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Orbit structure of quadratic maps and their bifurcation

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Abstract

We precisely study the structure of the orbit of critical point 1/2 and periodic points for $f_a = ax(1-x), (0 < a < 3.5699\ldots)$ on $[0,1]$ and as an application give the classification of those functions by topological conjugacy.

1 Introduction

Let f_a be a quadratic map of [0,1] into itself defined by $f_a(x) = ax(1-x)$ for x in [0,1], where a is a real number in (0,4]. We have already well known that the family $\{f_a : 0 < a \leq 4\}$ has a series of period-doubling bifurcations within the interval $[1, a(2^\infty)]$, where $a(2^\infty) = 3.5699\cdots$ (cf.[3]). Namely there is a sequence $\{a_n\}_{n=0}^{\infty}$ in $[1, a(2^\infty)]$ such that

(1) periodic points $P(2^n)$'s of prime period 2^n (a non-zero fixed point when $n = 0$) appear after a passes through $a_n(a > a_n)$, together with

(2) periodic points $P(2^{n-1})$'s of prime period 2^{n-1} (the fixed point 0 when $n = 0$) change from attractive to repelling periodic points.

The set of real numbers $\{a_n\}_{n=0}^{\infty}$ is a monotone increasing sequence such that $a_0 = 1, a_1 = 3, a_2 = 1 + \sqrt{6}, \ldots$ and it's limit point is $a(2^\infty)$. Let b_n be the number in $[0, a(2^\infty)]$ such that the critical point 1/2 becomes the periodic point of prime period 2^n for f_{b_n}. Then the numbers $\{b_n\}_{n=0}^{\infty}$ is an increasing sequence such that $b_0 = 2, b_1 = 1 + \sqrt{5}, b_2 = 3.499\ldots$ and we have

$$0 < a_0 < b_0 < a_1 < \ldots < a_n < b_n < a_{n+1} < \ldots < a(2^\infty).$$

Here, for the sake of convenience, we put $b_{-1} = 0$.

Now our purpose is to analyze the behavior of the orbit of critical point 1/2 for f_a and the periodic orbits when a passes through a_n or b_n. Especially we study the accurate order of those points in [0, 1]. Therefore, as an application, we can obtain an exact proof of the classification of the family $\{f_a : 0 < a < a(2^\infty)\}$ by topological conjugacy.

2 Bifurcations of periodic points and the orbit of the critical point

Considering the usual order in $[0,1]$, we denote by $P(2^n)_1$ the largest periodic point of prime period 2^n for f_a and put $P(2^n)_k = f_a^{k-1}(P(2^n)_1)$ for $k = 1, 2, \ldots, 2^n$, where f^k means the k-fold composition of f with itself. Moreover we put $x_0 = 1/2$ and $x_k = f_a^k(1/2)$ for $k = 1, 2, \ldots$.

2.1 Position of the points $P(2^n)_k$'s and x_k's

First we show the order of those points in $[0,1]$ for the initial 7 cases.

Case 1: $(0 = b_{-1} < a \le a_0 = 1)$

$$0 < \cdots < x_n < \cdots < x_0 < 1.$$

Case 2: $(1 = a_0 < a < b_0 = 2)$

$$0 < P(1)_1 < \cdots < x_n < \cdots < x_0 < 1.$$

Case 3: $(a = b_0 = 2)$

$$0 < x_0 = P(1)_1 < 1.$$

Case 4: $(2 = b_0 < a \le a_1 = 3)$

$$0 < x_0 < x_2 < \cdots < x_{2n} < \cdots < P(1)_1 < \cdots < x_{2n+1} < \cdots < x_3 < x_1 < 1.$$

Case 5: $(3 = a_1 < a < b_1 = 1 + \sqrt{5})$

$$0 < x_0 < x_2 < \cdots < x_{2n} < \cdots < P(2)_2 < P(1)_1 < P(2)_1 < \cdots < x_{2n+1} < \cdots < x_3 < x_1 < 1.$$

Case 6: $(a = b_1 = 1 + \sqrt{5})$

$$0 < x_0 = x_2 = x_{2n} = P(2)_2 < P(1)_1 < P(2)_1 = x_{2n+1} = x_3 = x_1 = 1.$$

