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CURVATURE–ADAPTED SUBMANIFOLDS

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Abstract

We show that the classification of curvature-adapted submanifolds in $\mathfrak{P}$-spaces can be reduced to that of curvature-adapted hypersurfaces by using tubes about submanifolds. Moreover, we treat the special case of non-flat complex and quaternionic space forms. This leads to a complete classification of the curvature-adapted submanifolds in quaternionic projective spaces.

1. Introduction

In this note we study a certain class of submanifolds whose extrinsic curvature is adapted in a natural way to the intrinsic curvature of the ambient Riemannian manifold. A general measure for the extrinsic curvature of a submanifold is provided by all the shape operators A_ξ with respect to normal vectors ξ. Given a normal vector ξ to a submanifold, the Jacobi operator $R_\xi := R(.,\xi)\xi$ measures the intrinsic curvature of the ambient Riemannian manifold $\bar{M}$ in the direction of ξ. Here, R denotes the Riemannian curvature tensor of $\bar{M}$. Both A_ξ and R_ξ are self-adjoint operators; their eigenvalues represent extremal curvatures, and their eigenspaces point out directions for which the curvature becomes extremal. We say that a submanifold M of a Riemannian manifold $\bar{M}$ is *curvature-adapted* if for every normal vector ξ to M, say at $p \in M$, the Jacobi operator R_ξ leaves the tangent space T_pM of M at p invariant, that is, if

(1) $$R_\xi(T_pM) \subset T_pM,$$

and if there exists a basis of T_pM consisting of eigenvectors both of A_ξ and $K_\xi := R_\xi|T_pM$, that is, if

(2) $$A_\xi \circ K_\xi = K_\xi \circ A_\xi.$$

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The operator K_ξ is called the normal Jacobi operator of M with respect to ξ. Clearly, in order to see whether a submanifold is curvature-adapted, it suffices to check (1) and (2) for *unit* normal vectors.

Obviously, every submanifold of a space of constant curvature is curvature-adapted. Also, every totally umbilical hypersurface of a Riemannian manifold is curvature-adapted. Note that condition (1) is always satisfied for hypersurfaces.

Special cases of curvature-adapted submanifolds already appeared in the literature in various contexts. For instance, J.E. D'Atri [9] studied isoparametric hypersurfaces in symmetric spaces which are *amenable*. For the rank one case this notion turns out to be equivalent to that of curvature-adaptedness. A. Gray [10] studied tubes about curvature-adapted (he calls them *compatible*) submanifolds in symmetric spaces. Further, the first author [3] classified all curvature-adapted real hypersurfaces in quaternionic projective spaces. And the authors [4] studied recently properties of Riemannian manifolds all of whose (sufficiently small) geodesic hyperspheres are curvature-adapted.

In this note we continue the study of curvature-adapted submanifolds. In Section 2 we discuss the relations between this notion for submanifolds and the tubes about them hereby showing that for the so-called $\mathfrak{P}$-spaces (see [4]) the classification of all curvature-adapted submanifolds is entirely determined by that of the curvature-adapted hypersurfaces. In Section 3 we discuss this for complex space forms and in Section 4 for the quaternionic space forms, hereby obtaining a complete classification of the curvature-adapted submanifolds in quaternionic projective spaces.

2. Tubes and curvature-adapted submanifolds

A locally symmetric space $\bar{M}$ can be characterized by the property that for every geodesic γ in $\bar{M}$ the associated Jacobi operator $R_\gamma := R(.,\dot{\gamma})\dot{\gamma}$ has constant eigenvalues and is diagonalizable by a parallel orthonormal frame field along γ. In [4] the authors studied Riemannian manifolds all of whose Jacobi operators R_γ have constant eigenvalues (so-called $\mathfrak{C}$-spaces) or are diagonalizable by a parallel orthonormal frame field along the affiliated geodesic γ (so-called $\mathfrak{P}$-spaces). In particular, we proved

Theorem 1. [4] *Let M be a curvature-adapted submanifold of a $\mathfrak{P}$-space $\bar{M}$. Then the tubes about M are also curvature-adapted in $\bar{M}$.*

Note that the tubes are always defined at least locally and for small radii.

