eta(Y)AX - g(AX, Y)\xi, \quad \nabla_X\xi = \phi AX, \tag{1.2}$$

where ∇ is the Riemannian connection of g and A denotes the shape operator with respect to N on M. Since the ambient swpace is of constant holomorphic sectional curvature $4c$, the equations of Gauss and Codazzi are respectively given as follows

$$\begin{aligned} R(X, Y)Z =& c\{g(Y, Z)X - g(X, Z)Y + g(\phi Y, Z)\phi X \\ & - g(\phi X, Z)\phi Y - 2g(\phi X, Y)\phi Z\} \\ & + g(AY, Z)AX - g(AX, Z)AY, \end{aligned} \tag{1.3}$$

$$\nabla_X A(Y) - \nabla_Y A(X) = c\{\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi\}, \tag{1.4}$$

where R denotes the Riemannian curvature tensor of M and $\nabla_X A$ denotes the covariant derivative of the shape operator A with respect to X.

The Ricci tensor S' of M is the tensor of type $(0,2)$ given by $S'(X,Y) = tr\{Z \to R(Z,X)Y\}$. Also it may be regarded as the tensor of type $(1,1)$ and denoted dby $S : TM \to TM$; it satisfies $S'(X,Y) = g(SX,Y)$. By the Gauss equation, (1.1) and (1.2), the Ricci tensor S is given by

$$SX = c\{(2n+1)X - 3\eta(X)\xi\} + hAX - A^2X, \tag{1.5}$$

$$\begin{aligned} \nabla_X S(Y) = &- 3c\{g(\phi AX, Y)\xi + \eta(Y)\phi AX\} \\ &+ (Xh)AY + h\nabla_X A(Y) - \nabla_X A^2(Y), \end{aligned} \tag{1.6}$$

where h is the trace of the shape operator A. A real hypersurface M of $M_n(c)$ is said to be *pseudo-Einstein* if the Ricci tensor S satisfies

$$SX = aX + b\eta(X)\xi$$

for any vector field X of tangent to M and some functions a and b on M. An eigenvector X of the shape operator A is called a *principal curvature vector*. Also an eigenvalue λ of A is called a *principal curvataure*. We denote by V_λ the eigenspace of A associated with eigenvalue λ. Now we introduce the notion of an *η-parallel shape operator* A(resp. *η-parallel Ricci tensor S*) of M in $M_n(c)$, $c \neq 0$, which is defined by $g(\nabla_X A(Y), Z) = 0$ (resp. $g(\nabla_X S(Y), Z) = 0$) for any X, Y and Z orthogonal to ξ.

In the following, we use the same terminology and notations as above unless otherwise stated. Now we prepare the following theorems in order to prove our results.

Theorem A. *Let M be a real hypersurface of $M_n(c)$, $c \neq 0$. Then M has constant principal curvatures and ξ is principal if and only if M is locally congruent to a tube of radius r over one of the following Kaehler submanifolds:*

In the case $c>0$([5], [14]),

(A_1) *hyperplane* $P_{n-1}C$, *where* $0<r<\frac{\pi}{2}$,
(A_2) *totally geodesic* $P_kC(1\le k\le n-2)$, *where* $0<r<\frac{\pi}{2}$,
(B) *complex quadric* Q_{n-1}, *where* $0<r<\frac{\pi}{4}$,
(C) $P_1C\times P_{(n-1)/2}C$, *where* $0<r<\frac{\pi}{4}$ $n(\ge 5)$ *is odd,*
(D) *complex Grassmann* $G_{2,5}C$ *where* $0<r<\frac{\pi}{4}$ *and* $n=9$,
(E) *Hermitian symmetric space* $SO(10)/U(5)$, *where* $0<r<\frac{\pi}{4}$ *and* $n=15$.

In the case $c<0$ ([1]),

(A_0) *horosphere (or Montiel tube) in* H_nC,
(A_1) *geodesic hypersphere* H_0C *or complex hyperbolic hyperplane* $H_{n-1}C$,
(A_2) *totally geodesic* $H_kC(1\le k\le n-2)$
(B) *totally real hyperbolic space* H_nR.

Theorem B. ([11], [12]). *Let* M *be a real hypersurface of* $M_n(c)$. *Then* M *satisfies* $\phi A=A\phi$ *if and only if* M *is locally congruent to one of type* A_1 *and* A_2 *when* $c>0$ *and of type* A_0, A_1 *and* A_2 *when* $c<0$.

Theorem C. ([2], [8], [11]). *Let* M *be a real hypersurface of* $M_n(c), n\ge 3$ *whose Ricci tensor is pseudo-Einstein. Then* M *is locally congruent to one of type* A_1, A_2 *and* B *when* $c>0$, *and of type* A_0 *and* A_1 *when* $c<0$.