Case 7: $(1 + \sqrt{5} = b_1 < a \le a_2 = 1 + \sqrt{6})$

$$0 < x_2 < x_6 < P(2)_2 < x_8 < x_4 < x_0 < P(1)_1 < x_3 < x_7 < P(2)_1 < x_5 < x_1 < 1.$$

In general we can see inductively the bifurcations of $P(2^n)_k$'s and x_k's as follows. For $n = 0, 1, 2, \ldots$ and $k = 1, 2, \ldots, 2^n$, let $r_n(k)$ and $s_n(k)$ be the indexes defined by

$$x_{r_n(k)} = \min\{x_k, x_{k+2^n}\} \qquad \text{and} \qquad x_{s_n(k)} = \max\{x_k, x_{k+2^n}\}.$$

We note that $s_n(1) = 1, r_n(2) = 2$ for every n. We begin induction with case of $(b_n, a_{n+1}]$.

(1) Now we suppose that for a in $(b_n, a_{n+1}]$ it follows that

$$x_{r_n(k)} < x_{r_n(k)+2^{n+1}} < \cdots < P(2^n)_k < \cdots < x_{s_n(k)+2^{n+1}} < x_{s_n(k)} \quad (2.1.1).$$

where each sequence in the left or right hand side converges to $P(2^n)_k$ and there are no $x_k (k \geq 1)$ outside the union $\bigcup_{k=1}^{2^n}[x_{r_n(k)}, x_{s_n(k)}]$. When $n = 0$, it is Case 4.

(2) When a moves into the open interval (a_{n+1}, b_{n+1}) the periodic points $\{P(2^{n+1})_k : k = 1, 2, \cdots, 2^{n+1}\}$ are born from $\{P(2^n)_k : k = 1, 2, \cdots, 2^n\}$ and the order is as follows:

$$x_{r_n(k)} < x_{r_n(k)+2^{n+1}} < \cdots < P(2^{n+1})_{r_n(k)} < P(2^n)_k < P(2^{n+1})_{s_n(k)} <$$

$$\cdots < x_{s_n(k)+2^{n+1}} < x_{s_n(k)} \quad (2.1.2),$$

where each sequence in the left and right hand side converges to $P(2^{n+1})_{r_n(k)}$ and $P(2^{n+1})_{s_n(k)}$ respectively and there are no $x_k (k \geq 1)$ outside the union

$$\bigcup_{k=1}^{2^n}([x_{r_n(k)}, P(2^{n+1})_{r_n(k)}] \cup [P(2^{n+1})_{s_n(k)}, x_{s_n(k)}]).$$

Here we remark that the order of $\{x_k\}_{k=0}^{\infty}$ does not change.

(3) When $a = b_{n+1}$, $x_{r_n}(k)$ and $x_{s_n}(k)$ coincide with $P(2^{n+1})_{r_n(k)}$ and $P(2^{n+1})_{s_n(k)}$ respectively and the order is shown as follows:

$$x_{r_n(k)} = P(2^{n+1})_{r_n(k)} < P(2^n)_k < P(2^{n+1})_{s_n(k)} = x_{s_n(k)}.$$

(4) When a moves into the semi-open interval $(b_{n+1}, a_{n+2}]$, though the order of position of $\{P(2^{n+1})_k : k = 1, 2, \ldots, 2^{n+1}\}$ does not change, $\{x_{r_n(k)+2^{n+1}}\}$ and $\{x_{s_n(k)+2^{n+1}}\}$ change the order of their position as follows:

$$x_{r_n(k)} < P(2^{n+1})_{r_n(k)} < x_{r_n(k)+2^{n+1}} < P(2^n)_k < x_{s_n(k)+2^{n+1}} < P(2^{n+1})_{s_n(k)} < x_{s_n(k)}.$$

This is rewritten as follows:

$$x_{r_{n+1}(r_n(k))} < x_{r_{n+1}(r_n(k)+2^{n+2})} < \ldots < P(2^{n+1})_{r_n(k)} < \cdots < x_{s_{n+1}(r_n(k)+2^{n+2})} <$$

$$x_{s_{n+1}(r_n(k))} < P(2^n)_k < x_{r_{n+1}(s_n(k))} < x_{r_{n+1}(s_n(k)+2^{n+2})} < \cdots <$$

$$P(2^{n+1})_{s_n(k)} < \cdots < x_{s_{n+1}(s_n(k)+2^{n+2})} < x_{s_{n+1}(s_n(k))}.$$

This phenomenon: $x_{r_{n+1}(j)} < P(2^{n+1})_j < x_{s_{n+1}(j)}$ corressponds to (2.1.1) in case of $(b_{n+1}, a_{n+2}]$, and these bifurcations occur repeatedly. Thus for every $n \geq 0$ we have (2.1.1) and (2.1.2) for a in $(b_n, a_{n+1}]$ and a in (a_{n+1}, b_{n+1}) respectively.