Theorem 2. [4] *A real analytic Riemannian manifold is a $\mathfrak{P}$-space if and only if all its (sufficiently small) geodesic hyperspheres are curvature-adapted.*

In the special case of locally symmetric spaces Theorem 1 has been proved by A. Gray [10, Theorem 6.14]. Theorem 2 provides us with particular examples of curvature-adapted submanifolds, namely geodesic hyperspheres in locally symmetric spaces and, more generally, in $\mathfrak{P}$-spaces (see [4] for examples of non-symmetric $\mathfrak{P}$-spaces). We shall now be concerned with a converse of Theorem 1 for general Riemannian manifolds.

Theorem 3. *Let M be a submanifold of a Riemannian manifold $\bar{M}$. If the (sufficiently small) tubes about M are curvature-adapted, then M is also curvature-adapted.*

Proof. Let p be any point in M and ξ a unit normal vector of M at p. We choose a geodesic γ in $\bar{M}$ defined on an open interval $I \subset \mathbb{R}$ such that $0 \in I$, $p = \gamma(0)$ and $\xi = \dot{\gamma}(0)$. Let D be the solution of the $\mathrm{End}(T\bar{M})$-valued Jacobi equation

$$Y'' + R_\gamma \circ Y = 0$$

along γ with initial values

$$Y(0) = \begin{pmatrix} id_{T_pM} & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad Y'(0) = \begin{pmatrix} -A_\xi & 0 \\ 0 & id_{\perp_p M} \end{pmatrix}$$

where the matrix decomposition is with respect to the orthogonal decomposition $T_p\bar{M} = T_pM \oplus \perp_p M$. For some sufficiently small $\epsilon \in \mathbb{R}_+$ we may define

$$B := D' \circ D^{-1} |]0, \epsilon[.$$

B is a solution of the $\mathrm{End}(T\bar{M})$-valued Riccati equation

$$B' + B^2 + R_\gamma = 0$$

along $\gamma |]0, \epsilon[$. It is known that $B(r)|(\mathbb{R}\dot{\gamma}(r))^\perp$ is the shape operator of the tube M_r of radius $r \in]0, \epsilon[$ about M with respect to $-\dot{\gamma}(r)$. Moreover, $R_\gamma(r)|(\mathbb{R}\dot{\gamma}(r))^\perp$ is the normal Jacobi operator of M_r with respect to $-\dot{\gamma}(r)$. Hence, by means of our assumption and as $\dot{\gamma}$ is an eigenvector of B and R_γ on $]0, \epsilon[$, we have

$$B \circ R_\gamma = R_\gamma \circ B.$$

Alas, B cannot be extended continuously to 0. Therefore we define

$$C(r) := \begin{cases} rB(r) & \text{, if } r \in]0, \epsilon[, \\ \begin{pmatrix} 0 & 0 \\ 0 & id_{\perp_p M} \end{pmatrix} & \text{, if } r = 0 \end{cases}$$

(see for instance [5, p. 161] for the special case where M is a single point and [11] for the general case). C is a differentiable tensor field of $T\bar{M}$ along $\gamma | [0, \epsilon[$ with

$$C'(0) = \begin{pmatrix} -A_\xi & 0 \\ 0 & 0 \end{pmatrix}$$

and, by means of the corresponding property of B,

$$C \circ R_\gamma = R_\gamma \circ C \quad \text{on }]0, \epsilon[.$$

We now write

$$C(r) = C(0) + rC'(0) + \text{remainder term},$$
$$R_\gamma(r) = R_\gamma(0) + rR'_\gamma(0) + \text{remainder term}.$$

From these expressions and the above commutativity condition we get by a standard argument

$$C(0)R_\gamma(0) = R_\gamma(0)C(0)$$

and

$$C(0)R'_\gamma(0) + C'(0)R_\gamma(0) = R'_\gamma(0)C(0) + R_\gamma(0)C'(0).$$

It can be seen easily that the first of these two conditions is equivalent to

$$R_\xi(T_pM) \subset T_pM,$$

and using this fact the second condition is equivalent to

$$A_\xi \circ K_\xi = K_\xi \circ A_\xi.$$

This proves that M is curvature-adapted. □

Combining now Theorem 1 and Theorem 3 we get

Corollary 1. *Let M be a submanifold of a $\mathfrak{P}$-space $\bar{M}$. Then M is curvature-adapted if and only if all the (sufficiently small and at least locally defined) tubes about M are curvature-adapted.*

