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Holomorphic Solutions of Some Functional Equations

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1. Introduction

Let $X(x, y)$ and $Y(x, y)$ be holomorphic functions in $|x|<t, |y|<t$, which are expanded there as

$$(1.1)\qquad \begin{cases} X(x, y)=\lambda x+\sum_{m+n\geq 2} p_{mn} x^{m} y^{n}=\lambda x+X_{1}(x, y) \\ \\ Y(x, y)=\mu y+\sum_{m+n\geq 2} q_{mn} x^{m} y^{n}=\mu y+Y_{1}(x, y). \end{cases}$$

We suppose that $\lambda \neq 0$. Our aim in this note is to show the following theorems.

Theorem 1. Suppose that $|\lambda|>1$ and $\lambda^{n} \neq \mu$ for any $n \in N$. Then there exists uniquely a function $\psi(x)$, which is holomorphic in some disc $|x|<\delta$ and satisfies the equation

$$(1.2)\qquad \psi(X(x, \psi(x)))=Y(x, \psi(x)).$$

Theorem 2. Suppose that $0<|\lambda|<1$ and $\mu \neq 0$ in (1.1). Further we suppose that $\lambda^{n} \neq \mu$ for $n \in N$. Then there exists uniquely a function $\psi(x)$, which is holomorphic in some disc $|x|<\delta$ and satisfies the equation (1.2).

Now we will consider the meaning of the equation (1.2).

Consider a simultaneous system of difference equations:

$$(1.4)\qquad \begin{cases} x(t+1) = X(x(t), y(t)), \\ y(t+1) = Y(x(t), y(t)). \end{cases}$$

Suppose (1.4) admits a solution $(x(t), y(t))$. If there is a function $\psi(x)$ such that $y(t)=\psi(x(t))$ for all t. Then $\psi(x)$ satisfies the equation (1.2). Conversely, suppose $\psi(x)$ satisfies (1.2). If $x(t)$ is a solution of

$$(1.5)\qquad x(t+1) = X(x(t), \psi(x(t))),$$

then $(x(t), y(t))$, where $y(t)=\psi(x(t))$, is a solution of (1.4).

A system of differential equations corresponding to (1.4) is

$$(1.4')\qquad \begin{cases} \dot{x}(t) = X(x(t), y(t)), \\ \dot{y}(t) = Y(x(t), y(t)). \end{cases}$$

The system (1.4') is equivalent to the equation

$$(1.6)\qquad \psi'(x)X(x, y) = Y(x, y),$$

or

$$(1.6')\qquad \frac{dy}{dx} = \frac{Y(x, y)}{X(x, y)}.$$

Thus we know that (1.2) is the equation corresponding to (1.6) for the system of differential equations.

Applications of theorems 1, 2 will appear in our forthcoming papers.

2. Proof of Theorem 1.

From the assume of $\lambda^n \neq \mu$ for any $n \in N$, we can determine formal power series $\psi(x)=\sum_{n=1}^{\infty} a_n x^n$ by (1.2) such as

$$\begin{cases} a_1 = 0 \\ a_n = \dfrac{f_n(\lambda, a_i, p_{i,j}, q_{i,j})}{(\lambda^n - \mu)}, \quad n \geqq 2, \end{cases}$$

where f_n is a polynomial in $\lambda, a_i, p_{i,j}, q_{i,j}$ $(i \leqq n-1, i+j \leqq n)$. Hence the solution $\psi(x)$ is unique if it exists.

Take an integer N so large that $\left|\frac{\mu}{\lambda^N}\right| < \frac{1}{2}$. Put $g_N(x) = a_2 x^2 + \cdots + a_N x^N$ and define the family F to be

$$F = \{\phi(x);\ \text{holomrphic and } |\phi(x)| \leqq K|x|^{N+1} \text{ in } |x| \leqq \delta\}$$

where δ and K are to be determined later.

Take $\phi(x) \in F$ and put

(2.1) $$u = \lambda x + X_1(x, g_N(x) + \phi(x)) = X(x, g_N(x) + \phi(x)).$$

We have that $|X_1(x, g_N(x) + \phi(x))| < \frac{|x|}{2}$ if δ is small, where δ can be chosen independently of $\phi(x)$. Thus we obtain inverse function $x = \eta(u)$ for $|u| < \delta'$, where δ' can be chosen independently of $\phi(x)$. We also have that

(2.2) $$|u| \geqq |\lambda x| - |X_1(x, g_N(x) + \phi(x))| > \lambda'|x|$$

for a λ', $1 < \lambda' < |\lambda|$. Thus $\phi(\eta(u))$ is defined if $\phi(u)$ is defined for $|u| \leqq \delta'$.

We may also assume that $\alpha = \frac{|\mu|}{\lambda'^N} < 1$.

