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Generalized Metrics For Second Order Equations Satisfying Huygens' Principle

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1. Introduction. Let M be an n-dimensional manifold without boundary of class C^∞ and $\pi : TM \to M$ the tangent bundle of M. A second order equation on M such that it is locally expressed by

$$\frac{d^2x^i}{dt^2} = F^i\left(x^1, \cdots, x^n, \frac{dx^1}{dt}, \cdots, \frac{dx^n}{dt}\right) \tag{1.1}$$

is considered to be a vector field V on TM with $\pi_* V(y) = y$, where $(U; x^1, \cdots, x^n)$ and $(TU; x^1, \cdots, x^n, y^1, \cdots, y^n)$ are local coordinate neighborhoods in TM, respectively. Restricting its domain to a hypersurface S in TM, we define second order equations on S satisfying the Huygens principle as follows ([4]). Let $f^t : TM \to TM$ be the local one-parameter group of diffeomorphisms generated by V. We assume that there exists a hypersurface $0 \notin S$ in TM such that S is f^t-invariant and each fibre S_p, $p \in M$, is a hypersurface in T_pM. We say that the local one-parameter group of diffeomorphisms $f^t : S \to S$ satisfies the Huygens principle if there exists a complementary f^t-invariant distribution D on S, where D is by definition such that

(1) $\dim D = \dim S - 1 = 2n - 2$,
(2) $V(y) \notin D(y)$ for any $y \in S$,
(3) $D(y) \supset T_y S_q$, $q = \pi(y)$, for any $y \in S$,
(4) ${f^t}_* D(y) = D(f^t y)$ for any $y \in S$, where ${f^t}_*$ is the differential map of f^t.

We proved in [4] that D is the natural almost contact structure of S. Further, we showed some conditions equivalent to the principle. In particular, we were suggested to use methods developed in Riemannian geometry for the investigation of second order equations satisfying the Huygens principle. The purpose of the present paper is to introduce a generalized metric and the connection of Rund type, and try to find out what condition on this connection allows us to use them in the same way as in Riemannian geometry.

In Section 2 we need to study the relation between the second order equation on S satisfying the Huygens principle and the equation of extremals of variational principle $\int L(x, \dot{x})\, dt$ in order to define a generalized metric we shall use, where L is given as follows. Since there exists a complementary f^t-invariant distribution D, each fibre S_p must be star-shaped around the origin for any $p \in M$ ([4]). Let $C = \{\lambda y | y \in S, \lambda > 0\}$. We can define a positively homogeneous function $L : C \to \mathbf{R}$ of degree 1 such that the indicatrices of L is S, i.e., $L(\lambda y) = \lambda$ for any $y \in S$ and $\lambda > 0$. We shall show that a local one-parameter group of diffeomorphisms on S satisfies the Huygens principle if and only if its orbits are extremals with unit speed of the variational problem for L. It should be noted that we suppose the equation (2.1) with $\frac{d^2x^i}{dt^2}$ as unknown variables has a solution of class C^∞ in the tangent bundle of S, instead of usually assuming that the fundamental tensor of L is positive definite or non-singular. Here the fundamental tensor of L is defined by

$$g_{ij}(y) = \frac{\partial^2 L^2}{\partial y^i \partial y^j}(y)$$

for $y \in TM$.

In Section 3 we shall introduce a generalized metric which plays the same roles as ones used in [4] to state the conditions equivalent to the Huygens principle, and the connection of Rund type (cf. [7]) which is compatible with the generalized metric. Theory of generalized metrics and connections has developed in a group of Finsler geometry ([6]). The theory begins with given generalized metrics which then define Finsler metrics. Conversely, the construction of generalized metrics here shows that any Finsler metric is represented with a generalized metric. Therefore, combined with the results in Section 2, the methods and results for Finsler geometry can be applied to the investigation for second order equations satisfying Huygens' principle.

In Section 4 the curvature tensor will be defined ala Finsler geometry (cf. [7]) and we shall give a condition that ensures its symmetric property with respect to the generalized metric. Further, we shall have a condition such that a variation through the projection of orbits yields a differential equation of Jacobi type as seen in the geometry of geodesics for Riemannian geometry (cf. [1]).

The generalization of the curvature tensor and the equation of Jacobi type are studied for second order equations which yield vector fields on the homogeneous (or projective) fibre bundles over manifolds in some papers (cf. [2],[3]). Here, the homogeneous fibre bundle is by definition a manifold whose points are all equivalence classes $[y] = \{\lambda y | y \in TM - 0, \lambda > 0\}$.

Starting from this point, our results in this paper would be applied if there exists a hypersurface S in TM such that second order equations on the homogeneous fibre bundle are realized on S. The distinguished point from those papers is that we introduce the metrics even if the indicatrices S are not convex. Instead, we assume Huygens' principle.

We shall find in [5] another way to use the generalized metrics given in this paper. There, for any point $p \in M$ a Riemannian metric $\bar{g}_p$ is constructed on $C_p = C \cap T_pM$ such that the envelope of the projection $\gamma_y(t) = \pi f^t y$ of orbits for any $t > 0$ and $y \in S_p$ which start from $p \in M$ in the tangent space T_pM at p are geodesics with respect to $\bar{g}_p$.