2.2 Positions of the periodic points $\{P(2^m)_k : 1 \leq k \leq 2^m\}$

From Section 2.1 we can see the following.

(2.2.1) For k and j with $0 < k, j < 2^m$, it follows that

$$\max\{P(2^{m+1})_k, P(2^{m+1})_{k+2^m}\} < \min\{P(2^{m+1})_j, P(2^{m+1})_{j+2^m}\}$$

if and only if $P(2^m)_k < P(2^m)_j$.

(2.2.2) For k with $0 < k < 2^m$, it follows that

$$P(2^{m+1})_k < P(2^{m+1})_{k+2^m} \text{ if and only if } P(2^m)_k < P(2^m)_{k_1}$$

where $k_1 \neq k$ but $k_1 \equiv k \pmod{2^{m-1}}$.

Statements (2.2.1) and (2.2.2) mean the following transition of bifurcations from the case: $a_0 < a < a_1$ to the case: $a_3 < a < a_4$ and so on.

$$\begin{gathered} P(1)_1 \\ P(2)_2 < P(2)_1 \\ P(4)_2 < P(4)_4 < P(4)_3 < P(4)_1 \\ P(8)_2 < P(8)_6 < P(8)_8 < P(8)_4 < P(8)_3 < P(8)_7 < P(8)_5 < P(8)_1. \\ \cdots\cdots\cdots\cdots\cdots\cdots. \end{gathered}$$

2.3 Position of the points $\{x_k : 1 \leq k \leq 2^n\}$ and $\{P(2^m)_k : 0 \leq m \leq n-1, 1 \leq k \leq 2^m\}$

By the preceding discussion, the order of those points is fixed for all $a > b_n$. In fact, the order of the orbit $\{x_k : 1 \leq k \leq 2^n\}$ and the periodic points of prime period $2^m(0 \leq m \leq n-1)$ does not change when a increases in $(b_n, a(2^\infty))$, and the order is expressed as follows. Suppose that a is in (b_n, b_{n+1}). For $m = 0, 1, 2, \ldots, n-1$ and $k = 1, 2, \ldots, 2^m$, let

$$x_{r(m,k)} = \min\{x_{2^{m+1}+k}, x_{2^{m+1}+k+2^m}\} \text{ and } x_{s(m,k)} = \max\{x_{2^{m+1}+k}, x_{2^{m+1}+k+2^m}\}$$

Here we note that $2^{m+1} + k = 2 \cdot 2^m + k$ and $2^{m+1} + k + 2^m = 3 \cdot 2^m + k$. Then it follows that

$$x_{r(m,k)} < P(2^m)_k < x_{s(m,k)} \qquad (2.3.1)$$

and each open interval $(x_{r(m,k)}, x_{s(m,k)})$ contains no points of $\{x_k : k \geq 1\}$.

2.4 Position of x_0 among the orbit $\{x_n\}$

Though the set $\{x_k : 1 \leq k \leq 2^n\}$ keeps their order for any $a \geq a_n$, the position of x_0 among $\{x_n\}$ changes as follows. Suppose that a is in (b_n, b_{n+1}). Then we obtain the following.

(1) If $n = 0$, then we have

$$x_0 < x_2 < P(1)_1 < x_3.$$

(2) If n is an odd number, we have

$$x_{4\cdot 2^{n-1}} < x_0 < P(2^{n-1})_{2^{n-1}} < x_{3\cdot 2^{n-1}} \qquad (2.4.1).$$

(3) If $n(>0)$ is an even number, we have

$$x_{3\cdot 2^{n-1}} < P(2^{n-1})_{2^{n-1}} < x_0 < x_{4\cdot 2^{n-1}} \qquad (2.4.2).$$

(4) There are no points of $\{x_k : k \geq 1\}$ between two points $x_{3\cdot 2^{n-1}}$ and $x_{4\cdot 2^{n-1}}$.