For $\phi(x) \in F$, we put

(2.3) $$\begin{aligned} T[\phi](u) &= Y(\eta(u), g_N(\eta(u)) + \phi(\eta(u))) - g_N(u) \\ &= \{Y(x, g_N(x) + \phi(x)) - Y(x, g_N(x))\} \\ &\quad + \{Y(x, g_N(x)) - g_N(X(x, g_N(x))\} \\ &\quad + \{g_N(X(x, g_N(x)) - g_N(u)\} \\ &= U + V + W. \end{aligned}$$

We have $|V| \leqq K_1'|x|^{N+1} \leqq K_1|u|^{N+1}$ for a constant K_1.

$$\begin{aligned} |W| &= \left| \int_0^{\phi(x)} \frac{d}{ds}\left(g_N(X(x, g_N(x) + s)\right) ds \right| \\ &\leqq K_2'|x| \cdot |\phi(x)| \\ &\leqq K_2|u| \cdot K|u|^{N+1} \end{aligned}$$

for a constant K_2, since $a_1=0$ as noted above. On the other hand,

$$U= \mu\phi(x)+Y_1(x, g_N(x)+\phi(x))-Y_1(x, g_N(x)),$$

in which

$$|\mu\phi(x)|\leqq|\mu|\cdot K|x|^{N+1}\leqq \alpha K|u|^{N+1}$$

and

$$|Y_1(x, g_N(x)+\phi(x))-Y_1(x, g_N(x))|\leqq K_3|u|\cdot K|u|^{N+1}$$

for constant K_3, which is seen as in the estimation of W.

δ' is so small that $\alpha + (K_2+K_3)\delta'=A<1$. Take K so large that

$$\frac{K_1}{1-A}<K,$$

and δ is taken as $\delta\leqq\delta'$ and further small if necessary.

Thus the family F is obtained, and we get that the operator T defined in (2.4) maps F into F. F is clearly convex, and a normal family by the theorem of Montel. Since T is obviously continuous, we get a fixed point $\phi_N(x)$ by the theorem of Tychonoff, see [1] and [2].

Then $\psi(x)=g_N(x)+\phi_N(x)$ obviously satisfies (1.2). Q.E.D.

3. Proof of Theorem 2.

Formal power series $\psi(x)=\sum_{n=2}^{\infty}a_n x^n$ is determined as in § 2.

Put $u=Y(x, y)=\mu y + Y_1(x, y)$. Since $\mu\neq 0$, we have

$$y=H(x, u)=\left(\frac{1}{\mu}\right)u + H_1(x, u).$$

Thus the equation (1.2) is equivalent to

(3.1) $$\psi(x)=H(x, \psi(X(x, \psi(x)))).$$

Take integer N so large that $\left|\frac{\lambda^N}{\mu}\right|<\frac{1}{2}$, define $g_N(x)$, and put

$$F=\{\phi(x);\ \text{holomorphic and } |\phi(x)|\leqq K|x|^{N+1} \text{ in } |x|\leqq\delta\}$$

where δ and K are to be determined later.

For $\phi(x)\in F$, we get

$$|X(x,\ g_N(x)+\phi(x))|\leqq|\lambda x| + |X_1(x,\ g_N(x)+\phi(x))|\leqq\lambda'|x|$$

with $|\lambda|<\lambda'<1$.

We may assume that $\alpha=\dfrac{\lambda'^N}{|\mu|}<1$.

Take $\phi(x)\in F$ and put

$$\begin{aligned}
(3.2)\quad T[\phi](x) &= H(x,\ g_N(X(x,\ g_N(x)+\phi(x)))+\phi(X(x,\ g_N(x)+\phi(x)))) \\
&\qquad - g_N(x) \\
&= \{H(x,\ g_N(X(x,\ g_N(x)+\phi(x)))+\phi(X(x,\ g_N(x)+\phi(x)))) \\
&\qquad -H(x,\ g_N(X(x,\ g_N(x)+\phi(x))))\} \\
&\quad + \{H(x,\ g_N(X(x,\ g_N(x)+\phi(x)))) -H(x,\ g_N(X(x,\ g_N(x))))\} \\
&\quad + \{H(x,\ g_N(X(x,\ g_N(x)))) - g_N(x)\} \\
&= U + V + W.
\end{aligned}$$

Obviously we have

$$|W|\leqq K_1|x|^{N+1} \text{ for a constant } K_1.$$

As in the proof of Theorem 1, we have, with a constant K_2,

$$\begin{aligned}
|V| &= \left|\int_0^{\phi(x)} \frac{d}{ds}\Big(H(x,\ g_N(X(x,\ g_N(x)+s)))\Big)\, ds\right| \\
&\leqq K_2|x|\cdot|\phi(x)| \\
&\leqq K_2|x|\cdot K|x|^{N+1}.
\end{aligned}$$

$$\begin{aligned}
|U| &= |\mu|^{-1}|\phi(X(x,\ g_N(x)+\phi(x))))| \\
&\qquad +|H_1(x,\ g_N(X(x,\ g_N(x)+\phi(x)))+\phi(X(x,\ g_N(x)+\phi(x)))) \\
&\qquad\qquad -H(x,\ g_N(X(x,\ g_N(x)+\phi(x))))\}| \\
&= U_1 + U_2,
\end{aligned}$$

in which

$U_1 \leqq |\mu|^{-1} |\phi(X(x, g_N(x)+\phi(x)))|$

$U_2 \leqq K_3 |x| \cdot K |x|^{N+1}$.

Thus, if δ is so small that

$$A = \alpha + (K_2 + K_3)\delta < 1$$

and K is taken so large that $\dfrac{K_1}{1-A} < K$, then the operator T in (2.6) maps F into F, and we know the existence of the fixed point as in the proof of Theorem 1. Q.E.D.

Reference

[1]. D.R. Smart, "Fixed point theorems", Cambridge Univ. Press, (1974).

[2]. T. Kimura, "Ordinary differential equation", Kyouritu shuppan Press, (1974).

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