2. Complementary invariant distribution. Let C be an open cone in TM and L a positively homogeneous function of degree 1 defined on C as in Section 1. Let

$$g_{ij}(y) = \frac{1}{2}\frac{\partial^2 L^2}{\partial y^i \partial y^j}(y)$$

for i, $j = 1, \cdots, n$ and any $y \in C$ which is called the fundamental tensor of L. Then, we have

$$\sum_{i,j=1}^{n} g_{ij}(y) y^i y^j = L(y)^2$$

for any $y \in C$. We do not assume that the matrix (g_{ij}) is positive definite or non-singular. We see in [7] that the extremals with unit speed of a variational problem

$$\int_a^b L(\dot{x}(t))dt$$

satisfy the equation

$$\sum_{j=1}^{n} g_{ij}\frac{d^2x^j}{dt^2} = -\sum_{a,b=1}^{n}\frac{1}{2}\left(\frac{\partial g_{ib}}{\partial x^a} + \frac{\partial g_{ai}}{\partial x^b} - \frac{\partial g_{ab}}{\partial x^i}\right)\frac{dx^a}{dt}\frac{dx^b}{dt}. \tag{2.1}$$

We assume that a vector field V on S with $\pi_* V(y) = y$ is associated with the equation (2.1), namely all integral curves $(x(t), \dot{x}(t))$ satisfy the equation (2.1). The V may be different from the one in the introduction. We denote by $f^t : S \to S$ the local one-parameter group of diffeomorphisms generated by V. If S has a parametrization

$$(x^1, \cdots, x^n, y^1, \cdots, y^{n-1}, H(x^1, \cdots, x^n, y^1, \cdots, y^{n-1}))$$

in $(TU; x^1, \cdots, x^n, y^1, \cdots, y^n)$, namely

$$L\left(x^1, \cdots, x^n, y^1, \cdots, y^{n-1}, H(x^1, \cdots, x^n, y^1, \cdots, y^{n-1})\right) = 1,$$

then $T_y S$ is generated by

$$(2.2) \qquad \begin{cases} X^i = \dfrac{\partial}{\partial x^i} + \dfrac{\partial H}{\partial x^i}\dfrac{\partial}{\partial y^n}, & i = 1, \cdots, n, \\ X^{n+j} = \dfrac{\partial}{\partial y^j} + \dfrac{\partial H}{\partial y^j}\dfrac{\partial}{\partial y^n}, & j = 1, \cdots, n-1, \end{cases}$$

Let $D = \{X \in T_y S | g(y, \pi_* X) = 0\}$. Then, $D(y)$ is generated by

$$(2.3) \qquad \begin{cases} X^i + \dfrac{\partial H}{\partial y^i} X^n, & i = 1, \cdots, n-1, \\ X^{n+j}, & j = 1, \cdots, n-1. \end{cases}$$

We first prove the following.

Proposition 2.1. *The distribution D is f^t-invariant. In particular, D is a complementary f^t-invariant distribution on S, and, hence, the local one-parameter group of diffeomorphisms on S whose orbits are extremals with unit speed satisfies the Huygens principle.*

Proof. We have only to prove that $[X, V] \in D$ for any vector field X contained in D because of Proposition 2.3 in [4]. Let

$$(2.4) \qquad V = \sum_{i=1}^{n-1} y^i \frac{\partial}{\partial x^i} + H \frac{\partial}{\partial x^n} + \sum_{j=1}^{n} b^j \frac{\partial}{\partial y^j}.$$

Then, by (2.2),

$$(2.5) \qquad [X^{n+j}, V] = X^j + \frac{\partial H}{\partial y^j} X^n + \sum_{k=1}^{n-1} \frac{\partial b^k}{\partial y^j} X^{n+k},$$

for $j = 1, \cdots, n-1$, and, therefore, by (2.3), we see that $[X^{n+j}, V] \in D$ for $j = 1, \cdots, n-1$. It remains to prove that

$$(2.6) \qquad g\left(y, \pi_* \left[X^i + \frac{\partial H}{\partial y^i} X^n, V\right]\right) = 0$$

for $i = 1, \cdots, n-1$. Since

$$[X^i, V] = \frac{\partial H}{\partial x^i} X^n + \sum_{k=1}^{n-1} \frac{\partial b^k}{\partial x^i} X^{n+k}, \quad i = 1, \cdots, n, \tag{2.7}$$

by (2.2), we have that

$$\pi_* \left[X^i + \frac{\partial H}{\partial y^i} X^n, V \right] = \left(\frac{\partial H}{\partial x^i} + \frac{\partial H}{\partial y^i} \frac{\partial H}{\partial x^n} - V \frac{\partial H}{\partial y^i} \right) \frac{\partial}{\partial x^n} \tag{2.8}$$

for $i = 1, \cdots, n-1$. Hence

(2.9)

$$g \left(y, \pi_* \left[X^i + \frac{\partial H}{\partial y^i} X^n, V \right] \right) = \frac{1}{2} \left(-\frac{\partial L^2}{\partial x^i} - \frac{\partial L^2}{\partial x^n} \frac{\partial H}{\partial y^i} - \frac{\partial L^2}{\partial y^n} \left(V \frac{\partial H}{\partial y^i} \right) \right)$$

for $i = 1, \cdots, n-1$, because

$$\sum_{i=1}^{n} g_{in} y^i = \sum_{i=1}^{n} \frac{1}{2} \frac{\partial^2 L^2}{\partial y^i \partial y^n} y^i = \frac{1}{2} \frac{\partial L^2}{\partial y^n} \tag{2.10}$$

and

$$\frac{\partial L^2}{\partial x^i} + \frac{\partial L^2}{\partial y^n} \frac{\partial H}{\partial x^i} = 0. \tag{2.11}$$