Theorem D. ([3]). *Let* M *be a real hypersurface in* $M_n(c), n\ge 3$. *Then* M *is pseudo-Einstein if and only if* M *satisfies*

$$(R(X,Y)S)Z+(R(Y,Z)S)X+(R(Z,X)S)Y=0$$

for any $X,Y,Z\in TM$ *in* $M_n(c)$.

Theorem E. ([11], [13]). *Let* M *be a real hypersurface of* $M_n(c), n\ge 2$. *Then the Ricci tensor* S *is* η*-parallel and the structure vector* ξ *is principal if and only if* M *is locally congruent to one of type* A_1, A_2 *and* B *when* $c>0$ *and of type* A_0, A_1, A_2 *and* B *when* $c<0$.

Theorem F. ([9]). *Let* M *be a real hypersurface of* $P_nC, n\ge 3$. *Then the following are equivalent:*

(a) M *is locally congruednt to one of type* A_1 *and* A_2,
(b) $\nabla_X A(Y)=-g(\phi X,Y)\xi-\eta(Y)\phi X$ *for any* $X,Y\in TM$.

Theorem G. ([6], [7]). *Let M be a real hypersurface of $P_nC, n \geq 3$. Then the following are equivalent:*

(c) *M is locally congruent to a geodesic hypersphere,*

(d) *$\nabla_X S(Y) = \kappa\{g(\phi X, Y)\xi + \eta(Y)\phi X\}$ for any $X, Y \in TM$, where κ is a function on M.*

Proposition A. *Let M be a real hypersurface of $M_n(c), c \neq 0$. If ξ is a principal curvature vector, then the corresponding principal curvature α is locally constant.*

Proposition B. *Let M be a real hypersurface of $M_n(c), c \neq 0$ and let $A\xi = \alpha\xi$. If $AX = \lambda X$ for $X \perp \xi$, then we have $A\phi X = (\alpha\lambda + 2c)/(2\lambda - \alpha)\phi X$.*

For the case $c = 1$ in $M_n(c)$ Y. Maeda [9], and Ki and Suh [4] proved the Propositions A and B. By using their methods we can simply obtain the above Propositions.

2. Supppliement theorems in $M_n(c)$

In this section, we will use later on the following lemma 2.1 that has been proved by Y. Maeda [9], Ki and Suh [4].

Lemma 2.1. *There exist no open sets O in M of $M_n(c), c \neq 0$ such that $\phi A + A\phi = 0$ at any point of O.*

Now we assume that a real hypersurface M of $M_n(c)$ satisfies

$$\nabla_X A(Y) = -\{g(\phi X, Y)\xi + \eta(Y)\phi X\} \tag{2.1}$$

for any vector fields X and Y tangent to M. Using the Ricci identity for (2.1) and making use of (1.2) we find

$$\begin{aligned} g(AY, W)g(LY, W) + g(AY, Z)g(LX, W) \\ -g(AX, W)g(LY, Z) - g(AX, Z)g(LY, W) \\ +g(\phi Y, W)g(BX, Z) + g(\phi Y, Z)g(BX, W) \\ -g(\phi Y, W)g(BY, Z) - g(\phi X, Z)g(BY, W) - 2g(\phi X, Y)g(BW, Z) = 0, \end{aligned} \tag{2.2}$$

where L and B are (1,1)type tensor fields defined by the following:

$$LX = cX - c\eta(X)\xi - A^2X, \quad BX = c(\phi A - A\phi)X. \tag{2.3}$$

Therefore L and B are symmetric operators. If $B = 0$, then $\phi A = A\phi$.

Let $e_1, \cdots, e_{2n-1}$ be local vector fields of orthonormal frames on M and contract (2.2) with X and W, we find

$$(2.4)\qquad \begin{aligned}&(trA)g(LY, Z) - \{(2n+2)c - trA^2\}g(AY, Z)\\&+ 2c\eta(AY)\eta(Z) + 2c\eta(AZ)\eta(Y) - 4cg(\phi A\phi Y, Z) = 0.\end{aligned}$$

Replacing Y by ξ in (2.4) and using (1.1), we have

$$(2.5)\qquad (trA)\eta(A^2X) = 2c\alpha\eta(X) - (2nc - trA^2)\eta(AX),$$

where $\alpha = \eta(A\xi)$.

On the other hand, putting $X = Z = \xi$ in (2.2) and exchanging Y and W, we get by taking skew symmetric parts

$$(2.6)\qquad \eta(AY)\eta(A^2W) = \eta(AW)\eta(A^2Y),$$

from which implies, for some scalar a,

$$(2.7)\qquad g(A^2X, \xi) = ag(AX, \xi),$$

where we have used Schwarz's inequality. From (2.4) and (2.7) we have

$$(2.8)\qquad b\eta(AX) = 2c\alpha\eta(X),$$

where $b = 2nc + atrA - trA^2$.