The change of position of x_0 from (2) to (3) can be recognized by the following inductive observation.

When $n = 1$, we have

$$x_2 < P(2)_2 < x_4 < x_0 < P(1)_1 < x_3 \qquad \text{(cf. Case 7 in 2.1)}.$$

Next we suppose that, for an odd number n and a in (b_n, b_{n+1}),

$$x_{2^n} < P(2^n)_{2^n} < x_{4\cdot 2^{n-1}} < x_0 < P(2^{n-1})_{2^{n-1}} < x_{3\cdot 2^{n-1}}.$$

When a is in (a_{n+1}, b_{n+1}), f_a has the periodic points $P(2^{n+1})_{2^{n+1}}$'s and the order is as follows:

$$x_{2^n} < P(2^{n+1})_{2^n} < P(2^n)_{2^n} < P(2^{n+1})_{2^{n+1}} < x_{2^{n+1}} < x_0 < x_{3\cdot 2^{n-1}}.$$

When a reaches to b_{n+1}, the order changes as follows:

$$x_{2^n} = P(2^{n+1})_{2^n} < P(2^n)_{2^n} < P(2^{n+1})_{2^{n+1}} = x_{2^{n+1}} = x_0 < x_{3\cdot 2^{n-1}}.$$

Moreover when a moves into (b_{n+1}, a_{n+2}), we have

$$x_{2^n} < P(2^{n+1})_{2^n} < x_{2^n+2^{n+1}} < P(2^n)_{2^n} < x_0 < x_{2^{n+1}+2^{n+1}} < P(2^{n+1})_{2^{n+1}} < x_{2^{n+1}} < x_{3\cdot 2^{n-1}}.$$

This is rewritten as follows:

$$x_{3\cdot 2^n} < P(2^n)_{2^n} < x_0 < x_{4\cdot 2^n} < P(2^{n+1})_{2^{n+1}} < x_{2^{n+1}}.$$

The change from (3) to (2) can be recognized in the same fashion. Thus we have Statements (1)~(4) for $n > 0$.

3 Applications

3.1 Conjugate classes

Two maps f and g on $[a,b]$ are said to be topologically conjugate if there exists a homeomorphism $h:[a,b]\to[a,b]$ such that $h\circ f=g\circ h$, and h is said to be a conjugacy for f and g. Suppose that two quadratic maps f_a and f_b on $[0,1]$ are topologically conjugate by a conjugacy h. Since the critical point $1/2$ for f_a and f_b is invariant for h, the forward orbits of $1/2$ for f_a and f_b have the same order in $[0,1]$. In addition, these are combinatorially equivalent, namely the backward orbits for two maps have the same order, too [3: Chap.II. Theorem 3.1].

Now we put $F_n=\{f_a : a_n<a<b_n\}, G_n=\{f_a : a=b_n\}$ and $H_n=\{f_a : b_n<a\le a_{n+1}\}$. Then the results shown in the preceding section mean that distinct two maps are combinatorially equivalent if and only if the maps belong to $H_n\bigcup F_{n+1}$ for some $n\ge -1$. Moreover, each map in $F_n\bigcup G_n\bigcup H_n$ has a periodic orbit of prime period 2^n. Thus, if f_a and f_b are topologically conjugate then the two maps belong to the same family. In this section we prove that the converse statement is valid. Namely we have the following theorem.

Theorem 3.1. *Each family in $\{F_n, G_n, H_{n-1} : n\ge 0\}$ is a topologically conjugate class.*

Now, at first we consider cases of H_{-1} and F_0. Suppose that f_a and f_b are in H_{-1}. Let $x_n=f_a^n(1/2)$ and $y_n=f_b^n(1/2)$. Seeing the order of $\{x_n\}$ and $\{y_n\}$ (cf. Section 2.1, Case 1), we can take a monotone increasing homeomorphism h_0 of $[x_1,y_0]$ onto $[y_1,y_0]$. Since $f_a([x_{n+1},x_n])=[x_{n+2},x_{n+1}]$, $f_a([x_0,1])=[0,x_1]$ and $f_b([y_{n+1},y_n])=[y_{n+2},y_{n+1}]$, $f_a([y_0,1])=[0,y_1]$ respectively, h_0 can be extended to a homeomorphism of $[0,1]$ onto itself with $h\circ f_a=f_b\circ h$. In the case where f_a and f_b are in F_0, two homeomorphisms