Since

$$\frac{\partial L^2}{\partial y^i} + \frac{\partial L^2}{\partial y^n} \frac{\partial H}{\partial y^i} = 0 \tag{2.12}$$

for $i = 1, \cdots, n-1$, we see that

$$V \frac{\partial L^2}{\partial y^i} + \left(V \frac{\partial L^2}{\partial y^n} \right) \frac{\partial H}{\partial y^i} + \frac{\partial L^2}{\partial y^n} \left(V \frac{\partial H}{\partial y^i} \right) = 0 \tag{2.13}$$

for $i = 1, \cdots, n-1$. We shall compute $V \frac{\partial L^2}{\partial y^i}$ and $\left(V \frac{\partial L^2}{\partial y^n} \right) \frac{\partial H}{\partial y^i}$. Since all integral curves of V satisfy the equation (2.1), we have

$$V \frac{\partial L^2}{\partial y^i} = \frac{\partial L^2}{\partial x^i} \tag{2.14}$$

for $i = 1, \cdots, n-1$. We also have

$$\left(V\frac{\partial L^2}{\partial y^n}\right)\frac{\partial H}{\partial y^i} = \frac{\partial L^2}{\partial x^n}\frac{\partial H}{\partial y^i} \tag{2.15}$$

for $i = 1, \cdots, n-1$. At last we have from (2.13), (2.14) and (2.15),

$$\begin{aligned} 0 &= \frac{\partial L^2}{\partial y^n}\left(V\frac{\partial H}{\partial y^i}\right) + V\frac{\partial L^2}{\partial y^i} + \left(V\frac{\partial L^2}{\partial y^n}\right)\frac{\partial H}{\partial y^i} \\ &= \frac{\partial L^2}{\partial y^n}\left(V\frac{\partial H}{\partial y^i}\right) + \frac{\partial L^2}{\partial x^i} + \frac{\partial L^2}{\partial x^n}\frac{\partial H}{\partial y^i} \end{aligned} \tag{2.16}$$

for $i = 1, \cdots, n-1$. This implies that

$$g\left(y, \pi_*\left[X^i + \frac{\partial H}{\partial y^i}X^n, V\right]\right) = 0$$

for $i = 1, \cdots, n-1$. This completes the proof.

Combined with Theorem 2.6 in [4] it follows from Proposition 2.1 that integral curves of V satisfy the original second order equation (1.1), i.e., $b^j = F^j|S$, if $H_{ij} = \dfrac{\partial^2 H}{\partial y^i \partial y^j}(y)$ make a non-singular matrix for all $y \in S$, because the nonsingularity of the matrix (H_{ij}) implies the uniqueness of the local one-parameter group f^t which leaves D invariant. Conversely, the following proposition shows that the solutions of the original equation satisfy the equation (2.1).

Proposition 2.2. *It holds that on S*

$$\sum_{j=1}^{n} g_{ji}F^j = -\sum_{a,b=1}^{n}\frac{1}{2}\left(\frac{\partial g_{ib}}{\partial x^a} + \frac{\partial g_{ai}}{\partial x^b} - \frac{\partial g_{ab}}{\partial x^i}\right)y^a y^b \tag{2.17}$$

for $i = 1, \cdots, n$. Thus, the orbits of a second order equation on S satisfying the Huygens principle consist of extremals of the variational problem for L.

Proof. Since $L(x^1, \cdots, x^n, y^1, \cdots, y^{n-1}, H(x^1, \cdots, x^n, y^1, \cdots, y^{n-1})) = 1$, we have, by differentiating $\sum_{k,m=1}^{n} g_{km}y^k y^m = 1$ in y^i,

$$\sum_{k=1}^{n} g_{ik}y^k + \sum_{k=1}^{n} g_{nk}y^k\frac{\partial H}{\partial y^i} = 0 \tag{2.18}$$

for $i = 1, \cdots, n-1$ and

$$\sum_{k,m=1}^{n-1} \frac{\partial g_{km}}{\partial x^i} y^k y^m + \sum_{k=1}^{n} 2 g_{nk} y^k \frac{\partial H}{\partial x^i} = 0 \tag{2.19}$$

for $i = 1, \cdots, n$. Therefore, we get, by differentiating (2.18) in y^i,

$$g_{ji} + g_{ni} \frac{\partial H}{\partial y^j} + g_{jn} \frac{\partial H}{\partial y^i} + g_{nn} \frac{\partial H}{\partial y^j} \frac{\partial H}{\partial y^i} + \sum_{k=1}^{n} g_{nk} y^k \frac{\partial^2 H}{\partial y^i \partial y^j} = 0 \tag{2.20}$$

for $i, j = 1, \cdots, n-1$. Thus, we have, from (2.20),

$$\begin{aligned} \sum_{j=1}^{n-1} g_{ji} F^j + \sum_{j=1}^{n-1} g_{ni} \frac{\partial H}{\partial y^j} F^j + \sum_{j=1}^{n-1} g_{jn} \frac{\partial H}{\partial y^i} F^j \\ + \sum_{j=1}^{n-1} g_{nn} \frac{\partial H}{\partial y^j} \frac{\partial H}{\partial y^i} F^j + \sum_{k=1}^{n} \sum_{j=1}^{n-1} g_{nk} y^k \frac{\partial^2 H}{\partial y^j \partial y^i} F^j = 0. \end{aligned} \tag{2.21}$$

By making use of the equation

$$F^n = \sum_{j=1}^{n} y^j \frac{\partial H}{\partial x^j} + \sum_{j=1}^{n-1} F^j \frac{\partial H}{\partial y^j} \tag{2.22}$$

