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MARCUS–THOMPSON TYPE THEOREMS FOR HILBERT SPACE OPERATORS

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ABSTRACT. Marcus and Thompson proved that the spectrum of the Hadamard product of normal matrices is contained in the polygon spanned by the products of eigenvalues of the matrices. We give it an extension to hyponormal operators on a Hilbert space by virtue of a recently established extension of the Toyama-Marcus-Khan theorem due to J.I.Fujii.

1. Introduction. The Hadamard product $A * B$, the entrywise product of matrices A and B, is studied in detail. T.Ando [1] gave a beautiful perspective on inequalities involving Hadamard product : For operators A and B acting on a Hilbert space H, being expressed as infinite matrices for a fixed orthonormal base $\{e_n\}$, the Hadamard product $A * B$ is also defined by the entrywise product, however it is less studied than matrix case.

Main difference of them lies in the fact that the Hadamard product $A * B$ of matrices A and B obtained by filtering the tensor product $A \otimes B$ through a positive contractive linear mapping, being assured by the Toyama-Marcus-Khan theorem. Unfortunately, there was no corresponding theorem for operators.

Very recently, J.I.Fujii [8] gives an elegant constructive proof to the theorem for operators (Theorem B in the below), and he extends Ando's inequalities for operators, cf. [2]. In [10], we give also certain comments on it.

In the present note, we extend the Marcus-Thompson theorem for normal matrices to hyponormal operators (Theorem 4). In connection with graph theory and characters of operator algebras, we also observe around that theorem.

2. The Marcus-Thompson theorem. In the below, we follow mainly notations and terminologies due to Halmos [13]. An operator means a bounded linear operator acting on a Hilbert space. The spectrum, numerical range and closed numerical range of an operator A are denoted by $\sigma(A)$, $W(A)$ and $\overline{W}(A)$ respectively. The convex and closed convex hull of a subset X in the plane are denoted by co X and $\overline{\text{co}}\ X$ respectively. For subsets X and Y in the plane, XY stands for the subset $\{xy\ ;\ x \in X, y \in Y\}$.

We first cite the Marcus-Thompson theorem [15].

Theorem A. *Let A and B be normal matrices with eigenvalues $a_1, \cdots, a_n$ and $b_1, \cdots, b_n$ respectively. Then the eigenvalues of the Hadamard product $A * B$ lie in the convex polygon in $\mathbb{C}$ supported by $\{a_i b_j\ ;\ i, j = 1, 2, \cdots, n\}$.*

For the sake of convenience, we rephrase it as follows :

Theorem A'. *Let A and B be normal operators on a finite dimensional Hilbert space. Then $\sigma(A * B)$ is contained in co $\sigma(A)\sigma(B)$.*

Now, to extend Theorem A to that of operators, an operator version of the Toyama-Marcus-Khan theorem due to J.I.Fujii [8] is quite useful. For a fixed orthonormal base, it is done by the following way.

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Theorem B. *Let $\{e_n\}$ be an orthonormal base on a Hilbert space H and U an isometry of H into $H \otimes H$ such that $Ue_n = e_n \otimes e_n$. Then the Hadamard product of A and B (with respect to $\{e_n\}$) is expressed as*

$$A \ast B = U^*(A \otimes B)U.$$

He proved Theorem B very recently and afterwards heard that his method has been already employed by Paulsen [17].

Paying attention to the tensor product $A \otimes B$ of operators A and B, the subset $\sigma(A)\sigma(B)$ is nothing but $\sigma(A \otimes B)$ by Brown and Pearcy [5]. Hence the conclusion of Theorem A' can be replaced by

$$(*) \qquad \sigma(A \ast B) \subseteq \mathrm{co}\,\sigma(A \otimes B).$$

In the next section, we give a weaker condition than Theorem A' to enjoy the inclusion (*) by the help of Theorem B.

Here we remark that the celebrated Schur theorem is easily followed from Theorem B. Therefore the spectrum of the Hadamard product of positive operators is also positive. Theorem A is a generalization of the Schur theorem in this sense, which might be illustrated by the following simple example :

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad B = 1.$$

3. Main results. This section is devoted to applications of Theorem B for spectra of Hadamard products of operators.

The following theorem is a direct consequence of the Toyama-Marcus-Khan theorem in the form of Paulsen-Fujii :

Theorem 1. *The numerical range $W(A \ast B)$ is contained in $W(A \otimes B)$ for arbitrary operators A and B, i.e.,*

$$(\dagger) \qquad W(A \ast B) \subseteq W(A \otimes B).$$

Proof. Since U in Theorem B is an isometry, we have

$$((A \ast B)x, x) = (U^*(A \otimes B)Ux, x) = ((A \otimes B)Ux, Ux) \in W(A \otimes B)$$

for all unit vectors $x \in H$, so that ($\dagger$) is obtained.