Lemma 2.2. *The structure vector ξ is a principal curvature vector for any point in a real hypersurface M of $M_n(c)$ safisfying* (2.1).

Proof. If $b \neq 0$, then ξ is a principal curvature vector by (2.8). If $b = 0$, then $\alpha = \eta(A\xi) = 0$. Putting $Y = \xi$ in (2.6), we get $A\xi = 0$. □

Lemma 2.3. *Let M be a realhypersurface of $M_n(c)$ satisfying* (2.1). *Then ϕ and A are commutative.*

Proof. Lemma 2.2 shows that we can put $A\xi = \alpha\xi$ for any point in M. Then by Proposition A we see that α is constant. Differentiating this equation and using (2.1), we get

$$(2.9)\qquad \alpha g(\phi AX, Y) = -cg(\phi X, Y) + g(A\phi AX, Y).$$

Exchanging X and Y in (2.9), we have $\alpha g((\phi A - A\phi)X, Y) = 0$. If $\alpha \neq 0$, it is clear. If $\alpha = 0$, we replace W by ϕW in (2.2) and contract X and W. Then we have

$$(2.10) \qquad (2n-2)g(BY, Z) + g(\phi AY, LZ) + G(\phi aZ, LY) \\ - g(\phi^2 Y, BZ) - g(\phi^2 Z, BY) = 0.$$

Substituting (2.3) into (2.10), we find

$$(2n+1)g(BY, Z) - g(\phi AY, A^2 Z) - g(\phi AZ, A^2 Y) = 0,$$

from which implies

$$(2.11) \qquad g(\{(2n+1)c(\phi A - A\phi) + A(\phi A - A\phi)A\}Y, Z) = 0$$

for any $Y, Z \in TM$. This means $\phi A = A\phi$ because of (2.9).□

From Lemma 2.3 and Theorem B we have the following supplement Theorem of Theorem F.

Theorem 2.1. *Let M be a real hypersurface of $M_n(c), n \geq 3$. Then the following are equivalent:*

(a) *M is locally congruent to one of type A_1 and A_2 when $c > 0$ and of A_0, A_1 and A_2 when $c < 0$,*
(b) *$\nabla_X A(Y) = -c\{g(\phi X, Y)\xi + \eta(Y)\phi X\}$ for any $X, Y \in TM$.*

Remark. We can prove Theorem 2.1 by using the condition of cyclic Ryan (cf. [3]) that is given by Ki, Nakagawa and Suh. The Riemannian manifold M is said to be *cyclic Ryan* if it satisfies $\mathfrak{S}(R(X,Y)S)(Z) = 0$ for any vector fields, where R, S and $\mathfrak{S}$ denote the Riemannian curvature tensor, the Ricci tensor and the cyclic sum with respect to X, Y and Z, respectively.

Now we prove the following supplement Theorem of Theorem G:

Theorem 2.2. *Let M be a real hypersurface of $M_n(c), n \geq 3$. Then the following are equivalent:*

(c) *M is locally congruent to type A_1 when $c > 0$, and one of type A_0 and A_1 when $c < 0$*
(d) *$\nabla_X S(Y) = k\{g(\phi X, Y)\xi + \eta(Y)\phi X\}$ for any $X, Y \in TM$, where k is locally non-zero constant.*

Proof. Suppose that the condition (d) holds. From this condition (d) and (1.2) we have

$$\begin{aligned}(2.12)\qquad &\nabla_W(\nabla_X S)(Y) - \nabla_{\nabla_W X} S(Y)\\ &= k\{\eta(X)g(AW,Y)\xi - 2\eta(Y)g(AW,X)\xi + g(\phi X,Y)\phi AW\\ &+ g(\phi AW,Y)\phi X + \eta(X)\eta(Y)AW\},\end{aligned}$$

from which yields

$$\begin{aligned}(2.13)\qquad (R(W,X)S)Y = {} & k\{\eta(X)g(AW,Y)\xi - \eta(W)g(AX,Y)\xi\\ &+ g(\phi X,Y)\phi AW - g(\phi W,Y)\phi AX + g(\phi AW,Y)\phi X\\ &- g(\phi AX,Y)\phi W + \eta(Y)(\eta(X)AW - \eta(W)AX)\}.\end{aligned}$$

Let $e_1,\cdots,e_{2n-1}$ be local vector fields of orthonormal frames on M. From (1.1) and (2.13) we find

$$\begin{aligned}(2.14)\qquad &\sum g((R(e_i,X)S)Y,e_i) = k\{\eta(X)\eta(AY)\\ &-2\eta(Y)\eta(AX) - g(A\phi Y,\phi X) + (trA)\eta(X)\eta(Y)\}.\end{aligned}$$

Since the left hand side of (2.14) is symmetric with respective to X and Y, the equation (2.14) implies

$$k\{\eta(X)\eta(AY) - 2\eta(Y)\eta(AX)\} = k\{\eta(Y)\eta(AX) - 2\eta(X)\eta(AY)\}.$$

Since $k(\neq 0)$ is constant, the above equation shows that

$$(2.15)\qquad \eta(X)\eta(AY) = \eta(Y)\eta(AX)$$

for any $X,Y \in TM$. The equation (2.15) tell us that ξ is principal. Moreover, the condition (d) shows that the Ricci tensor S of our real hypersurface M is pseudo-parallel. Therefore Theorem E assert that M is locally congruent to one of type A_1, A_2 and B when $c > 0$ and of type A_0, A_1, A_2 and B when $c < 0$.

