

CHARACTERIZATIONS OF REAL HYPERSURFACES
IN COMPLEX SPACE FORMS
IN TERMS OF WEINGARTEN MAP

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ABSTRACT. This paper consists of two parts. One is to give the notion of the ruled real hypersurfaces in a complex space form $M_n(c)$, $c \neq 0$ and to calculate the expression of the covariant derivative of its Weingarten map. Moreover, we know that real hypersurfaces of type A also satisfy this expression. The other is to show that ruled real hypersurfaces or real hypersurfaces of type A are the only real hypersurfaces in $M_n(c)$ which satisfy this expression.

1. Introduction

A complex n -dimensional Kaehlerian manifold of constant holomorphic sectional curvature c is called a complex space form, which is denoted by $M_n(c)$. A complete and simply connected complex space form consists of a complex projective space P_nC , a complex Euclidean space C^n or a complex hyperbolic space H_nC , according as $c > 0, c = 0$ or $c < 0$. The induced almost contact metric structure of a real hypersurface M of $M_n(c)$ is denoted by (ϕ, ξ, η, g) .

There exist many studies about real hypersurfaces of $M_n(c)$. One of the first researches is the classification of homogeneous real hypersurfaces of a complex projective space P_nC by Takagi [14], who showed that these hypersurfaces of P_nC could be divided into six types which are said to be of type A_1, A_2, B, C, D , and E , and in [3] Cecil-Ryan and [7] Kimura proved that they are realized as the tubes of constant radius over Kaehlerian submanifolds if the structure vector field ξ is principal. Also Berndt [2] showed recently that all real hypersurfaces with constant principal curvatures of a complex hyperbolic space H_nC are realized as the tubes of constant radius over certain submanifolds when the structure vector field ξ is principal. Nowadays in H_nC they are said to be of type A_0, A_1, A_2 , and B .

1991 *Mathematics Subject Classification.* Primary 53C40; Secondary 53C15.

This paper was supported by NON DIRECTED RESEARCH FUND, Korea Research Foundation, 1994 and partly by TGRC-KOSEF and BSRI-94-1404 .

Now, let us consider the following conditions that the second fundamental tensor A of M in $M_n(c), c \neq 0$ satisfies

$$(1.1) \quad (\nabla_X A)Y = -\frac{c}{4}\{\eta(Y)\phi X + g(\phi X, Y)\xi\},$$

for any tangent vector fields X and Y of M .

Maeda [10] investigated the condition (1.1) and used it to find a lower bound of $\|\nabla A\|$ for real hypersurfaces in $P_n C$. In fact, it was shown that $\|\nabla A\|^2 \geq \frac{c^2}{4}(n-1)$ for such hypersurfaces and the equality holds if and only if the condition (1.1) holds. Moreover, in this case it was known that M is locally congruent to one of homogeneous real hypersurfaces of type A_1 , and A_2 . Also Chen, Ludden and Montiel [4] generalized this inequality to real hypersurfaces in $H_n C$ and showed that the equality holds if and only if M is congruent to one of type A_0, A_1 , and A_2 .

Now let T_0 be a distribution defined by a subspace $T_0(x) = \{X \in T_x M : X \perp \xi(x)\}$ in the tangent subspace $T_x M$ of a real hypersurface M of $M_n(c), c \neq 0$, which is orthogonal to the structure vector field ξ and holomorphic with respect to the structure tensor ϕ . If we restrict the formula (1.1) to the orthogonal distribution T_0 , then for any vector fields X and Y in T_0 the shape operator A of M satisfies the following conditions

$$(1.2) \quad (\nabla_X A)Y = -\frac{c}{4}g(\phi X, Y)\xi.$$

Thus the above condition (1.2) is weaker than the condition (1.1). Then it is natural that real hypersurfaces of type A in $M_n(c), c \neq 0$, should satisfy the condition (1.2).

On the other hand, as an example of special real hypersurfaces of $P_n C$ different from the above ones, we can give a ruled real hypersurface. Kimura [8] and the present author [1] obtained some properties about ruled real hypersurfaces of $P_n C$ and $H_n C, n \geq 3$, respectively. In particular, examples of minimal ruled real hypersurfaces of $P_n C$ and $H_n C, n \geq 3$, are constructed respectively.