At once, we have the following weak form of the Marcus-Thompson theorem for arbitrary operators :

Theorem 2. *The spectrum $\sigma(A \ast B)$ is contained in $\overline{W}(A \otimes B)$ for arbitrary operators A and B, i.e.,*

$$(\ddagger) \qquad \sigma(A \ast B) \subseteq \overline{W}(A \otimes B).$$

Proof. Since $\sigma(T) \subseteq \overline{W}(T)$ in general, we have $\sigma(A \ast B) \subseteq \overline{W}(A \ast B)$ and consequently it is contained in $\overline{W}(A \otimes B)$ by Theorem 1.

Remark. We cannot replace (‡) in Theorem 2 by the stronger inclusion (*). In fact, the following example due to H.Choda in [7] is available as a counterexample of this.

$$A = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \quad \text{and} \quad B = 1.$$

Then $\sigma(A * 1) = \{1, -1\}$ whereas $\sigma(A \otimes 1) = \{0\}$ since $A^2 = 0$.

The above remark suggests us the requirement of additional assumption for (*). An operator T is called convexoid if $\overline{W}(T) = \text{co } \sigma(T)$, [13 ; Prob. 219]. Hence we have the following corollary by (‡).

Corollary 3. *If $A \otimes B$ is convexoid, then (*) holds for A and B.*

However it is insufficient that we regard Corollary 3 as an extension of the Marcus-Thompson theorem since $A \otimes B$ is not always convexoid even if both A and B are convexoid by Saito [18 ; Addendum].

On the other hand, an operator T is hyponormal if $T^*T \geq TT^*$. It is known that every hyponormal operator is convexoid, cf. Halmos [13 ; Prob. 205]. Thus the following theorem might be an operator version of the Marcus-Thompson theorem :

Theorem 4. *If A and B are hyponormal operators, then (*) holds for A and B.*

Proof. As a matter of fact, noting Corollary 3 and the fact that every hyponormal operator is convexoid, it is enough to check that if A and B are hyponormal operators, then so is $A \otimes B$. It is shown in [18 ; Corollary].

Investigating the proof of Theorem 4, we actually find some conditions on A and B implying

$$\overline{W}(A \otimes B) \subseteq \text{co } (\sigma(A) \cdot \sigma(B)). \tag{1}$$

In connection with (1), Saito [18] discussed some conditions implying

$$\overline{W}(A \otimes B) = \overline{\text{co}} \ (W(A) \cdot W(B)). \tag{2}$$

He showed that (2) holds for hyponormal operators A and B [18 ; Cor.]. Afterwards, Furuta and one of the authors [12] gave an equivalent condition to (2) under the assumption that both A and B are convexoid, which is closely related to Corollary 3 : If both A and B are convexoid, then $A \otimes B$ is convexoid if and only if (2) holds for A and B.

On the other hand, Shiu [19] proved the following theorem which is an operator version of a result by Johnson [14].

Theorem C. *If either A or B is normal, then (2) holds for A and B.*

Here we comment that a variant of Theorem 4 is obtained by Theorem C.

Theorem 5. *If A and B are convexoid operators and one of them is normal, then (*) holds for A and B.*

Proof. It follows from Theorems 1 and C that

$$\sigma(A * B) \subseteq \overline{W}(A * B) \subseteq \overline{W}(A \otimes B) = \overline{\mathrm{co}}\,(W(A) \cdot W(B)).$$

Moreover, since A and B are convexoid, we have

$$\overline{\mathrm{co}}\,(W(A) \cdot W(B)) = \overline{\mathrm{co}}\,(\mathrm{co}\,\sigma(A) \cdot \mathrm{co}\,\sigma(B)) = \mathrm{co}\,(\sigma(A) \cdot \sigma(B)) = \mathrm{co}\,\sigma(A \otimes B),$$

where the second equality is assured in [12 ; Lemma 2].

Remark. (1) Comparing with Theorems C and 5, we might be able to expect that if A and B are commuting operators and one of them is normal, then (*) holds. For this, Choda's example A and B stated in Remark after Theorem 2 is also suitable, that is, (*) does not hold for the matrices A and B. Actually, it is presented as a negative answer to the spectral inclusion $\sigma(\Phi(A)) \subseteq co\ \sigma(A)$ for an expectation Φ.

(2) Next we mention a generalization of Theorem 5. Shiu [19] showed that Theorem C has the following corollary by using dilation theorem : If Ω is a spectral set for A, then

$$\overline{W}(A \otimes B) = \overline{\mathrm{co}}\,(\Omega \cdot W(B)).$$

Therefore we have also generalizations of Theorem 5, e.g., if A is convexoid and $\sigma(B)$ is a spectral set for B, then (*) holds for A and B.