Hence, for a particular $\mathfrak{P}$-space, the determination of all the curvature-adapted submanifolds depends on that of the curvature-adapted hypersurfaces. In the two following sections we shall discuss this problem for complex and quaternionic space forms. Note that due to Theorem 2 the preceding corollary is not true for general Riemannian manifolds. This is easily seen by choosing M to be a single point.

We finish this section by recalling the notion of focal sets, which play an important role in the following. Let M be a submanifold of a complete Riemannian manifold $\bar{M}$ and $\xi \in \perp^1 M$ a unit normal vector of M at some point $p \in M$. Moreover, let $\gamma_\xi : [0, \infty[\to \bar{M}$ be the geodesic in $\bar{M}$ with $\gamma_\xi(0) = p$ and $\dot{\gamma}_\xi(0) = \xi$. The point $\gamma_\xi(r)$ $(r > 0)$ is said to be a *focal point* of M along γ_ξ if the differential of the normal exponential map of M is singular at $r\xi$. It might happen that there are no focal points along γ_ξ. But if there are any, we put $r_\xi := \min\{r > 0 \mid \gamma_\xi(r)$ is a focal point of M along $\gamma_\xi\}$ and call $\gamma_\xi(r_\xi)$ the *first focal point* of M along γ_ξ. By the *focal set* of M we mean the set of the first focal points of M along all the orthogonally emanating geodesics $\gamma_\xi, \xi \in \perp^1 M$.

3. Curvature-adapted submanifolds in complex space forms

Let $(\bar{M}, g, J)$ be a Kähler manifold of constant holomorphic sectional curvature $c \neq 0$ and of complex dimension $n > 1$. The Riemannian curvature tensor R of $\bar{M}$ is given by

$$R(X,Y)Z = \frac{c}{4}(g(Y,Z)X - g(X,Z)Y + g(JY,Z)JX - g(JX,Z)JY - 2g(JX,Y)JZ).$$

Hence we have for every unit tangent vector ξ of $\bar{M}$, say at $p \in \bar{M}$,

$$R_\xi X = \frac{c}{4}(X - g(X,\xi)\xi + 3g(X,J\xi)J\xi).$$

Thus R_ξ has three distinct constant eigenvalues, namely $0, c$ and $c/4$; the corresponding eigenspaces are $\mathbb{R}\xi, \mathbb{R}J\xi$ and the orthogonal complement of $\mathbb{R}\xi \oplus \mathbb{R}J\xi$ in $T_p\bar{M}$. So the shape operator A_ξ of a submanifold M of $\bar{M}$ and R_ξ satisfy (1) and (2) if and only if either $J\xi$ is normal to M or $J\xi$ is tangent to M and an eigenvector of A_ξ. Therefore we have

Proposition 1. *Let M be a connected submanifold of $\bar{M}$. Then M is curvature-adapted if and only if one of the following two statements is valid :*

(a) M is a complex submanifold of $\bar{M}$;
(b) J maps the normal bundle of M into the tangent bundle of M and for every unit normal vector ξ of M the vector $J\xi$ is a principal curvature vector of M with respect to ξ.

Let M be an orientable real hypersurface in $\bar{M}$ and ξ a unit normal field on M. The preceding proposition says that M is curvature-adapted if and only if $J\xi$ is a principal curvature vector of M everywhere. Real hypersurfaces in complex space forms satisfying the latter condition appear frequently in the literature (see [1] and the references there). In [1] the first author introduced the notion of *Hopf hypersurface* for such a hypersurface. This notion can be motivated by the fact that $J\xi$ is a principal curvature vector everywhere if and only if the foliation on M induced by the integral curves of $J\xi$ is totally geodesic. In the special situation of the unit sphere in $\mathbb{C}^n$ this foliation is just the well-known Hopf foliation of the sphere by great circles. Combining now Proposition 1 and Corollary 1 we have