Conversely, we must to check the condition (d) one by one for the above model spaces. But Kimura and Maeda [7] checked for the case $c > 0$. So, the rest of the proof is to check for the case $c < 0$.

Let M be of type A_0 in H_nC. In this case M has two distinct constant principal curvatures $\alpha = 2$ with multiplicity 1 and $\lambda = 1$ with multiplicity

$2n-2$. Let X be a principal curvature unit vector orthogonal to ξ with principal curvature λ. So that the shape operator A can be defined by

$$AX = X + \eta(X)\xi \tag{2.16}$$

for $X \in TM$. Substituting the condition (b) of Theorem 2.1 and (2.16) into (1.6), it is easily seen that M satisfies the condition (d) of Theorm 2.2, that is,

$$\nabla_X S(Y) = -2nc\{g(\phi X, Y)\xi + \eta(Y)\phi X\}. \tag{2.17}$$

Let M be of type A_1 in H_nC. Then M has two distinct constant principal curvatures $\alpha = 2coth(2r)$ and $\lambda = tanh(r)$ if $0 < \lambda < 1$ or $\lambda = coth(r)$ if $\lambda > 1$. Let X be a principal curvature unit vector orthogonal to ξ with principal curvature λ. So that A can be expressed by (cf. Takagi [15])

$$AX = \lambda X + \frac{1}{\lambda}\eta(X)\xi \tag{2.18}$$

for $X \in TM$. Substituting the condition (b) of Theorem 2.1 and (2.18) into (1.6), we find

$$\nabla_X S(Y) = -2nc\, tanh(r)\{g(\phi X, Y)\xi + \eta(Y)\phi X\}. \tag{2.19}$$

Therefore M of type A_1 satisfies the condition (d) of Theorem 2.2.

Let M be of type A_2 in H_nC. Then M has three distinct constant principal curvatures $\alpha = 2coth(2r)$ with multiplicity $1, \lambda = tanh(r)$ with multiplicity $2p$ and $\mu = coth(r)$ with multiplicity $2(n-p-1)$, where $1 \le p \le n-1, 0 < \lambda < 1$ (cf. Berndt [1]). Let X be a principal curvature unit vector orthogonal to ξ with principal curvature λ. We note that $\phi X \in V_\lambda$ because of Proposition A. From (1.6) and the condition (b) of Theorem 2.1, we obtain

$$\nabla_X S(\phi X) = -2c\{(p+1)tanh(r) + (n-p-1)coth(r)\}\xi. \tag{2.20}$$

Now let Y be a principal curvature unit vector orthogonal to ξ with principal curvature $\mu = coth(r)$. By the similar computation we have

$$\nabla_Y S(\phi Y) = -2c\{p\, tanh(r) + (n-p)coth(r)\}\xi. \tag{2.21}$$

From (2.20) and (2.21), it is easily seen that M of type A_2 does not satisfy the condition (d) in Theorem 2.2.

Let M be of type B in H_nC. Then M has three distinct constant principal curvatures $\alpha = 2tanh(2r)$ with multiplicity 1, $\lambda = tanh(r)$ with multiplicity $n-1$ and $\mu = coth(r)$ with multiplicity $n-1$ (cf. Berndt [1]). Let X be a principal curvature unit vector orthogonal to ξ with principal curvataure λ. Then we note that $\phi X \in V_\mu$ because of Proposition B and $T_pM = V_\lambda \oplus V_\mu \oplus \{\xi\}_R$ at any point p in M.

Now we choose a local vector field $\{e_1, \cdots, e_{n-1}, \phi e_1, \cdots, \phi e_{n-1}, \xi\}$ of orthonormal frames around a fixed point p in M such that $e_1, \cdots, e_{n-1}$ (resp. $\phi e_1, \cdots, \phi e_{n-1}$) is an orthonormal basis of V_λ (resp. V_μ). From the Codazzi equation (1.4) we get

$$\nabla_{\phi e_i} A(e_j) - \nabla_{e_j} A(\phi e_i) = 2c\delta_{ij}\xi. \tag{2.22}$$