4. Examples by graphs. In this section, we employ the notation and terminologies due to Watatani and his followers, cf. [9] for further references, who extend Mohar's theory of infinite graphs.

In short, a directed graph G is a pair of the set $V(G)$ of vertices and $E(G)$ of edges or arrows. $V(G)$ is represented by a base of Hilbert space $\ell^2(G)$ and each arrow is expressed as a dyad $e_v \otimes e_u$ for $u, v \in V(G)$, where $(e_v \otimes e_u)x = (x, e_u)e_v$ for $x \in \ell^2(G)$. The adjacency operator $A(G)$ of G is defined by

$$A(G) = \sum_{(u,v) \in E(G)} e_v \otimes e_u.$$

If G has bounded valency, then $A(G)$ is an operator on $\ell^2(G)$. Several notion of operators are converted into that of graphs by their adjacency operators, for example, spectrum, numerical range, spectral radius, numerical radius, etc.

For graphs G and F with the same vertices, we introduce the Hadamard product $G * F$ of G and F by the adjacency operator ;

$$A(G * F) = A(G) * A(F). \tag{§}$$

It is obvious that the definition is equivalent to

$$E(G * F) = E(G) \cap E(F). \tag{§§}$$

We show by graphs the following theorem.

Theorem 6. *There are normal operators A and B such that $A * B$ is nilpotent and*

$$\overline{W}(A * B) \neq \overline{co}\,(W(A) \cdot W(B)).$$

Proof. Let G and F be graphs whose figures are

$$G: \begin{tikzcd} & \circ \arrow[dl] & \\ \circ \arrow[rr] & & \circ \arrow[ul] \end{tikzcd} \quad \text{and} \quad F: \begin{tikzcd} & \circ \arrow[dr, shift left] & \\ \circ & & \circ \arrow[ul, shift left] \end{tikzcd}$$

respectively. Then $A = A(G)$ is unitary and $B = A(F)$ are selfadjoint, whereas $A * B$ is nilpotent.

In addition, if we consider the tensor products $G \otimes U$ and $F \otimes U$ by the bilateral shift U, then we have examples of infinite graphs.

In the previous note [10], we discussed the diagonalization

$$\Phi(A) = A * 1$$

for operators A. In the remainder of this section, we give certain remarks on the case of $\Phi(A(G)) = 0$, that is, on graphs without selfloop. Let G be a finite graph without selfloop. Since the spectrum $\sigma(G)$ is contained in the closed numerical range $\overline{W}(G)$, the disk $w(G)\mathbb{D}$ and $r(G)\mathbb{D}$ contain $\sigma(G)$, where $w(G)$ (resp. $r(G)$) is the numerical (resp. spectral) radius of G and $\mathbb{D}$ is the closed unit disk in the plane. However, it is hard to calculate their radii, in general.

Let $\gamma(G)$ be the maximum of numbers of arrows terminate to a vertex, i.e.,

$$\gamma(G) = \max_{v \in V(G)} \sharp\{u \in V(G)\ ;\ (u,v) \in E(G)\}.$$

In other words, *the Gersgorin constant* $\gamma(G)$ is the maximum of numbers of 1 in a row. By the well-known Gersgorin theorem, we have the following theorem :

Theorem 7. *The spectrum $\sigma(G)$ of a graph G without selfloop is contained in $\gamma(G)\mathbb{D}$, the disk centered at the origin and radius $\gamma(G)$.*

The theorem is satisfactory for the complete graph $K(n)$ since $\gamma(K(n)) = n - 1$ is just the norm of $K(n)$, whereas in general it is unsatisfactory : For this, let

$$A(G) = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

Then we have $\gamma(G) = 2$ whereas $r(G) = 1$ and $\|G\| = \sqrt{3}$.

Remark. Suppose that a graph G has no selfloop, i.e., $\Phi(A(G)) = 0$. Then it follows from Theorems 2 and C that

$$\{0\} = \sigma(\Phi(A(G))) \subseteq \overline{W}(A(G) \otimes 1) = \overline{W}(G),$$

which has a slight extension. But it does not claim that $\Phi(A(G)) = 0$ implies $0 \in \sigma(G)$.

5. An approach via characters of C^*-algebras. A character χ of a (unital) C^*-algebra $\mathcal{A}$ is a multiplicative state, i.e., a positive linear functional on $\mathcal{A}$ with $\chi(1) = 1$ and

$\chi(AB) = \chi(A)\chi(B)$ for $A, B \in \mathcal{A}$. The set of all states and all characters are denoted by Σ and X respectively.