Corollary 2. *A submanifold M in $\bar{M}$ is curvature-adapted if and only if all (sufficiently small) tubes about M are Hopf hypersurfaces.*

So the whole classification of the curvature-adapted submanifolds in non-flat complex space forms is reduced to that of Hopf hypersurfaces. We shall now discuss this in more detail for the ambient spaces $\mathbb{C}P^n$ and $\mathbb{C}H^n$, where $\mathbb{C}P^n$ and $\mathbb{C}H^n$ denote the n-dimensional complex projective and hyperbolic space of constant holomorphic sectional curvature $+4$ and -4, respectively.

1. Complex submanifolds. The zero set of one or several homogeneous polynomials on $\mathbb{C}^{n+1}$ always determines a complex submanifold in $\mathbb{C}P^n$. Conversely, every compact complex submanifold of $\mathbb{C}P^n$ can be realized in this way, that is, is an algebraic submanifold of $\mathbb{C}P^n$ (see [8]). As $\mathbb{C}H^n$ is biholomorphically equivalent to an open ball in $\mathbb{C}^n$, every complex submanifold in $\mathbb{C}^n$ determines one in $\mathbb{C}H^n$. Clearly, there are no compact complex submanifolds in $\mathbb{C}H^n$.

2. Hopf hypersurfaces. (See [1] for more details.) By means of Proposition 1 and Corollary 2 every tube about a complex submanifold in $\mathbb{C}P^n$ or $\mathbb{C}H^n$ is a Hopf hypersurface. Conversely, a result of T.E. Cecil and P.J. Ryan [7] says that every tubular

Hopf hypersurface in $\mathbb{C}P^n$ can be realized as a tube about a complex submanifold in $\mathbb{C}P^n$ (and perhaps also as a tube about another kind of submanifold; that depends on the structure of the focal set of the hypersurface). This is not true in $\mathbb{C}H^n$. In fact, the focal set of a tube M about the n-dimensional totally geodesic real hyperbolic subspace $\mathbb{R}H^n$ in $\mathbb{C}H^n$ is $\mathbb{R}H^n$ itself, which implies that M cannot be realized as a tube about a complex submanifold. Note that in the corresponding projective situation a tube about $\mathbb{R}P^n$ in $\mathbb{C}P^n$ can also be regarded as a tube about the complex quadric in $\mathbb{C}P^n$ (see [7]). Moreover, there are Hopf hypersurfaces in $\mathbb{C}H^n$ which are not tubular. For instance, a horosphere in $\mathbb{C}H^n$ is a Hopf hypersurface without any focal points. All its parallel hypersurfaces are also horospheres. However, we do not know any examples of non-tubular Hopf hypersurfaces in $\mathbb{C}P^n$.

3. Non-complex curvature-adapted submanifolds of codimension greater than one. It is easy to convince oneself that the totally geodesic real hyperbolic subspace $\mathbb{R}H^n$ is a curvature-adapted submanifold in $\mathbb{C}H^n$. (Note that condition (1) does not hold for the lower-dimensional totally geodesic real hyperbolic subspaces $\mathbb{R}H^k, 0 < k < n$). To our knowledge this is the only known example of a curvature-adapted submanifold in $\mathbb{C}H^n$ belonging to the class considered here. In case of $\mathbb{C}P^n$ we may use Corollary 2 and the classification (see [13]) of all Hopf hypersurfaces in $\mathbb{C}P^n$ with constant principal curvatures to get a few examples. (Note that the corresponding classification for $\mathbb{C}H^n$ in [2] does not provide any examples except the above mentioned $\mathbb{R}H^n$.) From this classification and the results in [6] we know that the Hopf hypersurfaces in $\mathbb{C}P^n$ with constant principal curvatures are precisely the tubes about the complex normally homogeneous submanifolds in $\mathbb{C}P^n$. Let N be complex normally homogeneous submanifold in $\mathbb{C}P^n$. Every tube about N can also be regarded as a tube about its focal set P. Thus, by means of Corollary 2, P is a curvature-adapted submanifold of $\mathbb{C}P^n$. The focal sets of the complex normally homogeneous submanifolds in $\mathbb{C}P^n$ have been computed explicitly in [6]. Selecting the non-complex ones we get the following further examples of curvature-adapted submanifolds in $\mathbb{C}P^n$:

- the focal set of the complex quadric in $\mathbb{C}P^n$; this is precisely $\mathbb{R}P^n$ (codimension n) ;
- the focal set of the Segre embedding of $\mathbb{C}P^1 \times \mathbb{C}P^m$ in $\mathbb{C}P^{2m+1}, m \geq 2$ (codimension three);
- the focal set of the Plücker embedding of the complex Grassmann manifold $\mathbb{C}G_{2,5}$ (of all two-dimensional linear subspaces in $\mathbb{C}^5$) in $\mathbb{C}P^9$ (codimension five);
- the focal set of the half spin embedding of the Hermitian symmetric space $SO(10)/U(5)$ in $\mathbb{C}P^{15}$ (codimension seven);

(see also [1] for more details). We do not know any other examples of curvature-adapted submanifolds in $\mathbb{C}P^n$ belonging to this class.

4. Curvature-adapted submanifolds in quaternionic space forms

Let $(\bar{M}, g, \mathfrak{J})$ be a quaternionic Kähler manifold of constant quaternionic sectional curvature $c \neq 0$ and of quaternionic dimension $n \geq 2$. Here, $\mathfrak{J}$ denotes the quaternionic

Kähler structure of $\bar{M}$. The Riemannian curvature tensor R of $\bar{M}$ is locally of the form

$$R(X,Y)Z = \frac{c}{4}(g(Y,Z)X - g(X,Z)Y + \sum_{i=1}^{3}(g(J_iY,Z)J_iX - g(J_iX,Z)J_iY - 2g(J_iX,Y)J_iZ)),$$

where J_1, J_2, J_3 is a canonical local basis of $\mathfrak{J}$ (see [12]). Hence, if ξ is a unit tangent vector of $\bar{M}$ at some point $p \in \bar{M}$, we have

$$R_\xi X = \frac{c}{4}(X - g(X,\xi)\xi + 3\sum_{i=1}^{3} g(X, J_i\xi)J_i\xi).$$

We see that R_ξ has three distinct eigenvalues, namely $0, c$ and $c/4$ of multiplicity $1, 3$ and $4(n-1)$, respectively; the corresponding eigenspaces are $\mathbb{R}\xi, \mathfrak{D}_0(\xi)$ and $\mathfrak{D}(\xi)$, respectively, where

$$\mathfrak{D}_0(\xi) := \{J\xi \mid J \in \mathfrak{J}\}$$

and $\mathfrak{D}(\xi)$ is the orthogonal complement of $\mathbb{R}\xi \oplus \mathfrak{D}_0(\xi)$ in $T_p\bar{M}$. Analogous to the complex case we now get

Proposition 2. *Let M be a connected submanifold of $\bar{M}$. Then M is curvature-adapted if and only if M satisfies one of the following two conditions :*

(a) M is a quaternionic submanifold of $\bar{M}$;
(b) there exists a k-dimensional ($k \in \{1,2,3\}$) subbundle $\mathfrak{J}_0$ of $\mathfrak{J}$ such that the endomorphisms in $\mathfrak{J}_0$ map normal spaces of M into tangent spaces of M and for every unit normal vector ξ of M the shape operator A_ξ maps the space $\{J\xi \mid J \in \mathfrak{J}_0\}$ into itself.

Combined with Corollary 1 this yields

Corollary 3. *A submanifold M of $\bar{M}$ is curvature-adapted if and only if the shape operator A of every (sufficiently small) tube about M maps $\mathfrak{D}$ into itself (or equivalently, $\mathfrak{D}_0$ into itself).*

In [3] the first author classified all curvature-adapted real hypersurfaces in the quaternionic projective space $\mathbb{H}P^n$, endowed with the Fubini-Study metric of constant quaternionic sectional curvature $+4$, that is, all real hypersurfaces with the property $A\mathfrak{D} \subset \mathfrak{D}$. Using this classification we shall now deduce a complete classification of all curvature-adapted submanifolds in $\mathbb{H}P^n$.

