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CHARACTERIZATIONS OF SOME REAL HYPERSURFACES IN $P_n(C)$ IN TERMS OF RICCI TENSOR*

Dedicated to Professor Hisao Nakagawa on his sixtieth birthday

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§0. Introduction

Let $P_n(C)$ be an n-dimensional complex projective space equipped with the Fubini-Study metric of constant holomorphic sectional curvature 4, and let us denote by M a real hypersurface of $P_n(C)$. Then M admits a natural almost contact structure (ϕ, ξ, η, g) induced from the almost complex structure J of $P_n(C)$.

Recently many differential geometers ([1],[3],[5],[7],[13]) have studied several characterizations of homogeneous real hypersurfaces which are said to be of type A_1, A_2, B, C, D and E,introduced as model hypersurfaces in the works of Takagi[13], Cecil-Ryan[1] and Kimura and Maeda[7], in terms of tensor equations.

On the other hand, Tashiro-Tachibana[15] proved that there does not exist a real hypersurface in $P_n(C)$ with the parallel second fundamental tensor. Thus there can not be existed totally umblical or totally geodesic hypersurfaces in $P_n(C)$. From this point of view, Y.Maeda[10] calculated the norm of the second fundamental tensor and showed that it is estimated by $\|\nabla A\|^2 \geq 4(m-1)$,where the equality holds if and only if M is of type A_1 and A_2.

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Also Ki[3] proved that there does not exist a real hypersurface of $P_n(C)$ with the parallel Ricci tensor. From this it seems to be natural to consider some problems concerned with the estimation of the Ricci tensor for the real hypersurfaces of $P_n(C)$. Untill now it has not been well known to us for these problems. But among of them Kimura and Maeda[9] have characterized a geodesic hypersphere which is said to be of type A_1 by estimating the norm of the covariant derivative of the Ricci tensor.

In this paper we will find a new tensorial formula concerned with the parallel Ricci tensor by using the Hopf-fibration $\tilde{\pi} : S^{2m+1} \to P_n(C)$ and give it another characterization of type A_1 and A_2 by the following.

Theorem A. *Let M be a real hypersurface in P_nC $(n \geq 3)$ with constant mean curvature. Then M satisfies*

$$
\begin{aligned}
(\nabla_X S)Y =& h\{g(\phi Y, X)\xi - \eta(Y)\phi X\} - \{g(\phi Y, X)A\xi \\
& - g(A\xi, Y)\phi X\} - 2\{\eta(Y)\phi AX - g(AX, \phi Y)\xi\}
\end{aligned}
$$

if and only if M is of type A_1 and A_2 provided that $\eta(A\xi)$ is constant.

Finally as an application of this characterization we will estimate the norm of the covariant derivative of the Ricci tensor for this type as follows.

Theorem B. *Let M be a real hypersurface of $P_n(C)$ $(n \geq 3)$ with constant mean curvature and ξ is principal. Then the following inequality holds*

$$
\begin{aligned}
\|\nabla S\|^2 \geq & 4\alpha(\alpha - h)^3 + 4(2-n)(h-\alpha)^2 + 8(2 - \alpha(h-\alpha))(TrA^2 - \alpha^2) \\
& + 4(h-\alpha)\{Tr\phi A^2\phi A - hTr\phi A\phi A\} \\
& + 8\{TrA\phi A^2\phi A - hTrA\phi A\phi A\},
\end{aligned}
$$

where h means the trace of the Weingarten map A and $\alpha = \eta(A\xi)$. *Moreover, the above equality holds if and only if M is locally congruent to of type* A_1,*and* A_2.

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§1. Preliminaries

Let M be a real hypersurface of a complex projective space $P_n(C)$, and let C be a unit vector field on a neighborhood of a point x in M. Let us denote by J the almost complex structure of $P_n(C)$.

For any local vector field X on a neighborhood of x in M, the transformation of X and C under J can be given by

$$JX = \phi X + \eta(X)C, \qquad JC = -\xi,$$

where ϕ defines a skew-symmetric transformation on the tangent bundle TM of M, while η and ξ denote a 1-form and a vector field on a neighborhood of X in M respec tively. Then it is seen that $g(\xi, X) = \eta(X)$, where g denotes the induced Riemannian metric on M.