On the other hand, we have

$$\begin{aligned} &\nabla_{\phi e_i} A(e_j) - \nabla_{e_j} A(\phi e_i) \\ &= \nabla_{\phi e_i}(Ae_j) - A\nabla_{\phi e_i} e_j - \nabla_{e_j}(A\phi e_i) + A\nabla_{e_j}(\phi e_i) \\ &= (\lambda I - A)\nabla_{\phi e_i} e_j - (\mu I - A)\nabla_{e_j}(\phi e_i). \end{aligned} \tag{2.23}$$

Then from (2.22) and (2.23) we obtain

$$\begin{aligned} &g((\lambda I - A)\nabla_{\phi e_i} e_j, e_k) - g((\mu I - A)\nabla_{e_j}(\phi e_i), e_k) \\ &= (\lambda - \mu) g(\nabla_{e_j}(\phi e_i), e_k) = 0. \end{aligned}$$

Thus we have

$$g(\nabla_{e_j}(\phi e_i), e_k) = 0$$

for $1 \leq i, j, k \leq n-1$, which implies

$$\begin{aligned} g(\nabla_{e_j} A(\phi e_i), e_k) &= g((\mu I - A)\nabla_{e_j}(\phi e_i), e_k) \\ &= (\mu - \lambda) g(\nabla_{e_j}(\phi e_i), e_k) = 0. \end{aligned} \tag{2.24}$$

Moreover, we find

$$\begin{aligned} g(\nabla_{e_j} A(\phi e_i), \xi) &= g((\mu I - A)\nabla_{e_j}(\phi e_i), \xi) \\ &= (\mu - \alpha) g(\nabla_{e_j}(\phi e_i), \xi \\ &= \lambda(\alpha - \mu) g(\phi e_i, \phi e_j). \end{aligned} \tag{2.25}$$

Therefore from (2.24) and (2.25) we have

$$\nabla_{e_j} A(\phi e_i) = \lambda(\alpha - \mu)\delta_{ij}\xi \tag{2.26}$$

for $1 \le i, j \le n-1$. By the similar computation,

$$\nabla_{\phi e_i} A(e_j) = \mu(\lambda - \alpha)\delta_{ij}\xi \tag{2.27}$$

for $1 \le i, j \le n-1$. From (1.6) and (2.26) we find

$$\nabla_X S(\phi X) = [-3c\lambda + \{(n-1)(\lambda+\mu) - \mu\}\lambda(\alpha - \mu)]\xi. \tag{2.28}$$

Next let Y be a principal curvature unit vector orthogonal to ξ with principal curvature μ. Using the similar computation we get, from (1.6) and (2.27),

$$\nabla_Y S(\phi Y) = [-3c\mu + \{(n-1)(\lambda+\mu) - \lambda\}\mu(\alpha - \lambda)]\xi. \tag{2.29}$$

Therefore we get $\nabla_X S(\phi X) \neq \nabla_Y S(\phi Y)$. In fact, if we assume that the equations (2.28) and (2.29) have the same coefficients of ξ, that is,

$$\{-3c + (n-1)(\lambda+\mu)\alpha - \lambda\mu\}(\lambda - \mu) = 0,$$

then we have

$$-3c + 2(n-1)(1-c) = 1 \quad \text{or} \quad \lambda = \mu,$$

because of $\lambda\mu = 1$ and $\lambda + \mu = 2(1-c)/\alpha$. This contradicts. Thus M does not satisfy the condition (d) of Theorem 2.2.□

3. Real hypersurfaces in terms of Ricci tensor and curvature operator

In this section, we are concerned with the condition about the covariant derivative of the Ricci tensor in $M_n(c)$. Next we consider the curvature operator $R(X, Y)$ in $M_n(c)$. First of all we introduce the following.

Theorem 3.1. *Let M be a real hypersurface of $M_n(c), n \ge 3$. Then M satisfies*

$$\nabla_X S(Y) = \kappa\{g(\phi X, Y)\xi + \eta(Y)\phi X\} \tag{3.1}$$

for any $X, Y \in TM$, where κ is a function on M if and only if M is locally congruent to a geodesic hypersphere in P_nC, and M is locally congruent to a horosphere, a geodesic hypersphere, or a complex hyperbolic hyperplane in H_nC.

Proof. For the real hypersurface M of P_nC, Kimura and Maeda ([6], [7]) proved this Theorem under the same condition by using the Ricci identity. If we use the same method in $M_n(c)$ as used by them, we can also obtain this Theorem 3.1. Thus we omit the proof. □

Moreover, by Theorem 3.1 we find the following.