(see [4]), we get

$$\sum_{j=1}^{n-1} g_{ni} \frac{\partial H}{\partial y^j} F^j = g_{ni} F^n - \sum_{j=1}^{n} g_{ni} y^j \frac{\partial H}{\partial x^j} \tag{2.23}$$

and

$$\sum_{j=1}^{n-1} g_{nn} \frac{\partial H}{\partial y^i} \frac{\partial H}{\partial y^j} F^j = g_{nn} F^n \frac{\partial H}{\partial y^i} - \sum_{j=1}^{n} g_{nn} \frac{\partial H}{\partial y^i} y^j \frac{\partial H}{\partial x^j} \tag{2.24}$$

for $i = 1, \cdots, n-1$. Hence, by substituting (2.23) and (2.24) into (2.21), we get

$$\begin{aligned} \sum_{j=1}^{n} \left(g_{ji} + g_{jn} \frac{\partial H}{\partial y^i} \right) F^j - \left(g_{ni} + g_{nn} \frac{\partial H}{\partial y^i} \right) \sum_{j=1}^{n} y^j \frac{\partial H}{\partial x^j} \\ + \sum_{k=1}^{n} \sum_{j=1}^{n-1} g_{nk} y^k \frac{\partial^2 H}{\partial y^j \partial y^i} F^j = 0 \end{aligned} \tag{2.25}$$

for $i = 1, \cdots, n-1$. On one hand, we have, by differetiating (2.18) along $\sum_{a=1}^{n} y^a \dfrac{\partial}{\partial x^a}$,

$$(2.26)\quad \sum_{k,a=1}^{n} \frac{\partial g_{ik}}{\partial x^a} y^k y^a + \sum_{a=1}^{n} g_{in} \frac{\partial H}{\partial x^a} y^a + \sum_{k,a=1}^{n} \frac{\partial g_{nk}}{\partial x^a} y^k y^a \frac{\partial H}{\partial y^i} + \sum_{a=1}^{n} g_{nn} \frac{\partial H}{\partial x^a} y^a \frac{\partial H}{\partial y^i} + \sum_{k,a=1}^{n} g_{nk} y^k \frac{\partial^2 H}{\partial x^a \partial y^i} y^a = 0,$$

and, from Proposition 2.6 in [4] and (2.26), we have

$$\sum_{k=1}^{n} \sum_{j=1}^{n-1} g_{nk} y^k F^j \frac{\partial^2 H}{\partial y^j \partial y^i} = \sum_{k=1}^{n} g_{nk} y^k \frac{\partial H}{\partial x^i} + \sum_{k=1}^{n} g_{nk} y^k \frac{\partial H}{\partial y^i} \frac{\partial H}{\partial x^n} + \sum_{k,a=1}^{n} \frac{\partial g_{ik}}{\partial x^a} y^k y^a + \sum_{k,a=1}^{n} \frac{\partial g_{nk}}{\partial x^a} y^k y^a \frac{\partial H}{\partial y^i} + \sum_{a=1}^{n} \left(g_{in} + g_{nn} \frac{\partial H}{\partial y^i} \right) \left(y^a \frac{\partial H}{\partial x^a} \right)$$

for $i = 1, \cdots, n-1$. Therefore, we get, by making use of (2.19) and (2.25),

$$(2.27)\quad \sum_{j=1}^{n} \left(g_{ji} + g_{jn} \frac{\partial H}{\partial y^i} \right) F^j + \sum_{k,a=1}^{n} \frac{1}{2} \left(\left(\frac{\partial g_{ik}}{\partial x^a} + \frac{\partial g_{ia}}{\partial x^k} - \frac{\partial g_{ak}}{\partial x^i} \right) + \left(\frac{\partial g_{nk}}{\partial x^a} + \frac{\partial g_{na}}{\partial x^k} - \frac{\partial g_{ak}}{\partial x^n} \right) \frac{\partial H}{\partial y^i} \right) y^k y^a = 0$$

for $i = 1, \cdots, n-1$. Let

$$(2.28)\quad \Gamma_i(x, y) = \sum_{j=1}^{n} g_{ji} F^j + \sum_{k,a=1}^{n} \frac{1}{2} \left(\frac{\partial g_{ik}}{\partial x^a} + \frac{\partial g_{ia}}{\partial x^k} - \frac{\partial g_{ak}}{\partial x^i} \right) y^a y^k$$

for $i = 1, \cdots, n$. Then, the above equation (2.27) becomes

$$(2.29)\quad \Gamma_i = -\Gamma_n \frac{\partial H}{\partial y^i}$$

for $i = 1, \cdots, n-1$. We want to show that $\Gamma_i = 0$ for $i = 1, \cdots, n$. To do this we first have that $\sum_{k=1}^{n} \Gamma_k y^k = 0$. In fact, since L^2 is constant along both solutions of the original equation (1.1) and of (2.1), we have

$$(2.30) \qquad \sum_{j=1}^{n} y^j \frac{\partial L^2}{\partial x^j} + \sum_{j=1}^{n} F^j \frac{\partial L^2}{\partial y^j} = 0 = \sum_{j=1}^{n} y^j \frac{\partial L^2}{\partial x^j} + \sum_{j=1}^{n} \frac{d^2 x^j}{dt^2} \frac{\partial L^2}{\partial y^j}$$

and, therefore, by $\dfrac{\partial L^2}{\partial y^j} = \sum_{k=1}^{n} 2 g_{jk} y^k$ and (2.1), we get

$$(2.31) \qquad \sum_{j=1}^{n} 2 g_{j\ell} F^j y^\ell = - \sum_{j,\ell,a=1}^{n} y^j \left(\frac{\partial g_{ja}}{\partial x^\ell} + \frac{\partial g_{j\ell}}{\partial x^a} - \frac{\partial g_{a\ell}}{\partial x^j} \right) y^a y^\ell,$$

namely, $\sum_{j=1}^{n} \Gamma_k y^k = 0$. If $\Gamma_n = 0$, then $\Gamma_i = 0$ for $i = 1, \cdots, n-1$ because of the above equation (2.29). Suppose for indirect proof that $\Gamma_n \neq 0$ for some $y \in S$. Then,

$$0 = \sum_{k=1}^{n} \Gamma_k y^k = \Gamma_n \left(y^n - \sum_{k=1}^{n-1} \frac{\partial H}{\partial y^k} y^k \right).$$

By assumption, we have

$$y^n = H(x^1, \cdots, x^n, y^1, \cdots, y^{n-1}) = \sum_{k=1}^{n-1} \frac{\partial H}{\partial y^k} y^k$$

on a neighborhood of y. We work in a coordinate neighborhood $(U; x^1, \cdots, x^n)$ such that $x(\pi(y)) = 0$, $\left. \dfrac{\partial}{\partial x^n} \right|_0 = y$. Then, the y has the coordinate $(0, \cdots, 0, 1)$. On the other hand,

$$y^n = H(0, \cdots, 0) = \sum_{k=1}^{n-1} \frac{\partial H}{\partial y^k}(0, \cdots, 0) y^k = 0,$$

contradicting $y = 0 \notin S$. Hence, we claim that $\Gamma_n = 0$, and, therefore, $\Gamma_i = 0$ for $i = 1, \cdots, n$. This completes the proof.