The set of tensors (ϕ, ξ, η, g) is called an *almost contact structure* on M. Then they satisfy the following

$$\phi^2 = -I + \eta \otimes \xi, \phi\xi = 0, \eta(\phi X) = 0, \eta(\xi) = 1, \tag{1.1}$$

where I denotes the identity transformation. Furthermore, the covariant derivatives of the structure tensors are given by

$$(\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi, \nabla_X \xi = \phi AX, \tag{1.2}$$

where ∇ is the Riemannian connection of g and A denotes the shape operator with respect to the unit normal C on M.

Since the ambient space is of constant holomorphic sectional curvature 4, the equations of Gauss and Codazzi are respectively given as follows

$$(1.3) \qquad R(X,Y)Z = g(Y,Z)X - g(X,Z)Y + g(\phi Y,Z)\phi X - g(\phi X,Z)\phi Y$$

$$-2g(\phi X,Y)\phi Z + g(AY,Z)AX - g(AX,Z)AY,$$

$$(1.4) \qquad (\nabla_X A)Y - (\nabla_Y A)X = \eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X,Y)\xi,$$

where R denotes the Riemannian curvature tensor of M and $\nabla_X A$ denotes the covariant derivative of the shape operator A with respect to X.

The Ricci tensor S' of M is the tensor of type $(0,2)$ given by $S'(X,Y) = trZ \rightarrow R(Z,X)Y$. Also it may be regarded as the tensor of type $(1,1)$ and denote by $S : TM \rightarrow TM$; it satisfies $S'(X,Y) = g(SX,Y)$. From (1.3) we see that the Ricci tensor S of M is given by

$$(1.5) \qquad S = (2n+1)I - 3\eta \otimes \xi + hA - A^2,$$

where we have put $h = trA$. The covariant derivative of (1.5) are given as follows

$$(1.6) \quad (\nabla_X S)Y = -3(\nabla_X \eta)(Y)\xi - 3\eta(Y)\nabla_X \xi + (Xh)AY + h(\nabla_X A)Y - (\nabla_X A^2)Y.$$

§2. Some properties concerned with parallel Ricci tensor

Let us now consider a fibration $\pi : \bar{M} \to M$ which is compatible with the Hopf-fibration $\tilde{\pi} : S^{2m+1} \to P_n(C)$, where M is a real hypersurface of $P_n(C)$ and $\bar{M} = \tilde{\pi}^{-1}(M)$ is a hypersurface of a $(2n+1)$-dimensional unit sphere S^{2m+1}. More precisely speaking $\pi : \bar{M} \to M$ is a fibration with totally geodesic fibres such that the following digram commutative;

$$\begin{array}{ccc} \bar{M} & \xrightarrow{\bar{i}} & S^{2m+1} \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \tilde{\pi}} \\ M & \xrightarrow{i} & P_n(C) \end{array} \tag{2.1}$$

where $\bar{i} : \bar{M} \to S^{2m+1}$ and $i : M \to P_n(C)$ are isometric immersions.

Prior to state main results of this section, let us describe one of model spaces which are said to be of type A_1, A_2 in Theorem A by using the Hopf-fibration $\tilde{\pi}$. Denoting by $S^{2p+1}(a)$ a hypersphere of radius a centered at the origin in a $2(p+1)$-dimensional complex number space C^{p+1}, we can consider the product space $M^C_{p,q}(a,b) = S^{2p+1}(a) \times S^{2q+1}(b)$ as a submanifold in $C^{p+q+2} = C^{p+1} \times C^{q+1}$. Thus,if $a^2 + b^2 = 1$, for any portion (p,q) of an integer $m-1$ such that $p+q = m-1, p \geq 0, q \geq 0$, $\bar{M}^C_{p,q}(a,b)$ may be considered as a hypersurface of $S^{2m+1}(1) \subset C^{n+1}$. Thus by using the Hopf-fibration $\tilde{\pi}$ we put $M^C_{p,q}(a,b) = \tilde{\pi}(\bar{M}^C_{p,q}(a,b))$, which gives an example for the real hypersurface of $P_n(C)$.

Let S^{2m+1} be covered by a system of coordinate neighborhoods $\{\hat{U}; y^\kappa\}$ such that $\tilde{\pi}(\hat{U}) = \tilde{U}$ are coordinate neighborhoods of $P_n(C)$ with local coordinate (y^j).