Theorem 4. *A connected submanifold M in $\mathbb{H}P^n$ ($n \geq 2$) is curvature-adapted if and only if it is congruent to an open part of one of the following submanifolds in $\mathbb{H}P^n$:*

(I) the k-dimensional totally geodesic quaternionic projective subspace $\mathbb{H}P^k, k \in \{0,...,n-1\}$;
(II) the n-dimensional totally geodesic complex projective subspace $\mathbb{C}P^n$;

(III) $Q^{n-1} := \{[z+vj] \mid z,v \in \mathbb{C}^{n+1}$ *Hermitian orthonormal* $\}$;
(IV) *a tube of some radius* $r \in]0,\pi/2[$ *about* $\mathbb{H}P^k$ *for some* $k \in \{0,...,n-1\}$;
(V) *a tube of some radius* $r \in]0,\pi/4[$ *about* $\mathbb{C}P^n$.

Remarks

1. In the definition of Q^{n-1} we have identified $\mathbb{H}^{n+1}$ with $\mathbb{C}^{n+1} + \mathbb{C}^{n+1}j$. The brackets denote projective coordinates. Q^{n-1} is a $(4n-3)$-dimensional submanifold of $\mathbb{H}P^n$ and diffeomorphic to $SU(n+1)/SU(2) \times SU(n-1)$ (see [14]).
2. The focal set of $\mathbb{H}P^k$ is a suitably embedded $(n-k-1)$-dimensional totally geodesic quaternionic projective subspace of $\mathbb{H}P^n$. So the tube of radius r about $\mathbb{H}P^k$ can also be considered as a tube of radius $\pi/2 - r$ about $\mathbb{H}P^{n-k-1}$. Similarly, the focal set of $\mathbb{C}P^n$ in $\mathbb{H}P^n$ is Q^{n-1}. Thus the tube of radius r about $\mathbb{C}P^n$ is just the tube of radius $\pi/4 - r$ about Q^{n-1}.

Proof. For a connected real hypersurface M in $\mathbb{H}P^n$ it has been proved in [3] that M is curvature-adapted if and only if it is congruent to an open part of one of the model spaces (IV) and (V). As all these model spaces are tubes, it suffices by means of the above results to determine their focal sets. First we consider the tubes about $\mathbb{H}P^k$. An entirely analogous argumentation to the complex analogue studied in [7, p. 493] shows that the focal set of such a tube is the disjoint union of $\mathbb{H}P^k$ and a suitably embedded $(n-k-1)$-dimensional totally geodesic quaternionic projective subspace $\mathbb{H}P^{n-k-1}$. Moreover, the tube of radius $r \in]0,\pi/2[$ about $\mathbb{H}P^k$ is the tube of radius $\pi/2 - r$ about this $\mathbb{H}P^{n-k-1}$. Hence the model spaces (IV) give us the totally geodesic quaternionic projective subspaces as further curvature-adapted submanifolds in $\mathbb{H}P^n$. Next, we consider the tubes about $\mathbb{C}P^n$. It follows from the results in [14, pp. 362-364] that the focal set of a tube about $\mathbb{C}P^n$ is the disjoint union of $\mathbb{C}P^n$ and Q^{n-1} and that the tube of radius $r \in]0,\pi/4[$ about $\mathbb{C}P^n$ is the tube of radius $\pi/4 - r$ about Q^{n-1}. According to Theorem 3 the spaces $\mathbb{C}P^n$ and Q^{n-1} are also curvature-adapted. Finally, from Corollary 1 we see that there are no further curvature-adapted submanifolds in $\mathbb{H}P^n$ of codimension greater than two. □

In quaternionic hyperbolic space $\mathbb{H}H^n$ we know of the following examples of curvature-adapted submanifolds :

- the k-dimensional totally geodesic quaternionic hyperbolic subspace $\mathbb{H}H^k, k \in \{0,..., n-1\}$, and the tubes about it;
- the n-dimensional totally geodesic complex hyperbolic subspace and the tubes about it;
- the horospheres in $\mathbb{H}H^n$.

Concerning this and further remarks on the problem of curvature-adapted real hypersurfaces in $\mathbb{H}H^n$ we refer to [3].

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