Proposition 3.2. *Let M be a real hypersurface of $M_n(c), n \geq 3$. Then the Ricci tensor S satisfies*

$$\|\nabla S\|^2 \geq \frac{1}{n-1}\{2nc(h - \eta(A\xi)) + \phi A\xi h - tr(\phi A \nabla_\xi A)\}^2. \tag{3.2}$$

Moreover, the equality of (3.2) *holds if and only if M is locally congruent to type A_1 when $c > 0$, and one of type A_0 or A_1 when $c < 0$.*

Proof. We define the tensor T on M as

$$T(X, Y) = \nabla_X S(Y) - \kappa g(\phi X, Y)\xi - \kappa\eta(Y)\phi X,$$

where κ is a function on M. Here we choose a local vector field $\{e_i\}$ of orthonormal frames of M. Calculating the length of T we have

$$0 \leq \|T\|^2 = \|\nabla S\|^2 - 4\kappa \sum g(\nabla_{e_i} S(\xi), \phi e_i) + 4(n-1)\kappa^2. \tag{3.3}$$

Since (3.3) is an inequality for any real number κ, taking the discriminant of (3.3) we have

$$\|\nabla S\|^2 \geq \frac{1}{n-1}\{\sum g(\nabla_{e_i} S(\xi), \phi e_i)\}^2. \tag{3.4}$$

On the other hand, from (1.1) and (1.6) we have

$$\begin{aligned}\sum g(\nabla_{e_i} S(\xi), \phi e_i) = &\sum g(-3c\phi A e_i + (e_i h)A\xi \\ &+ (hI - A)\nabla_{e_i} A(\xi) - \nabla_{e_i} A(A\xi), \phi e_i).\end{aligned}$$

Using the Codazzi equation (1.4), the above equation becomes

$$\sum g(\nabla_{e_i} S(\xi), \phi e_i) = -2nc(h - \eta(A\xi)) - \phi A\xi h + tr(\phi A \nabla_\xi A). \tag{3.5}$$

Thus the equations (3.4) and (3.5) show that (3.2) holds. □

Now we consider the curvature operator $R(X, Y)$ in the complex space form $M_n(c)$. Here we shall prove the following.

Theorem 3.3. *Let M be a real hypersurface in $M_n(c), n \geq 3$. If M satisfies*

$$(R(W,X)S)Y = \kappa\{\eta(X)(g(W,Y)\xi + \eta(Y)W - \eta(W)(g(X,Y)\xi + \eta(Y)X)\}, \tag{3.6}$$

where κ is a function on M and $W, X, Y \in TM$. Then M is locally congruent to a tube of radius r over the following Kaehlerian submanifolds:

In case $c > 0$ ([7]),

(1) *hyperplane $P_{n-1}C$, where $0 < r < \frac{\pi}{2}$,*
(2) *totally geodesic $P_{(n-1)/2}C$, when $r = \frac{\pi}{4}$.*

In case $c < 0$,

(1) *horosphere in H_nC,*
(2) *geodesic hypersphere or complex hyperbolic hyperplane in H_nC.*

Proof. For the case P_nC, Theorem 3.3. was proved by Kimura and Maeda [7]. Here we shall prove this Theorem in the case H_nC. Since M of Theorem 3.3 is a cyclic Ryan (cf. [3]), Our real hypersurface M must be pseudo-Einstein because of Theorem D. And hence Theorem C shows that M is locally congruent to one of type A_0 and A_1 when $c < 0$.

Conversely, let M be one of type A_0 and A_1. Theorem 3.1 asserts that M satisfies the condition (b). By making use of the Ricci identity and using (1.2), we find that M satisfies the equation (2.13) in the proof of Theorem 2.2.

Now let M be of type A_0 in H_nC. Then M has two distinct constant principal curvatures $\alpha = 2$ and $\lambda = 1$. Thus the shape operator A of M can be expressed as (2.16). Substituting (2.16) nto (2.13), we get (3.6).

Next let M be of type A_1 in H_nC. Then M has two distinct constant principal curvatures $\alpha = 2coth(2r)$ and $\lambda = tanh(r)$ (or $coth(r)$). Thus the shape operator A can be defined by (2.18). Substituting (2.18) into (2.13), we have (3.6). □

Now we put the tensor T on M of $M_n(c)$ as the following:

$$\begin{aligned} T(W,X,Y) =&(R(W,X)S)Y - \kappa\{\eta(X)(g(W,Y)\xi + \eta(Y)W) \\ &- \eta(W)(g(X < Y)\xi + \eta(Y)X)\}, \end{aligned}$$

where κ is a function on M. By the same computation as in Proposition 3.2 where we have used the equations (1.1), (1.3), (1.5) and (1.6), we find the length of the curvature tensor for the Ricci tensor as the following.

Proposition 3.4. *Let M be a real hypersurface of $M_n(c), n \geq 3$. Then the curvature tensor satisfies*

$$(3.7)\quad \|RS\|^2 \geq \frac{2}{n-1}\{\|S\xi\|^2 - c\rho + c\eta(S\xi) - \eta(A\xi)tr(AS) + \eta(ASA\xi)\}^2,$$

where ρ is the scalar curvature of M. Moerover, the equality of (3.7) holds if and only if M is locally congruent to one of type A_1 and totally geodesic $P_{(n-1)/2}C, r = \frac{\pi}{4}$ when $c > 0$, and of type A_0 or A_1 when $c < 0$.