3. Generalized metrics and connections. We call a function $f : TM \to \mathbf{R}$ *a generalized function on* M and a map $X : TM \to TM$ with $X(y) \in T_{\pi(y)}M$ *a generalized vector field on* M. Hereafter let Q be a generalized distribution on M such that $y \in Q(y) \subset T_{\pi(y)}M$, $Q(\lambda y) = Q(y)$ for $\lambda > 0$, and the fundamental tensor $g(y) = (g_{ij}(y))$ is positive definite on $Q(y)$ for any $y \in C$. Let $E_1 = \dfrac{y}{L(y)}, \cdots, E_m$ be an orthonormal basis of Q with respect to g, namely, $g(y)(E_i(y), E_j(y)) = \delta_{ij}$ and $\{E_1(y), \cdots, E_m(y)\}$ spans $Q(y)$ for any $y \in C$. Define a generalized metric h as follows, by making use of a Riemannian metric $\tilde{g}$ on M,

$$h(y)(X,Y) = \tilde{g}(\pi(y))(X - P(X), Y - P(Y)) + g(P(X), P(Y)) \tag{3.1}$$

for any X, $Y \in T_{\pi(y)}M$, where $P(X) = \sum_{j=1}^{m} g(X, E_j(y))E_j(y)$. We should notice that the function L is represented with this generalized metric, i.e.,

$$\sum_{i,j=1}^{n} h_{ji}(y)y^j y^i = L(y)^2 \tag{3.2}$$

for any $y \in C$. Since $h(X, \cdot) = g(X, \cdot)$ for any $X \in Q(y)$, we have the following.

Lemma 3.1. *It holds that*

$$\sum_{j=1}^{n} h_{ji}(y)y^j = \sum_{j=1}^{n} g_{ji}(y)y^j, \tag{3.3}$$

$$\sum_{j=1}^{n} \frac{\partial h_{ji}}{\partial x^k} y^j = \sum_{j=1}^{n} \frac{\partial g_{ji}}{\partial x^k} y^j, \tag{3.4}$$

$$\sum_{j=1}^{n} \frac{\partial h_{ji}}{\partial y^k} y^j = g_{ki} - h_{ki} \tag{3.5}$$

for any $y \in C$ *and* $k, i = 1, \cdots, n$.

We omit the proof.

Let an associated vector field V be

$$V = \sum_{j=1}^{n} y^j \frac{\partial}{\partial x^j} + \sum_{j=1}^{n} b^j(y) \frac{\partial}{\partial y^j} \tag{3.6}$$

on S. We may think $b^j = F^j|S$ for $j = 1, \cdots, n$ in virtue of Proposition 2.1 and 2.2. We extend the vector field V on S to C by putting $b^j(\lambda y) = \lambda^2 b^j(y)$ for any $y \in S$ and $\lambda > 0$. As seen before for $L^2(y)$, we put

$$\gamma_j{}^k{}_i(y) = -\frac{1}{2} \frac{\partial^2 b^k}{\partial y^j \partial y^i}(y)$$

(cf. [4]). We define the coefficients $N^k{}_i$ of a non-linear connection by

$$N^k{}_i(y) = \sum_{j=1}^{n} y^j \gamma_j{}^k{}_i(y), \tag{3.7}$$

and a differential operator (see [7])

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - \sum_{k=1}^{n} N^k{}_i \frac{\partial}{\partial y^k} \tag{3.8}$$

for $i = 1, \cdots, n$. For a generalized function $f : C \to \mathbf{R}$ we put

$$Xf = \sum_{j=1}^{n} X^j \frac{\delta f}{\delta x^j} \tag{3.9}$$

if a generalized vector field X is expressed as $X = \sum_{j=1}^{n} X^j \frac{\partial}{\partial x^j}$ in a coordinate neighborhood $(U; x^1, \cdots, x^n)$, where $X^j : C \cap \pi^{-1}(U) \to \mathbf{R}$. We also define the coefficients $\Gamma_j{}^i{}_k$ of a V-connection (see [7]) by

$$\Gamma_j{}^i{}_k(y) = \frac{1}{2} \sum_{a=1}^{n} h^{ia} \left(\frac{\delta h_{ak}}{\delta x^j} + \frac{\delta h_{ja}}{\delta x^k} - \frac{\delta h_{jk}}{\delta x^a} \right), \tag{3.10}$$

and a covariant differentiation ∇ (see [7]) by

$$\nabla_X Y(y) = \sum_{j,i=1}^{n} X^i \left(\frac{\delta Y^j}{\delta x^i} + \sum_{k=1}^{n} \Gamma_i{}^j{}_k(y) Y^k \right) \frac{\partial}{\partial x^j} \tag{3.11}$$

for any generalized vector fields $X = \sum_{j=1}^{n} X^j \frac{\partial}{\partial x^j}$ and $Y = \sum_{j=1}^{n} Y^j \frac{\partial}{\partial x^j}$ on M. From definition we have the following.