Now let M be a ruled real hypersurface in $M_n(c), c \neq 0$, which will be constructed in section 3. Then it is well known that its structure vector field ξ is not principal. Thus we can put $A\xi = \alpha\xi + \beta U$, where U is a unit vector field in T_0 and α and β are smooth functions on M . Then it will be shown explicitly in section 3 that the shape operator A of the ruled real hypersurface M in $M_n(c)$ satisfies

$$(1.3) \quad (\nabla_X A)Y = f(X, Y)\xi,$$

where

$$(1.4) \quad f(X, Y) = \beta^2 \{g(X, U)g(Y, \phi U) + g(X, \phi U)g(Y, U)\} - \frac{c}{4}g(\phi X, Y)$$

for any vector field X and Y in the distribution T_0 . When the function β is identically vanishing, the condition (1.3) reduces to the condition (1.2). From this point of view the purpose of this paper is to treat the converse problem with the condition (1.3) and to generalize the main result in [6]. Namely we give a characterization of real hypersurfaces of type A or ruled real hypersurfaces in $M_n(c)$, $c \neq 0$, as the following

Theorem. *Let M be a real hypersurface of $M_n(c)$, $c \neq 0$, $n \geq 3$. If it satisfies (1.3), then M is locally congruent to a real hypersurface of type A or a ruled real hypersurface.*

In section 2 we recall some fundamental properties of real hypersurface of $M_n(c)$, $c \neq 0$. In section 3 let us calculate explicitly the component of $(\nabla_X A)Y$ for any vector fields X and Y in T_0 in the direction of the structure vector field. Finally in section 4 by using the new method of *mod* ξ we prove Lemmas 4.1 and 4.2, which will be useful to get the complete proof of the above Theorem.

The present author would like to express his sincere gratitude to the referee for his valuable suggestions and comments.

§2. Preliminaries.

Let M be a real hypersurface of $n(\geq 2)$ -dimensional complex space form $M_n(c)$ of constant holomorphic sectional curvature $c(\neq 0)$ and let C be a unit normal field on a neighborhood of a point x in M . We denote by J an almost complex structure of $M_n(c)$. For a local vector field X on a neighborhood of x in M , the images of X and C under the linear transformation J can be represented as

$$JX = \phi X + \eta(X)C, \quad JC = -\xi,$$

where ϕ defines a skew-symmetric transformation on the tangent bundle TM of M , while η and ξ denote a 1-form and a vector field on a neighborhood of x in M , respectively. Moreover, it is seen that $g(\xi, X) = \eta(X)$, where g denotes the induced Riemannian metric on M . By properties of the almost complex structure J , the set (ϕ, ξ, η, g) of tensors satisfies

$$\phi^2 = -I + \eta \otimes \xi, \quad \phi \xi = 0, \quad \eta(\phi X) = 0, \quad \eta(\xi) = 1,$$

where I denotes the identity transformation. Accordingly, the set is so called an *almost contact metric structure*. Furthermore the covariant derivative of the structure tensors are given by

$$(2.1) \quad (\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi, \quad \nabla_X \xi = \phi AX,$$

where ∇ is the Riemannian connection of g and A denotes the shape operator with respect to the unit normal C on M .

Since the ambient space is of constant holomorphic sectional curvature c , the equations of Gauss and Codazzi are respectively given as follows

$$(2.2) \quad R(X, Y)Z = \frac{c}{4} \{g(Y, Z)X - g(X, Z)Y + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y - 2g(\phi X, Y)\phi Z\} + g(AY, Z)AX - g(AX, Z)AY,$$

$$(2.3) \quad (\nabla_X A)Y - (\nabla_Y A)X = \frac{c}{4} \{\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi\},$$

where R denotes the Riemannian curvature tensor of M and $\nabla_X A$ denotes the covariant derivative of the shape operator A with respect to X .

Next we suppose that the structure vector field ξ is principal with corresponding principal curvature α . Then it is seen in [5] and [10] that α is constant on M and it satisfies

$$(2.4) \quad A\phi A = \frac{c}{4}\phi + \frac{1}{2}\alpha(A\phi + \phi A),$$

and hence, by (2.1) and (2.3), we get

$$(2.5) \quad \nabla_\xi A = -\frac{1}{2}\alpha(A\phi - \phi A).$$

§3. Ruled real hypersurfaces.

This section is concerned with necessary properties about ruled real hypersurfaces. First of all, we define a ruled real hypersurface M of $M_n(c)$, $c \neq 0$. Let $\gamma : I \rightarrow M_n(c)$ be any regular curve. For any $t \in I$ let $M_{n-1}^{(t)}(c)$ be a totally geodesic complex hypersurface through the point $\gamma(t)$ of $M_n(c)$ which is orthogonal to a holomorphic plane spanned by $\gamma'(t)$ and $J\gamma'(t)$. Set $M = \{x \in M_{n-1}^{(t)}(c) : t \in I\}$. Then, by the construction, M becomes a real hypersurface of $M_n(c)$, which is called

a ruled real hypersurface. Under this construction the ruled real hypersurface M of $M_n(c)$, $c \neq 0$, has some fundamental properties.