Theorem 3.5. *Let M be a real hypersurface of $M_n(c), n \geq 2$. If M satisfies*

$$(3.8)\qquad (R(X,Y)A)Z + (R(Y,Z)A)X + (R(Z,X)A)Y = 0.$$

Then M is locally congruent to type $A_1, n \geq 3$, and a real hypersurface in P_2C on which ξ is principal when $c > 0$, and of type A_0, A_1 and a real hypersurface in H_2C on which ξ is principal when $c < 0$.

Proof. From (1.2), (1.3) and (3.8) we find

$$(3.9)\qquad \begin{aligned} &g((\phi A + A\phi)X, Y)\phi Z + g((\phi A + A\phi)Y, Z)\phi X \\ &+ g((\phi A + A\phi)Z, X)\phi Y - 2g(\phi X, Y)\phi AZ \\ &- 2g(\phi Y, Z)\phi AX - 2g(\phi Z, X)\phi AY = 0, \end{aligned}$$

because of constant $c \neq 0$. Putting $X = e_i, Y = \phi e_i$, we have

$$(3.10)\quad (h - \eta(A\xi))\phi Z - (2n-3)\phi AZ - A\phi W + \eta(A\phi Z)\xi - 2\eta(Z)\phi A\xi = 0.$$

Replacing Z by ξ in (3.10) we get $\phi A\xi = 0$, which is that ξ is principal. Hence the equation (3.10) becomes

$$(3.11)\qquad (h - \eta(A\xi))\phi Z - (2n-3)\phi AZ - A\phi Z = 0.$$

For any $X, Y \in TM$, the equation (3.11) yields

$$(3.12)\quad (h - \eta(A\xi))g(\phi X, Y) - (2n-3)g(\phi AX, Y) - g(A\phi X, Y) = 0.$$

Exchanging the role of X and Y, we have also

$$(3.13)\quad (h - \eta(A\xi))g(\phi Y, X) - (2n-3)g(\phi AY, X) - g(A\phi Y, X) = 0.$$

From (3.12) and (3.13) we have

$$(2n-4)g((\phi A - A\phi)X, Y) = 0.$$

Hence we have $\phi A = A\phi$ in the case of $n \geq 3$. Thus, in the case of $n \geq 3$, by virtue of Theorem B our real hypersurface M is locally congruent to one of type A_1 and A_2 when $c > 0$, and of type A_0, A_1 and A_2 when $c < 0$.

Conversely, we must check the equation (3.8) for the above model spaces. But in case $c > 0$ Kimura and Maeda [7] have checked. So, let us check (3.8) for the three model spaces of type A_0, A_1 and A_2 one by one in case $c < 0$.

Let M be of type A_0 in H_nC. From (1.3) and (2.16) we find

$$(R(W,X)A)Y = (2+c)\{\eta(X)\eta(Y)W + \eta(X)g(W,Y)\eta - \eta(W)\eta(Y)X - \eta(W)g(X,Y)\xi\}, \tag{3.14}$$

from which satisfies (3.8).

Let M be of type A_1 in H_nC. From (1.3) and (2.18) we find

$$\begin{aligned}(R(W,X)A)Y =& (\lambda + \frac{1}{\lambda} + \frac{c}{\lambda})\{\eta(Y)(\eta(Z)X + g(Z,X)\xi) \\ &- \eta(X)(\eta(Z)Y + g(Z,Y)\xi)\}.\end{aligned} \tag{3.15}$$

This equation (3.15) satisfies (3.8).

Let M be of type A_2 in H_nC. Set $X \in V_\lambda, Y \in V_\mu$ and $\|X\| = \|Y\| = 1$. We note that $\phi X \in V_\lambda$ because of Proposition B. Hence from the Gauss equation (1.3) we find

$$(R(X,\phi X)A)Y + (R(\phi X, Y)A)X + (R(Y,X)A)\phi X = 2c(\lambda - \mu)\phi Y \neq 0.$$

Therefore in case $n \geq 3$ we assert that M satisfying (3.8) must be of one of type A_0 and A_1.

In case $n = 2$, let $A\xi = \alpha\xi$ and X be a principal curvature vector orthogonal to ξ with principal curvature r in $M_n(c)$. From (1.3) and Proposition B we find

$$\begin{aligned}&(R(X,\xi)A)\phi X + (R(\xi,\phi X)A)X + (R(\phi X, X)A)\xi \\ &= \frac{r\alpha + 2c}{2r - \alpha}R(X,\xi)\phi X + rR(\xi,\phi X)X + \alpha R(\phi X, X)\xi = 0.\end{aligned}$$

Thus the equation (3.8) is equivalent to the condition that ξ is principal. So, this proof is completed. □

Theorem 3.6. *Let M be a real hypersurface of $M_n(c), n \geq 2$. If M satisfies*

$$(R(W,X)A)Y = \kappa\{\eta(W)(\eta(Y)X + g(X,Y)\xi) - \eta(X)(\eta(Y)W + g(W,Y)\xi)\}, \tag{3.16}$$

where κ is a function on M and $W, X, Y \in TM$. Then M is locally congruent to type A_1 when $c > 0$, and one of type A_0 and A_1 when $c < 0$.