LEMMA 3.2. *It holds that*

$$(3.12) \qquad Xh(Y,Z) = h(\nabla_X Y, Z) + h(Y, \nabla_X Z)$$

for any generalized vector fields X, Y, Z on M.

As usual we can define the covariant differentiation of a vector field Y along a curve $c : I \to M$ with $\dot{c}(t) \in C$ (see [7]);

$$(3.13) \qquad \nabla_{\dot{c}(t)} Y = \sum_{j=1}^{n} \left(\frac{dY^j}{dt} + \sum_{i,k=1}^{n} \Gamma_i{}^j{}_k(\dot{c}(t)) \frac{dx^i}{dt} Y^k \right) \frac{\partial}{\partial x^j}.$$

We call a curve $c : I \to M$ *a geodesic* if $\nabla_{\dot{c}(t)} \dot{c} = 0$ for any $t \in I$.

LEMMA 3.3. *Let $\alpha_y(t) = \pi f^t y$ for t and $y \in S$. Then, α_y is a geodesic.*

PROOF. We suppose $\alpha_y(t) = (x^1(t), \cdots, x^n(t))$ in a coordinate neighborhood $(U; x^1, \cdots, x^n)$. Since $f^t y$ is an integral curve of V and

$$(3.14) \qquad b^i(y) = -\sum_{j,k=1}^{n} \gamma_j{}^i{}_k(y) y^j y^k,$$

we see from (3.4) that

$$(3.15) \qquad \frac{d^2 x^i}{dt^2} = b^i(\dot{x}(t)) = -\sum_{j,k=1}^{n} \gamma_j{}^i{}_k(\dot{x}(t)) \dot{x}^j(t) \dot{x}^k(t).$$

Hence, it suffices to prove that

$$(3.16) \qquad b^i = -\sum_{j,k=1}^{n} \Gamma_j{}^i{}_k y^j y^k$$

for any $y \in S$. By (3.10), (3.8), (3.5), (3.14), (3.4), (3.3) and (2.17), we have

$$(3.17) \quad \sum_{i,j,k=1}^{n} h_{it}\Gamma_j{}^i{}_k y^j y^k = \frac{1}{2}\sum_{j,k=1}^{n}\left(\left(\frac{\partial h_{tk}}{\partial x^j}+\frac{\partial h_{jt}}{\partial x^k}-\frac{\partial h_{jk}}{\partial x^t}\right)\right.$$

$$\left.-\sum_{a,b=1}^{n}\left(y^b\gamma_b{}^a{}_j\frac{\partial h_{tk}}{\partial y^a}+y^b\gamma_b{}^a{}_k\frac{\partial h_{jt}}{\partial y^a}-y^b\gamma_b{}^a{}_t\frac{\partial h_{jk}}{\partial y^a}\right)\right)y^j y^k$$

$$=\frac{1}{2}\left(\sum_{j,k=1}^{n}\left(\frac{\partial h_{tk}}{\partial x^j}+\frac{\partial h_{jt}}{\partial x^k}-\frac{\partial h_{jk}}{\partial x^t}\right)y^j y^k+\sum_{a=1}^{n}2b^a(g_{ta}-h_{ta})\right)$$

$$=-\sum_{a=1}^{n}g_{at}b^a+\sum_{a=1}^{n}b^a(g_{ta}-h_{ta})=-\sum_{a=1}^{n}b^a h_{ta}.$$

Since h_{ta} make a non-singular matrix, we get the desired equation.

LEMMA 3.4. *Let α_y be a geodesic and let X, Y be any generalized vector fields along α_y. Then,*

$$(3.18) \quad \frac{d}{dt}h(\dot\alpha_y(t))(X,Y)=h(\dot\alpha_y(t))\left(\nabla_{\dot\alpha_y(t)}X,Y\right)+h(\dot\alpha_y(t))\left(X,\nabla_{\dot\alpha_y(t)}Y\right)$$

for any t.

PROOF. Let $\alpha_y(t)=(x^1(t),\cdots,x^n(t))$ in a coordinate neighborhood $(U;x^1,\cdots,x^n)$. Then, we have

$$(3.19) \quad \frac{d}{dt}h_{ij}\left(x(t),\frac{dx}{dt}(t)\right)=\sum_{k=1}^{n}\left(\frac{dx^k}{dt}\frac{\partial h_{ij}}{\partial x^k}+\frac{d^2x^k}{dt^2}\frac{\partial h_{ij}}{\partial y^k}\right)$$

$$=\sum_{k=1}^{n}\left(\frac{dx^k}{dt}\frac{\partial h_{ij}}{\partial x^k}-\sum_{a,b=1}^{n}\gamma_a{}^k{}_b\frac{dx^a}{dt}\frac{dx^b}{dt}\frac{\partial h_{ij}}{\partial y^k}\right)$$

$$=\sum_{k=1}^{n}\frac{dx^k}{dt}\left(\frac{\partial h_{ij}}{\partial x^k}-\sum_{a,b=1}^{n}\frac{dx^a}{dt}\gamma_a{}^b{}_k\frac{\partial h_{ij}}{\partial y^b}\right)$$

$$=\sum_{k=1}^{n}\frac{dx^k}{dt}\frac{\delta h_{ij}}{\delta x^k}=\sum_{t,k=1}^{n}\frac{dx^k}{dt}\left(h_{tj}\Gamma_i{}^t{}_k+h_{it}\Gamma_j{}^t{}_k\right).$$

By using (3.19) and (3.11) we get (3.18). This completes the proof.