Let us put $A\xi = \alpha\xi + \beta U$, where U is a unit vector orthogonal to ξ and α and β ($\beta \neq 0$) are smooth functions on M . As is seen in [9], the shape operator A satisfies

$$(3.1) \quad AU = \beta\xi, \quad AX = 0$$

for any vector field X orthogonal to ξ and U . Then it turns out to be

$$(3.2) \quad A\phi X = -\beta g(X, \phi U)\xi, \quad \phi AX = 0, X \in T_0.$$

Next the covariant derivative $(\nabla_X A)Y$ with respect to X and Y in T_0 is explicitly expressed. The equation (2.3) of Codazzi gives us

$$(\nabla_X A)\xi - (\nabla_\xi A)X = -\frac{c}{4}\phi X.$$

By the direct calculation of the left hand side of the above relation and using the second equation of (3.2) we get

$$d\alpha(X)\xi + d\beta(X)U + \frac{c}{4}\phi X + \beta\nabla_X U - \nabla_\xi(AX) + A\nabla_\xi X = 0, X \in T_0.$$

Let T_1 be a distribution defined by a subspace $T_1(x) = \{u \in T_0(x) : g(u, U(x)) = g(u, \phi U(x)) = 0\}$. Since AX is expressed as the linear combination of ξ and U by (3.1), we can derive from (3.1), (3.2) and the above equation the following relations:

$$\beta\nabla_X U = \begin{cases} (\beta^2 - \frac{c}{4})\phi X, & X = U \\ 0, & X = \phi U \\ -\frac{c}{4}\phi X, & X \in T_1, \end{cases}$$

$$d\beta(X) = \begin{cases} 0, & X = U \\ \beta^2 + \frac{c}{4}, & X = \phi U \\ 0, & X \in T_1. \end{cases}$$

Using these relations we can obtain the component of $(\nabla_X A)Y$, $X, Y \in T_0$ in the direction of ξ . In fact, we have

$$\begin{aligned} g((\nabla_X A)Y, \xi) &= g((\nabla_X A)\xi, Y) = g(\nabla_X(A\xi) - A\nabla_X\xi, Y) \\ &= d\beta(X)g(Y, U) + \beta g(\nabla_X U, Y), \end{aligned}$$

from which together with the above equations it follows

$$(3.5) \quad (\nabla_X A)Y = f(X, Y)\xi, \quad X, Y \in T_0,$$

where we put

$$(3.6) \quad \begin{aligned} f(X, Y) \\ = \beta^2 \{g(X, U)g(Y, \phi U) + g(X, \phi U)g(Y, U)\} - \frac{c}{4}g(\phi X, Y). \end{aligned}$$

Accordingly it is seen that for any X and Y in T_0

$$g((\nabla_\xi A)X, Y) = \beta^2 \{g(X, U)g(Y, \phi U) + g(X, \phi U)g(Y, U)\},$$

from which it follows that

$$(3.7) \quad (\nabla_\xi A)X = \beta^2 \{g(X, \phi U)U + g(X, U)\phi U\} + g((\nabla_X A)\xi, \xi)\xi,$$

where $g((\nabla_X A)\xi, \xi) = d\alpha(X) + 2\beta g(AX, \phi U)$. When the function β in (3.6) vanishes, by virtue of (1.1) and (1.2) we know that real hypersurfaces of type A also satisfy the formula (3.5).

§4. Proof of the Theorem.

In this section we are only concerned with the proof of Theorem. Let M be the real hypersurface of $M_n(c)$, $c \neq 0$, $n \geq 3$. Throughout this section, unless otherwise stated, we assume that the structure vector field ξ is not principal. Then we can put $A\xi = \alpha\xi + \beta U$, where U is a unit vector field in the holomorphic distribution T_0 and α and β are smooth functions on M . First assume that the following condition on M :

$$(4.1) \quad (\nabla_X A)Y = f(X, Y)\xi, \quad X, Y \in T_0,$$

where

$$(4.2) \quad f(X, Y) = \beta^2 \{g(X, U)g(Y, \phi U) + g(Y, U)g(X, \phi U)\} - \frac{c}{4}g(\phi X, Y).$$