Proof. First we note that (3.16) satisfies (3.8). Therefore, in case $n \geq 3$, our real hypersurface M satisfying (3.18) must be of type A_1 because of Theorem 3.5. So, the rest of the proof is to study in case $n = 2$. Now we shall show that M must be homogeneous in $M_n(c)$. Let $A\xi = \alpha\xi$(see Proof of Theorem 3.5) and X be a principal curvature unit vector orthogonal to ξ with principal curvature λ. Then the Gauss equation (1.3) gives

$$g((R(X,\xi)A)\xi, X) = c\alpha + \alpha^2\lambda - c\lambda - \alpha\lambda^2, \tag{3.17}$$

$$g((R(\phi X,\xi)A)\xi, \phi X) = c\alpha + \alpha^2\frac{\alpha\lambda + 2c}{2\lambda - \alpha} - c\frac{\alpha\lambda + 2c}{2\lambda - \alpha} - \alpha\left(\frac{\alpha\lambda + 2c}{2\lambda - \alpha}\right)^2. \tag{3.18}$$

On the other hand, (3.16) implies

$$g((R(X,\xi)A)\xi, X) = g((R(\phi X,\xi)A)\xi, \phi X) = -\kappa. \tag{3.19}$$

From (3.17), (3.18) and (3.19) we have

$$\alpha^2\lambda - c\lambda - \alpha\alpha^2 = \alpha^2\frac{\alpha\lambda + 2c}{2\lambda - \alpha} - c\frac{\alpha\lambda + 2c}{2\lambda - \alpha} - \alpha\left(\frac{\alpha\lambda + 2c}{2\lambda - \alpha}\right)^2,$$

from which implies

$$(\lambda^2 - \alpha\lambda - c)\{2\alpha\lambda^2 - 2(\alpha^2 - c)\lambda + \alpha(\alpha^2 + c)\} = 0. \tag{3.20}$$

By virtue of Proposition A and (3.20) it is easily seen that λ is constant. Hence our real hypersurface M must be homogeneous in $M_n(c)$ because of Theorem F. So, we have only to prove that M of type B in H_nC does not satisfy (3.20). If M of type B is a tube of radius r over H_2R, then

T has three distinct constant principal curvatures $\alpha = 2tanh(2r), \lambda_1 = tanh(r)$ and $\lambda_2 = coth(r)$. It follows from these principal curvatures that $\lambda_1 + \lambda_2 = 4/\alpha$, which implies that the quadratic equation $\lambda^2 - \alpha\lambda - c = 0$ does not have solutions λ_1 and λ_2. Moreover, the quadratic equation $2\alpha\lambda^2 - 2(\alpha^2 - c)\lambda + \alpha(\alpha^2 + c) = 0$ does not have the solutions λ_1 and λ_2. In fact, we assume that λ_1 and λ_2 are solutions of this equation. Then this equation shows that

$$\lambda_1 + \lambda_2 = (\alpha^2 - c)/\alpha, \ \lambda_1\lambda_2 = (\alpha^2 + c)/2. \tag{3.21}$$

On the other hand, it follows from the principal curvatures λ_1 and λ_2 that

$$\lambda_1 + \lambda_2 = tanh(r) + coth(r) = 4/\alpha, \quad \lambda_1\lambda_2 = tanh(r)coth(r) = 1. \tag{3.22}$$

From (3.21) and (3.22) we have $c = \alpha^2 - 4$ and $c = \alpha^2 - 2$. We get a contradiction. This means that there is no real hypersurface M of type B in H_2C. □

Now we define the tensor T on M as:

$$\begin{aligned} T(W, X, Y) =& (R(W, X)A)Y - \kappa\{\eta(W)(\eta(Y)X + g(X, Y)\xi) \\ & - \eta(X)(\eta(Y)W + g(W, Y)\xi)\}. \end{aligned}$$

By the same discussion as Proposition 3.2 we find the following:

Proposition 3.7. *Let M be a real hypersurface of $M_n(c), n \geq 2$. Then we have*

$$\|RA\|^2 \geq \frac{2}{n-1}\{(2nc - c - trA^2)\eta(A\xi) + h(\|A\xi\|^2 - c)\}^2. \tag{3.23}$$

Moreover, the equality of (3.23) *holds if and only if M is locally congruent to type A_1 when $c > 0$, and one of type A_0 and A_1 when $c < 0$.*

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