For an operator A, let $\mathcal{A} = C^*(A)$ be the C^*-algebra generated by A and the identity 1. The set $\pi_n(A)$ of all normal approximate propervalues of A is identical with the image $X(A) = \{\varphi(A)\ ;\ \varphi \in X(\mathcal{A})\}$, [11] and [6].

If A is normal, then $\pi_n(A) = \sigma(A)$ and $C^*(A) = C(\sigma(A))$, the algebra of all complex-valued continuous functions on the spectrum $\sigma(A)$ of A. Furthermore, if $\mathcal{B}$ is a unital C^*-algebra, then the tensor product $\mathcal{C} = C^*(A) \otimes_\alpha \mathcal{B}$ is identical with the algebra $C(\sigma(A); \mathcal{B})$ of all continuous $\mathcal{B}$-valued functions on $\sigma(A)$, cf. [20].

If χ is a character of the tensor product $\mathcal{C}$, then it induces a character of $\mathcal{B}$ and that of $C^*(A)$, so that χ is the product character : $\chi = \chi_1 \otimes \chi_2$, where χ_1 and χ_2 are characters of $C^*(A)$ and $\mathcal{B}$ respectively.

Replacing $\mathcal{B}$ by $C^*(B)$, we have the following theorem, which might be implicitly motivated by Theorem C due to Shiu. Incidentally the reverse inclusion $\pi_n(A \otimes B) \supseteq \pi_n(A) \cdot \pi_n(B)$ holds without any assumption, [11].

Theorem 8. *If A is a normal operator, then*

$$\pi_n(A \otimes B) = \pi_n(A) \cdot \pi_n(B)$$

for an operator B.

As a consequence, it follows that if A is normal, then

$$\mathrm{co}\ \pi_n(A \otimes B) = \mathrm{co}\ [\pi_n(A) \cdot \pi_n(B)]$$

for an operator B. Assume furthermore that $A \otimes B$ is convexoid, that is,

$$\overline{W}(A \otimes B) = \mathrm{co}\ \pi_n(A \otimes B)$$

by [11]. Thus we will arrive at a version of the Marcus-Thompson theorem. A sufficient condition for which $A \otimes B$ is convexoid is given by [12].

The discussion in Theorem 8 is also applicable to prove Theorem C by vertue of the Berberian-Orland theorem [3].

Theorem D. *The closed numerical range $\overline{W}(A)$ of A is identical with the image $\Sigma(A) = \{\varphi(A)\ ;\ \varphi \in \Sigma(\mathcal{A})\}$.*

Related to Theorem C, Shiu [19] gave the following theorem on numerical ranges of products of operators.

Theorem E. *If A and B are commuting operators such that one of them is normal, then*

$$\overline{W}(AB) \subseteq \overline{\mathrm{co}}\ (W(A) \cdot W(B)).$$

Theorem E might be a generalization of the following theorem due to Bouldin [4] :

Theorem F. *If A and B are commuting operators such that one of them is positive, then*

$$W(AB) \subseteq W(A) \cdot W(B).$$

In [16], Nakamoto presented a proof of Theorem F by a simple calculation, which is actually shown by rephrasing as follows : If A is an operator and B is positive, then

$$W(B^{1/2}AB^{1/2}) \subseteq W(A) \cdot W(B),$$

cf. [14 ; Theorem 1]. This obviously implies that

$$\sigma(AB) \subseteq \overline{W}(A) \cdot \overline{W}(B)$$

under the same assumption as above.

Finally we give an intermediate result between Theorems E and F as follows :

Theorem 9. *If A and B are commuting operators such that one of them is selfadjoint, then*

$$W(AB) \subseteq co\,(W(A) \cdot W(B)).$$

Proof. We assume that A is selfadjoint. Let $A = A_+ - A_-$ be the Hahn decomposition and P the associated projection, i.e., $PA = AP = -A_-$ and $(1-P)A = A(1-P) = A_+$. Then, putting $y = A_+^{1/2}x/\|A_+^{1/2}x\|$ and $z = A_-^{1/2}x/\|A_-^{1/2}x\|$, we have

$$\begin{aligned}(ABx, x) &= (A_+Bx, x) - (A_-Bx, x)\\ &= (A_+x, x)(By, y) - (A_-x, x)(Bz, z)\\ &= ((1-P)Ax, x)(By, y) + (PAx, x)(Bz, z).\end{aligned}$$

We here put $\alpha = \|(1-P)x\|$, $\beta = \|Px\|$ and $u = (1-P)x/\alpha$, $v = Px/\beta$. Then u and v are unit vectors and $\alpha^2 + \beta^2 = 1$. Finally we have

$$(ABx, x) = \alpha^2(Au, u)(By, y) + \beta^2(Av, v)(Bz, z) \in \text{co}\,(W(A) \cdot W(B)).$$

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