Then we can express the projection $\tilde{\pi}$ by $y^j = y^j(y^\kappa)$ and put $E_\kappa{}^j = \partial_\kappa y^j$ ($\partial_\kappa = \partial/\partial y^\kappa$) with the rank of matrix $(E_\kappa{}^j)$ being always $2n$. Let us denote by $\tilde{\xi}^\kappa$ components of the unit Sasakian structure vector $\tilde{\xi}$ of S^{2m+1} induced from C^{n+1}. Then $\{E_\kappa{}^j, \tilde{\xi}_\kappa\}$ becomes a local coframe in $\hat{U}$, where $\tilde{\xi}_\kappa = \tilde{\xi}^\mu g_{\mu\kappa}, g_{\mu\kappa}$ being components of the metric tensor on S^{2n+1}.

Next we define $E^\kappa{}_j$ by $(E^\kappa{}_j, \tilde{\xi}^\kappa) = (E_\kappa{}^j, \tilde{\xi}_\kappa)^{-1}$. Then $\{E^\kappa{}_j, \tilde{\xi}^\kappa\}$ is a local frame in $\hat{U}$ and $\{E_\kappa{}^j, \tilde{\xi}_\kappa\}$ the frame dual to $\{E^\kappa{}_j, \tilde{\xi}^\kappa\}$, where the indices $\kappa, \mu, \nu, \ldots,$ and $i, j, k, \ldots,$ run over the range $\{1, 2, \ldots, 2n+1\}$, and $\{1, 2, \ldots, 2n\}$ respectively.

Let us take coordinate neighborhoods $\{\bar{U}; x^\alpha\}$ of $\tilde{\pi}^{-1}(M)$ such that $\pi(\bar{U}) = U$ are coordinate neighborhoods of M with local coordinates (x^a). Moreover, let the isometric immersion $\tilde{i} : \tilde{\pi}^{-1}(M) \to S^{2m+1}(1)$ be locally expressed by $y^\kappa = y^\kappa(x^\alpha)$, then the commutativity of diagram (2.1), that is, $\tilde{\pi} \circ \tilde{i} = i \circ \pi$ implies that

$$B_a{}^j E_\alpha{}^a = E_\kappa{}^j B_\alpha{}^\kappa, \quad E_\kappa{}^j B_b{}^j = B_\alpha{}^\kappa E^\alpha{}_b,$$

where $E_\alpha^a = \partial_\alpha x^a$ and $B_\alpha{}^\kappa = \partial_\alpha y^\kappa$, indices $\alpha, \beta, \gamma, \ldots,$ and $a, b, c, \ldots,$ run over the range $1, 2, \ldots, 2n$ and $1, 2, \ldots, 2n-1$ respectively. Hence the Sasakian structure vector $\tilde{\xi}$ is always tangent to $\bar{M}$.

If we denote by ξ^α component of $\tilde{\xi}$ in a coordinate neighborhood $\{\bar{U}; x^\alpha\}$ of $\bar{M}$, Similarly we obtain a local frame $\{E^\alpha{}_a, \xi^\alpha\}$ and its dual coframe $\{E_\alpha{}^a, \xi_\alpha\}$ in $\bar{U}$, where ξ_α is the associated vector field of ξ^α with respect to the metric tensor $g_{\beta\alpha} = g_{\mu\kappa} B_\beta{}^\mu B_\alpha{}^\kappa$ of $\bar{M}$.

Since the metrics $g_{\alpha\beta}$ on $\tilde{\pi}^{-1}(M)$ are invariant with respect to the submersion $\tilde{\pi}$, the van der Waerden-Bortolotti covariant derivative of $E_\alpha{}^a, E^\alpha{}_a, \xi^\alpha$ *and* ξ_α are given by(See Ishihara and Konish [2])

$$(2.2) \qquad \bar{\nabla}_\beta E_\alpha{}^a = -\phi_b{}^a(E_\beta{}^b\xi_\alpha + \xi_\beta E_\alpha^b),\ \bar{\nabla}_\beta E^\alpha{}_a = -\phi_{ba}E_\beta{}^b\xi_\beta E^\alpha{}_a,$$

$$(2.3) \qquad \bar{\nabla}_\beta\xi_\alpha = \phi_{cb}E_\beta{}^c E_\alpha{}^b,\ \bar{\nabla}_\beta\xi^\alpha = \phi_c{}^a E_\beta{}^c E^\alpha a,$$

respectively, where $\bar{\nabla}_\beta$ denote the operators of covariant differentiations with respect to $g_{\beta\alpha}$ and $\phi_{ba} = \phi_b{}^c g_{ca}, \phi_b{}^a$ is a component of an almost contact metric structure tensor ϕ on M.