$$h_0:[x_1,x_0]\to[y_1,y_0] \text{ and } k_0:[z_0,z_1]\to[w_0,w_1]\ ,$$

can be extended to a conjugacy for f_a and f_b, where $0<z_0<\lim_{n\to\infty}x_n, z_1=f_a(z_0)$ and $0<w_0<\lim_{n\to\infty}y_n, w_1=f_b(w_0)$ (cf. Section 2.1, Case 2).

For other cases we can obtain a conjugacy for f_a and f_b in the same family similarly to the cases above, though we have to check more complex phenomena shown in Section 3. However in this note we show that two maps in the same family in $\{F_n, G_n\}$ satisfy a condition for those maps to be topologically conjugate. Such conditions are mentioned in [4:Chap.II, Theorem 3.1] and [1:Theorem II.6.3]. Here we use the latter condition since it is more useful directly in our case. First we confirm the definition of kneading sequence used in [1]. The kneading sequence $I_a(1/2)$ is a sequence consisting of three alphabets R, C, and L, where $j(\ge 1)$-th element of $I_a(1/2)$ is R, C, or L if $x_j>1/2$, $x_j=1/2$ or $x_j<1/2$ respectively, where x_n is n-th image of critical (turning) point $1/2$. Then by the results in Section 2, we have the following.

(1) $I_a(1/2)$ is invariant for f_a in $H_n \cup F_{n+1}$.

(2) For f_a in F_n (resp. H_n), $I_a(1/2) = \mathrm{D}^\infty$ (= the infinite repeat of D), where D is a finite sequence with period 2^{n-1} (resp. 2^n) and consists of two kinds of letters R and L, where each number is odd.

(3) For f_a in $F_n \cup G_n \cup H_n$, the stable orbit of f_a is the set $\{P(2^n)_k : 1 \le k \le 2^n\}$ of periodic points with prime period 2^n.

Especially we note that Section 2.4 implies that for a in (b_n, b_{n+1}), the number d_n of R's in (2) in the statements above is determined as follows:

$$d_0 = 1 \quad \text{and} \quad d_n = \begin{cases} 2d_{n-1} - 1 & \text{if } n \text{ is odd,} \\ 2d_{n-1} + 1 & \text{if } n \text{ is even.} \end{cases}$$

Now the family $f_a(x) = ax(1-x)$ on $[1-x_1, x_1]$ $(2 < a \le 4)$ corresponds to those functions $f_\mu(x) = 1-\mu x^2$ $(0 < \mu = a^2/4 - a/2 \le 2)$ which belong to a family of functions discussed in [1:Theorem II.6.3]. Thus by 2.a of Theorem II.6.3, it follows that two maps f_a and f_b in the same family, F_n or H_n, are topologically conjugate as functions on $[f_a^2(1/2), f_a(1/2)]$ and $[f_b^2(1/2), f_b(1/2)]$. From this we can easily drive that f_a and f_b are topologically conjugate as functions on $[0, 1]$ in the same way as in case of H_{-1}. Consequently we complete the proof.

3.2 Covariant representation of topological dynamics

We consider that the topological dynamical systems $([0,1], f_a)$ have fruitful non-commutative structure as well as the case of homeomorphisms (cf. [5]). The non-commutative structure would be represented as covariant representation. Those representations of $([0,1], f_a)$ are studied in [2] and the orbit structure described in Section 2 of this paper plays an important role.

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References

[1] P.Collet and J.P.Eckmann, *Iterated maps on the intervals as dynamical systems*, Birkhäuser, Boston, 1980.

[2] S. Kawamura, Covariant representations of one-dimensional unimodal maps (in preparation).

[3] R.M.May, Simple mathematical models with very complicated dynamics, Nature, 261(1976), 459-467.

[4] W. de Melo and S. van Strien, *One-dimensional dynamics, Ergebnisse der Mathematik und ihrer Grenzgebiete* 25, Springer, Berlin, 1993.

[5] J. Tomiyama, *Invitation to C^*-algebras and topological dynamics*, World Scientific, Singapore, 1987.

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