4. Curvature tensor. We have already defined a covariant differentiation by generalized vector fields. Here we also define a covariant differentiation by vertical vector fields. Let $f : C \to \mathbf{R}$ be a generalized function on M and Y a vertical vector field on C, namely $Y = \sum_{j=1}^{n} Y^j \frac{\partial}{\partial y^j}$ in a coordinate neighborhood. Then,

$$Yf = \sum_{j=1}^{n} Y^j \frac{\partial f}{\partial y^j}. \tag{4.1}$$

If X is a generalized vector field, then

$$\nabla_Y X = \sum_{i,j=1}^{n} Y^j \frac{\partial X^i}{\partial y^j} \frac{\partial}{\partial x^i} \tag{4.2}$$

in a coordinate neighborhood $(U; x^1, \cdots, x^n)$. To define a curvature tensor we need the generalized Lie bracket $\{X, Y\}$ for generalized vector fields X, Y on M, which is by definition

$$\{X, Y\}f = X(Yf) - Y(Xf)$$

for any generalized function $f : TM \to \mathbf{R}$. By direct computation we get the following.

LEMMA 4.1. *It holds that*

$$\{X, Y\} = \sum_{i,j=1}^{n} \left(X^j \frac{\delta Y^i}{\delta x^j} - Y^j \frac{\delta X^i}{\delta x^j} \right) \frac{\partial}{\partial x^i} + \sum_{a,i,j=1}^{n} X^j Y^i \left(\frac{\delta N^a{}_j}{\delta x^i} - \frac{\delta N^a{}_i}{\delta x^j} \right) \frac{\partial}{\partial y^a} \tag{4.3}$$

where X, Y are generalized vector fields on M with $X = \sum_{i=1}^{n} X^i \frac{\partial}{\partial x^i}$ and $Y = \sum_{i=1}^{n} Y^i \frac{\partial}{\partial x^i}$ in a coordinate neighborhood $(U; x^1, \cdots, x^n)$.

LEMMA 4.2. *The following are true.*

$$\{X, Y\} = -\{Y, X\}. \tag{4.4}$$

$$\{fX, Y\} = f\{X, Y\} - (Yf)X. \tag{4.5}$$
$$\{X, fY\} = (Xf)Y + f\{X, Y\}.$$

Here f is any generalized function and X, Y are any generalized vector fields on M.

We can define the curvature tensor R by

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{\{X,Y\}} Z \tag{4.6}$$

for any generalized vector fields X, Y, Z on M as seen in [7].. The local expression is given in the following.

LEMMA 4.3. *It holds that*

$$R(X,Y)Z = \sum_{a,b,j,i=1}^{n} X^a Y^j Z^b \left(\frac{\delta \Gamma_j{}^i{}_b}{\delta x^a} - \frac{\delta \Gamma_a{}^i{}_b}{\delta x^j} + \sum_{k=1}^{n} \left(\Gamma_a{}^i{}_k \Gamma_j{}^k{}_b - \Gamma_j{}^i{}_k \Gamma_a{}^k{}_b \right) \right) \frac{\partial}{\partial x^i}. \tag{4.7}$$

For convenience we write $< \cdot, \cdot >= h(\cdot, \cdot)$.

LEMMA 4.4. *For any tangent vectors X, Y, Z, W of M, we have the following.*

$$< R(X,Y)Z, W > = - < R(Y,X)Z, W > \tag{4.8}$$

$$< R(X,Y)Z, W >+ < R(Y,Z)X, W > + < R(Z,X)Y, W >= 0 \tag{4.9}$$

We define a curvature tensor N of the non-linear connection by

$$N(X,Y) = \sum_{j,i=1}^{n} X^j Y^i \left(\frac{\delta N^a{}_j}{\delta x^i} - \frac{\delta N^a{}_i}{\delta x^j} \right) \frac{\partial}{\partial y^a} \tag{4.10}$$

for any tangent vectors X, Y of M as seen in [7]. The coefficients $N_j{}^a{}_i$ are given by

$$N_j{}^a{}_i = \frac{\delta N^a{}_j}{\delta x^i} - \frac{\delta N^a{}_i}{\delta x^j}.$$

LEMMA 4.5. *For any tangent vectors X, Y, Z, W of M, the following is true.*

$$< R(X,Y)Z, W >= - < R(X,Y)W, Z > + \left(\nabla_{N(X,Y)} h \right) (Z, W) \tag{4.11}$$

Proof. Let p be a point of M with X, Y, Z, $W \in T_pM$ and let $y \in C$ with $\pi(y) = p$. Extend X, Y, Z, W to generalized vector fields on a neighborhood of p such that they depend only on the underlying points p. We denote them by the same notation. From (3.12) we have

$$\begin{aligned} X(Y < Z,W >) =& < \nabla_X \nabla_Y Z, W > + < \nabla_Y Z, \nabla_X W > \\ &+ < \nabla_X Z, \nabla_Y W > + < Z, \nabla_X \nabla_Y W > \end{aligned} \tag{4.12}$$

$$\begin{aligned} Y(X < Z,W >) =& < \nabla_Y \nabla_X Z, W > + < \nabla_X Z, \nabla_Y W > \\ &+ < \nabla_Y Z, \nabla_X W > + < Z, \nabla_Y \nabla_X W > \end{aligned} \tag{4.13}$$

and

$$\begin{aligned} \{X,Y\} < Z,W > =& < \nabla_{\{X,Y\}} Z, W > \\ &+ < Z, \nabla_{\{X,Y\}} W > + (\nabla_{N(X,Y)} h)(Z,W). \end{aligned} \tag{4.14}$$

These formulas and the definition (4.6) of R give us this lemma.

We will get the symmetric property of the curvature tensor, assuming a condition.

Theorem 4.6. *If the (0,2)-tensor* $\sum_{i=1}^{n} y^i N_j{}^b{}_i (g_{tb} - h_{tb})$ *is zero at* $y \in S$, *then*

$$< R(X,y)y, Y > = < R(Y,y)y, X > \tag{4.15}$$

for any tangent vectors $X, Y \in T_{\pi(y)}M$.