The equation (4.1) is also equivalent to

$$(4.3) \quad (\nabla_X A)\xi \equiv \beta^2 \{g(X, \phi U)U + g(X, U)\phi U\} - \frac{c}{4}\phi X \pmod{\xi}$$

for any vector field X in T_0 . Moreover, from (3.7) it follows that

$$(4.4) \quad (\nabla_\xi A)X \equiv \beta^2 \{g(X, \phi U)U + g(X, U)\phi U\} \pmod{\xi}, \quad X \in T_0.$$

The ξ -component of the vector field $(\nabla_\xi A)X$ is given by

$$(4.5) \quad g((\nabla_\xi A)X, \xi) = d\alpha(X) + 2\beta g(AX, \phi U)$$

for any vector field X in T_0 .

For any vector field Z the orthogonal decomposition in the direction of ξ is expressed as

$$Z = (Z)_0 + g(Z, \xi)\xi,$$

where $(Z)_0$ denotes the T_0 -component of Z . Since the component of the vector field $\nabla_X Y$ in the direction of ξ is given by $-g(\phi AX, Y)$ by the second equation of (2.1), we have the following orthogonal decomposition

$$\nabla_X Y = (\nabla_X Y)_0 - g(\phi AX, Y)\xi.$$

Using the above orthogonal decomposition and taking account of (4.1), (4.3) and calculating directly $(\nabla_X \nabla_Y A)Z$ which is defined by

$$(\nabla_X \nabla_Y A)Z = \nabla_X((\nabla_Y A)Z) - (\nabla_{\nabla_X Y} A)Z - (\nabla_Y A)(\nabla_X Z),$$

we see that it satisfies

$$(4.6) \quad \begin{aligned} (\nabla_X \nabla_Y A)Z &\equiv \beta^2 g(\phi AX, Y) \{g(Z, \phi U)U + g(Z, U)\phi U\} \\ &\quad + g(\phi AX, Z) [\beta^2 \{g(Y, \phi U)U + g(Y, U)\phi U\} - \frac{c}{4}\phi Y] \\ &\quad + [\beta^2 \{g(Y, \phi U)g(Z, U) + g(Y, U)g(Z, \phi U)\} \\ &\quad - \frac{c}{4}g(\phi Y, Z)] \phi AX \\ &\quad (\text{mod } \xi) \end{aligned}$$

for any vector fields X, Y and Z in T_0 .

On the other hand, it is well known that the Ricci formula for the shape operator A is given by

$$(\nabla_X \nabla_Y A)Z - (\nabla_Y \nabla_X A)Z = R(X, Y)(AZ) - A(R(X, Y)Z)$$

for any vector fields X, Y and Z . Accordingly, taking X, Y and Z in the distribution T_0 in the above Ricci formula and taking account of the Gauss equation (2.2) and

(4.6) we obtain

(4.7)

$$\begin{aligned}
& \beta^2 g((A\phi + \phi A)X, Y) \{g(Z, \phi U)U + g(Z, U)\phi U\} \\
& + g(\phi AX, Z) [\beta^2 \{g(Y, \phi U)U + g(Y, U)\phi U\} - \frac{c}{4}\phi Y] \\
& - g(\phi AY, Z) [\beta^2 \{g(X, \phi U)U + g(X, U)\phi U\} - \frac{c}{4}\phi X] \\
& + [\beta^2 \{g(Y, \phi U)g(Z, U) + g(Y, U)g(Z, \phi U)\} - \frac{c}{4}g(\phi Y, Z)] \phi AX \\
& - [\beta^2 \{g(X, \phi U)g(Z, U) + g(X, U)g(Z, \phi U)\} - \frac{c}{4}g(\phi X, Z)] \phi AY \\
& \equiv \frac{c}{4} \{g(Y, AZ)X - g(X, AZ)Y + g(\phi Y, AZ)\phi X - g(\phi X, AZ)\phi Y \\
& - 2g(\phi X, Y)\phi AZ\} \\
& - \frac{c}{4} \{g(Y, Z)AX - g(X, Z)AY + g(\phi Y, Z)A\phi X - g(\phi X, Z)A\phi Y \\
& - 2g(\phi X, Y)A\phi Z\} \\
& - g(Y, AZ)A^2 X + g(X, AZ)A^2 Y + g(Y, A^2 Z)AX - g(X, A^2 Z)AY \\
& \pmod{\xi}
\end{aligned}$$

for any vector fields X, Y and Z in T_0 . Putting $Z = U$ in (4.7), we have

(4.8)