Proof. By (4.11), (4.10), (3.5), (4.9) and the assumption, we have

(4.16)

$$\begin{aligned} < R(X,y)y, Y > &= - < R(X,y)Y, y > + \sum_{j,i,b,k,t=1}^{n} X^j y^i N_j{}^b{}_i \frac{\partial h_{kt}}{\partial y^b} y^k Y^t \\ &= - < R(X,y)Y, y > + \sum_{j,i,b,t=1}^{n} X^j y^i N_j{}^b{}_i (g_{tb} - h_{tb}) Y^t \\ &= < R(y,Y)X, y > + < R(Y,X)y, y > . \end{aligned}$$

By (4.11), (3.5) and (3.3), we get

(4.17)

$$< R(Y,X)y,y > = - < R(Y,X)y,y > + \sum_{j,i,b,t=1}^{n} Y^j X^i N_j{}^b{}_i (g_{tb} - h_{tb}) y^t$$
$$= - < R(Y,X)y,y >,$$

and, therefore,

$$< R(Y,X)y,y >= 0. \tag{4.18}$$

and, hence

$$< R(X,y)y,Y >=< R(y,Y)X,y > \tag{4.19}$$

Combining (4.19), (4.8) and (4.11), we get, by the skew-symmrtric property of N,

(4.20)

$$< R(X,y)y,Y > =< R(Y,y)y,X > + \sum_{j,i,b,t=1}^{n} y^j Y^i N_j{}^b{}_i (g_{tb} - h_{tb}) X^t$$
$$=< R(Y,y)y,X >$$

This completes the proof.

We conclude this section to see that the curvature tensor $R(\cdot,y)y$ appears from the geodesic variation. To see it we need a lemma.

LEMMA 4.7. *Let Y be a generalized vector field on M and α a geodesic. Then*

$$\nabla_{\dot{\alpha}(t)} Y = \sum_{i=1}^{n} \left(\frac{dY^i}{dt} + \sum_{j,k=1}^{n} \Gamma_j{}^i{}_k \frac{dx^j}{dt} Y^k \right) \frac{\partial}{\partial x^i} \tag{4.21}$$

where $\alpha(t) = (x^1(t), \cdots, x^n(t))$ and $Y = \sum_{i=1}^{n} Y^i \dfrac{\partial}{\partial x^i}$ in a coordinate neighborhood $(U; x^1, \cdots, x^n)$.

PROOF. By definition, (3.13), (3.14), (3.7) and (3.8), we have

(4.22)

$$\begin{aligned}\nabla_{\dot\alpha(t)}Y &= \sum_{i,j=1}^{n}\frac{dx^j}{dt}\left(\frac{\delta Y^i}{\delta x^j}+\sum_{k=1}^{n}\Gamma_j{}^i{}_kY^k\right)\frac{\partial}{\partial x^i}\\ &=\sum_{i,j=1}^{n}\frac{dx^j}{dt}\left(\frac{\partial Y^i}{\partial x^j}-\sum_{a=1}^{n}N^a{}_j\frac{\partial Y^i}{\partial y^a}+\sum_{k=1}^{n}\Gamma_j{}^i{}_kY^k\right)\frac{\partial}{\partial x^i}\\ &=\sum_{i=1}^{n}\left(\frac{dY^i}{dt}(x(t),\dot x(t))+\sum_{k,j=1}^{n}\Gamma_j{}^i{}_k\frac{dx^j}{dt}Y^k\right)\frac{\partial}{\partial x^i}\end{aligned}$$

because α is a geodesic.

Let $\beta : I\times(-\varepsilon,\varepsilon)\to M$ be a geodesic variation, namely $\beta_s : I\to M$ is a geodesic for each $s\in(-\varepsilon,\varepsilon)$, and let $X(t,s)=\dfrac{\partial\beta}{\partial t}(t,s)$, $Y(t,s)=\dfrac{\partial\beta}{\partial s}(t,s)$ for any $(t,s)\in I\times(-\varepsilon,\varepsilon)$. We make the covariant derivative of Y along the geodesic $\alpha=\beta_0$ twice in a coordinate neighborhood. Then, we have, by (4.21), (3.14), (3.7), (3.15), (3.16),

$$\begin{aligned}\nabla_{\dot\alpha(t)}\nabla_{\dot\alpha(t)}Y &= R(\dot\alpha(t),Y)\dot\alpha(t)\\ &\quad-\sum_{i,j,a,k=1}^{n}\frac{\partial\Gamma_j{}^i{}_k}{\partial y^a}\left(\frac{\partial Y^a}{\partial t}X^k+\sum_{b=1}^{n}N^a{}_bY^kX^b\right)X^j\frac{\partial}{\partial x^i}\end{aligned}$$

In the computation we should notice that $\nabla_{\dot\alpha(t)}Y$ is a generalized vector field along α. We have just proved the following.

PROPOSITION 4.8. *If* $\dfrac{\partial\Gamma_j{}^i{}_k}{\partial y^a}(y)y^j=0$ *for any* $y\in S$, *then any geodesic variation vector field* Y *along any geodesic* α *satisfies the equation of Jacobi type*

$$\nabla_{\dot\alpha}\nabla_{\dot\alpha}Y+R(Y,\dot\alpha)\dot\alpha=0.$$

It should be noted that the assumptions of Theorem 4.6 and Proposition 4.8 are satisfied if $h=g$, namely, $y\to L(y)^2$ is strictly convex and $Q(y)=T_{\pi(y)}M$ for any $y\in S$ and we can see a systematic description for this case (cf. [7]). The properties in Theorem 4.6 and Proposition 4.8 play very important roles in the proof of Rauch's comparison theorem.

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