$$\begin{aligned}
& \beta^2 g((A\phi + \phi A)X, Y)\phi U \\
& + g(\phi AX, U) [\beta^2 \{g(Y, \phi U)U + g(Y, U)\phi U\} - \frac{c}{4}\phi Y] \\
& - g(\phi AY, U) [\beta^2 \{g(X, \phi U)U + g(X, U)\phi U\} - \frac{c}{4}\phi X] \\
& + (\beta^2 + \frac{c}{4}) \{g(Y, \phi U)\phi AX - g(X, \phi U)\phi AY\} \\
& \equiv \frac{c}{4} \{g(Y, AU)X - g(X, AU)Y + g(\phi Y, AU)\phi X - g(\phi X, AU)\phi Y \\
& - 2g(\phi X, Y)\phi AU\} \\
& - \frac{c}{4} \{g(Y, U)AX - g(X, U)AY + g(\phi Y, U)A\phi X - g(\phi X, U)A\phi Y \\
& - 2g(\phi X, Y)A\phi U\} \\
& - g(Y, AU)A^2 X + g(X, AU)A^2 Y + g(Y, A^2 U)AX - g(X, A^2 U)AY \\
& \pmod{\xi}
\end{aligned}$$

for any vector fields X and Y in T_0 . In particular, putting $Y = U$ and supposing

that X is orthogonal to U in (4.8), we have

$$\begin{aligned}
(4.9) \quad & \beta^2 g((A\phi + \phi A)X, U)\phi U \\
& + g(\phi AX, U)\left(\beta^2 - \frac{c}{4}\right)\phi U - g(\phi AU, U)\left\{\beta^2 g(X, \phi U)U - \frac{c}{4}\phi X\right\} \\
& - \left(\beta^2 + \frac{c}{4}\right)g(X, \phi U)\phi AU \\
& \equiv \frac{c}{4}\{g(U, AU)X - g(X, AU)U + g(\phi U, AU)\phi X - g(\phi X, AU)\phi U \\
& - 2g(\phi X, U)\phi AU\} \\
& - \frac{c}{4}\{AX + 3g(X, \phi U)A\phi U\} \\
& - g(U, AU)A^2 X + g(X, AU)A^2 U + g(U, A^2 U)AX - g(X, A^2 U)AU \\
& \pmod{\xi}
\end{aligned}$$

for any vector field X in T_0 orthogonal to U . Taking the inner product of (4.9) with U , we have

$$\begin{aligned}
& (\beta^2 + c)g(AU, \phi U)g(X, \phi U) + \frac{c}{4}g(X, AU) \\
& - g(AU, AU)g(X, AU) + g(AU, U)g(X, A^2 U) = 0,
\end{aligned}$$

which yields that

$$\begin{aligned}
(4.10) \quad & g(AU, U)A^2 U + \left\{\frac{c}{4} - g(AU, AU)\right\}AU \\
& + (\beta^2 + c)g(AU, \phi U)\phi U \equiv 0 \pmod{\xi, U}.
\end{aligned}$$

Now we can consider that there is a vector field V in the distribution T_0 in such a way that AU is expressed as a linear combination of the vector fields ξ, U and V , where ξ, U and V are orthonormal. Namely, since the shape operator A is symmetric, we may put $AU = \beta\xi + \gamma U + \delta V$, where γ and δ are smooth functions on M .

Let us denote by M_0 an open subset consisting of points x in M at which $\beta(x) \neq 0$. Then we shall first prove the following property.

Lemma 4.1. *If $n \geq 3$, then $\delta = 0$ on M_0 , that is, it satisfies*

$$(4.11) \quad AU = \beta\xi + \gamma U.$$

Proof. Let M_1 be an open subset consisting of points x in M_0 at which $\delta(x) \neq 0$. Suppose that M_1 is not empty and there is a vector field W in the distribution T_0

in such a way that AV is expressed as a linear combination of the vector fields U, V and W , where U, V and W are orthonormal. Namely, since the shape operator A is symmetric, we may put

$$AV = \delta U + \epsilon V + \sigma W,$$

where ϵ and σ are smooth functions on M_1 . Consequently we have

$$A^2U = \beta(\alpha + \gamma)\xi + (\beta^2 + \gamma^2 + \delta^2)U + \delta(\gamma + \epsilon)V + \delta\sigma W.$$

Substituting this into (4.10), we get

$$(4.12) \quad (\gamma\epsilon - \beta^2 - \delta^2 + \frac{c}{4})V + (\beta^2 + c)g(\phi U, V)\phi U + \gamma\sigma W = 0.$$

Let $L(\xi, U, \phi U)$ be a distribution defined by a subspace $L_x(\xi, U, \phi U)$ in the tangent space $T_x M$ spanned by vectors ξ_x, U_x and ϕU_x at any point x in M_0 . Let T_1 be an orthogonal complement in the tangent bundle TM of the distribution $L(\xi, U, \phi U)$. Then the equation (4.8) is reduced to

$$\begin{aligned} & \beta^2 g((A\phi + \phi A)X, Y)\phi U \\ & + g(\phi AX, U)[\beta^2\{g(Y, \phi U)U + g(Y, U)\phi U\} - \frac{c}{4}\phi Y] \\ & + \frac{c}{4}g(\phi AY, U)\phi X + (\beta^2 + \frac{c}{4})g(Y, \phi U)\phi AX \\ \equiv & \frac{c}{4}\{g(Y, AU)X - g(X, AU)Y + g(\phi Y, AU)\phi X \\ & - g(\phi X, AU)\phi Y - 2g(\phi X, Y)\phi AU\} \\ & - \frac{c}{4}\{g(Y, U)AX + g(\phi Y, U)A\phi X - 2g(\phi X, Y)A\phi U\} \\ & - g(Y, AU)A^2X + g(X, AU)A^2Y + g(Y, A^2U)AX - g(X, A^2U)AY \\ & (\text{mod } \xi) \end{aligned}$$

for any vector fields X in T_1 and Y in T_0 . Taking the inner product of this equation with U , we have

$$\begin{aligned} & (\beta^2 + \frac{c}{4})g(AX, \phi U)g(Y, \phi U) + \frac{c}{4}\{-g(X, AU)g(Y, U) \\ & + g(\phi X, AU)g(Y, \phi U) + 2g(AU, \phi U)g(Y, \phi X)\} \\ & - g(AX, AU)g(Y, AU) + g(X, AU)g(A^2Y, U) = 0, \\ & X \in T_1, Y \in T_0, \end{aligned}$$

from which it follows that

$$\begin{aligned}
(4.13) \quad & \delta g(X, V)A^2U - \delta\{(\gamma + \epsilon)g(X, V) + \sigma g(X, W)\}AU \\
& + \left\{(\beta^2 + \frac{c}{4})g(AX, \phi U) - \frac{c}{4}\delta g(X, \phi V)\right\}\phi U \\
& - \frac{c}{4}\delta g(X, V)U + \frac{c}{2}\delta g(\phi U, V)\phi X \\
& \equiv 0 \pmod{\xi}, \quad X \in T_1.
\end{aligned}$$

Now let us show $\sigma = 0$. Firstly for a case where $\beta^2 + c = 0$ in (4.12) we have

$$(4.14) \quad \begin{cases} \gamma\epsilon - \beta^2 - \delta^2 + \frac{c}{4} = 0, \\ \gamma\sigma = 0. \end{cases}$$

If $\gamma = 0$, then $\beta^2 + \delta^2 = \frac{c}{4} > 0$. From this together with the assumption $\beta^2 + c = 0$ it makes a contradiction. Thus $\gamma \neq 0$. Hence from (4.14) we have $\sigma = 0$.

Secondly let us consider for a case where $\beta^2 + c \neq 0$ in (4.12). If there is a point x at which $g(\phi U, V) \neq 0$, then (4.12) gives the fact that ϕU belongs to $L_x(V, W)$. Since AU belongs to the subspace $L_x(\xi, U, V)$ at x in M_1 and A^2U also belongs to the subspace $L_x(\xi, U, V, W)$, we see by (4.13) that the vector ϕX at x can be expressed as the linear combination of ξ, U, V and W . This means that the dimension of the tangent space $T_x M$ is not greater than 4. A contradiction, because we have assumed $n \geq 3$. So, it implies that $g(\phi U, V) = 0$ on M_1 . Accordingly, by (4.12) we also get the above equation (4.14). If $\gamma \neq 0$, then $\sigma = 0$. Thus we suppose that $\gamma = 0$. Putting $X = V$ in (4.13) and making use of (4.14), we have

$$\delta^2 \sigma W + (\beta^2 + \frac{c}{4})\sigma g(W, \phi U)\phi U = 0.$$

Thus we have

$$\sigma(\beta^2 + \delta^2 + \frac{c}{4})g(W, \phi U) = 0$$

on M_1 . On the other hand, by (4.14) we have $\beta^2 + \delta^2 + \frac{c}{4} \neq 0$ on M_1 . Consequently, $\sigma g(W, \phi U) = 0$ holds on M_1 , which implies that $\sigma = 0$ on M_1 .

Finally we want to show that $\delta = 0$. For this, putting $X = Z = V$ and $Y = U$ in (4.7), and using the expressions of AU and AV on M_1 , we obtain

$$(4.15) \quad \gamma\epsilon - \delta^2 + \frac{c}{4} = 0.$$

By the first equation of (4.14) and (4.15) we have $\beta^2 = 0$, a contradiction. Thus the open set M_1 should be empty, which means $\delta = 0$ on M_0 . It completes the proof of Lemma 4.1. \square

Next we shall prove the following

Lemma 4.2. If $n \geq 3$, then we have

$$(4.16) \quad AU = \beta\xi \quad \text{and} \quad AX = 0, \quad X \in T_0, \quad X \perp U \quad \text{on} \quad M_0.$$

Proof. For this proof let us continue our discussion on the open subset M_0 , unless otherwise stated. Putting $X, Y \in T_1$ in (4.8), we have

$$\begin{aligned} \frac{c}{2}g(\phi X, Y)A\phi U &= \frac{c}{2}rg(\phi X, Y)\phi U + \beta^2g((A\phi + \phi A)X, Y)\phi U \\ &\quad + \frac{c}{4}g(\phi AY, U)\phi X - \frac{c}{4}g(\phi AX, U)\phi Y \end{aligned}$$

From this, let us take a unit $Y = \phi X$ in T_1 , then it follows that

$$(4.17) \quad \begin{aligned} \frac{c}{2}A\phi U &= \frac{c}{2}\gamma\phi U + \beta^2\{g(AX, X) + g(A\phi X, \phi X)\}\phi U \\ &\quad + \frac{c}{4}g(\phi AX, U)X + \frac{c}{4}g(\phi A\phi X, U)\phi X. \end{aligned}$$

From the above formula (4.17) we can express the vector field $A\phi U$ as the following

$$(4.18) \quad A\phi U = \rho\phi U + \epsilon X + \theta\phi X,$$

for some orthogonal vector fields $X, \phi X$ in T_1 . Moreover, from (4.17) and (4.18) it follows

$$\begin{aligned} \epsilon &= g(A\phi U, X) \\ &= \frac{1}{2}g(\phi AX, U) \\ &= -\frac{1}{2}g(X, A\phi U) \\ &= -\frac{1}{2}\epsilon, \end{aligned}$$

where we have used (4.17) to the second equality. This means $\epsilon = 0$. Similarly we can also assert $\theta = 0$. So it follows

$$(4.19) \quad A\phi U = \rho\phi U.$$

If we put $Z = \phi U$ and a unit vector field $Y = \phi X \in T_1$ in (4.7), and take the inner product to the obtained equation with the vector field U , we have

$$(4.20) \quad \beta^2\{g(AX, X) + g(A\phi X, \phi X)\} = \frac{c}{2}(\rho - \gamma),$$

Also putting $X \in T_1$ in (4.9), we have

$$(4.26) \quad (\gamma - \lambda)(\gamma\lambda + \frac{c}{4}) + \beta^2\lambda = 0.$$

On the other hand, if we consider a principal vector $X \in T_1$ such that $AX = \lambda X$ in (4.22), then we get

$$(4.27) \quad (\lambda - \rho)(\lambda\rho + \frac{c}{4}) = 0.$$

Thus secondly, it remains only to show that $\gamma = \rho$. Then (4.24) gives $\lambda = 0$, from which together with (4.22) we get our assertion, that is, $\gamma = \rho = \lambda = 0$. For this we suppose $\gamma \neq \rho$. Then from (4.24) we know $\lambda \neq 0$. Now, substituting (4.24) into (4.25) and (4.26), we have the followings

$$(4.28) \quad \frac{c}{4}(2\gamma + 3\rho) - (\gamma\rho + \frac{5}{4}c)\lambda = 0,$$

$$(4.29) \quad (\gamma - \lambda)(\gamma\lambda + \frac{c}{4}) + \frac{c}{4}(\rho - \gamma) = 0,$$

respectively.

On the other hand, we know $\lambda \neq \rho$. In fact, if we suppose $\lambda = \rho$, then (4.29) gives $(\gamma - \rho)\gamma\rho = 0$. This implies $\gamma\rho = 0$. Thus substituting this into (4.28), we get $\gamma = \rho$. This is a contradiction. So, from (4.27) it follows

$$(4.30) \quad \lambda\rho + \frac{c}{4} = 0.$$

Accordingly, in order to get the complete proof of Lemma 4.2 it is sufficient to show that the above formulas (4.28), (4.29) and (4.30) do not hold on M_0 simultaneously. For this, let us suppose the above formulas hold on M_0 . Then, substituting (4.30) into (4.29), we have

$$\lambda(\gamma - \rho)(\gamma - \lambda + \rho) = 0.$$

From this together with $\lambda(\gamma - \rho) \neq 0$, we know $\lambda = \gamma + \rho$. Thus, substituting this into (4.28) and using (4.30) to the obtained equation, we can easily verify that

$$-3\lambda + \gamma + \rho = 0.$$

From this we have $\lambda = 0$, which is a contradiction. Thus the above formulas do not hold on M_0 simultaneously. So we have verified the second assertion holds on M_0 . Now we complete the proof of Lemma 4.2. \square

By Lemmas 4.1 and 4.2 we can obtain the following on M_0

$$(4.31) \quad \begin{cases} A\xi = \alpha\xi + \beta U, & AU = \beta\xi, & A\phi U = 0, \\ AX = 0, & X \in T_1. \end{cases}$$

Now, after the above preparation we are in a position to prove the Theorem. Let M_2 be a complement of the subset M_0 in M . Suppose that the interior of M_2 is not empty. On this subset the function β vanishes identically and the structure vector field ξ is principal. It is seen in [5] and [10] that the corresponding principal curvature α is constant on the interior of M_2 , because this is a local property. So, by (4.4) and (4.5), we have

$$(\nabla_\xi A)X = 0, \quad X \in T_0.$$

Suppose that $\alpha \neq 0$. Then by (2.5) and the fact that ξ is principal we have

$$(A\phi - \phi A) = 0 \quad \text{on} \quad \text{Int } M_2.$$

For any principal vector X in T_0 with corresponding principal curvature λ , we have

$$(2\lambda - \alpha)A\phi X = \left(\frac{c}{2} + \alpha\lambda\right)\phi X$$

by (2.4). Using the above two equations we get

$$(4.32) \quad 2\lambda^2 - 2\alpha\lambda - \frac{c}{2} = 0,$$

from which it follows that all principal curvatures are non-zero constant on the interior of M_2 . Next, suppose that $\alpha = 0$. Then $A\xi = 0$. By (2.4) we have $4A\phi A = c\phi$, which implies that $A\phi X = \frac{c}{4\lambda}\phi X$, if $AX = \lambda X$. Let X, Y and Z in T_0 are principal vectors corresponding principal curvatures λ, μ and σ , respectively. Then, calculating the components of the vector ϕZ in (4.7), we see

$$\left(\sigma - \frac{c}{4\sigma}\right)g(\phi X, Y) = 0.$$

Accordingly we have

$$4\sigma^2 = c.$$

This means that all principal curvatures except for α are non-zero constant on M_2 . By continuity of principal curvatures, M_2 is M itself and the subset M_0 is empty. Thus in this case M is congruent to a real hypersurface of type A.

When we suppose that the set $\text{Int } M_2$ is empty, then the open set M_0 becomes dense subset of M . By the continuity of principal curvatures again we see that the shape operator satisfies the condition (4.31) on the whole M . Accordingly we get $g(\nabla_X Y, \xi) = -g(\nabla_X \xi, Y) = -g(\phi AX, \xi) = 0$ by (2.1), which means that $\nabla_X Y - \nabla_Y X$ is also contained in T_0 . Hence the distribution T_0 is integrable on M . Moreover the integral manifold of T_0 can be regarded as the submanifold of codimension 2 in $M_n(c)$ whose normal vectors are ξ and C . Since we have $\bar{g}(\bar{\nabla}_X Y, \xi) = g(\nabla_X Y, \xi) = 0$ and $\bar{g}(\bar{\nabla}_X Y, C) = -\bar{g}(\bar{\nabla}_X C, Y) = g(AX, Y) = 0$ for any vector fields X and Y in T_0 by (1.1) and (4.22), where $\bar{\nabla}$ denotes the Riemannian connection of $M_n(c)$. It is seen that the submanifold is totally geodesic in $M_n(c)$. Since T_0 is J -invariant, its integral manifold is a complex submanifold and therefore it is a complex space form $M_{n-1}(c)$. Thus M is locally congruent to a ruled real hypersurface. It completes the proof of our Theorem. \square

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Received November 7, 1994, Revised March 23, 1995