Let us denote by $\tilde{K}$ and K be the Ricci tensors of $\tilde{\pi}^{-1}(M)$ and M respectively. Also denote by $\tilde{K}_{\gamma\beta}$ and K_{cb} components of $\tilde{K}$ and K respectively. Then putting

$$(2.4) \qquad \tilde{K}_{cb} = K_{\gamma\beta}E^\gamma{}_c E^\beta{}_b, \quad \tilde{K}_{co} = K_{\gamma\beta}E^\gamma{}_c\xi^\beta,$$

we get the following from the equations of co-Gauss and co-Codazzi

$$(2.5) \qquad \tilde{K}_{cb} = K_{cb} + 2\phi_c{}^e\phi_{eb},$$

$$(2.6) \qquad \tilde{K}_{co} = \nabla_e\phi_c{}^e.$$

Now let us suppose that the Ricci tensor is parallel on $\tilde{M}$. Then applying the operator $\nabla_e = E^\epsilon{}_e\nabla_\epsilon$ to (2.5), (2.6) and using (2.1) and (2.4), we can easily find

$$(2.7) \qquad \nabla_e\tilde{K}_{cb} = \phi_{ce}\tilde{K}_{ob} + \phi_{be}\tilde{K}_{oc},$$

(2.8) $$\phi_c{}^d \tilde{K}_{db} + \phi_b{}^d \tilde{K}_{cd} = 0$$

(2.9) $$\nabla_e \tilde{K}_{co} = 2(n-1)\phi_{ce} + \tilde{K}_{\dot{c}a}\phi_e{}^a,$$

(2.10) $$\tilde{K}_{do}\phi_c{}^d = 0.$$

For a compatible submersion $(\bar{M}, M, \pi)$ with the Hopf-fibration $\tilde{\pi}$, when M is a locally symmetric space or a Einstein space, the Ricci tensor is parallel on M. Thus $M = \tilde{\pi}(\bar{M})$ satisfies (2.7) $\sim$ (2.10). In particular, since the Ricci tensor is parallel on $\bar{M}^C_{p,q}(a,b)$, $M^C_{p,q}(a,b)$ also satisfies (2.7) $\sim$ (2.10).

§3. Certain Lemma

Substituting (2.5) and (2.6) into (2.7) and using (1.2), we obtain

$$(3.1) \qquad (\nabla_X S)Y = h\{g(\phi Y, X)\xi - \eta(Y)\phi X\} - \{g(\phi Y, X)A\xi - g(A\xi, Y)\phi X\}$$
$$-2\{\eta(Y)\phi AX - g(AX, \phi Y)\xi\},$$

where we have put $h = TrA$. Then we give a Lemma as follows

Lemma 3.1. *Let M be a real hypersurface of $P_n(C)$ $(n \geq 3)$ with constant mean curvature. If M satisfies (3.1) and $\eta(A\xi)$ is constant, then the structure vector field ξ is principal, that is, the trajectories of ξ is geodesic.*

Proof. Now let us suppose that M satisfies the condition. In order to use the formula $(R(W,X)S)Y = (\nabla_W \nabla_X S - \nabla_X \nabla_W S - \nabla_{[W,X]}S)Y$, firstly we differentiate (3.1) and use (1.2) as follows

$$(\nabla_W(\nabla_X S))Y - (\nabla_{\nabla_W X} S)Y = (Wh)\{g(\phi Y, X)\xi - \eta(Y)\phi X\}$$
$$+ h[\eta(Y)g(AW, X)\xi - g(AW, Y)\eta(X)\xi + g(\phi Y, X)\phi AW - g(\phi AW, Y)\phi X$$
$$- \eta(Y)\{\eta(X)AW - g(AW, X)\xi\}] - [\eta(Y)g(AW, X)A\xi - \eta(X)g(AW, Y)A\xi$$
$$+ g(\phi Y, X)(\nabla_W A)\xi + g(\phi Y, X)A\phi AW - g((\nabla_W A)\xi + A\phi AW, Y)\phi X$$
$$- g(A\xi, Y)\{\eta(X)AW - g(AW, X)\xi\}] - 2[g(\phi AW, Y)\phi AX + \eta(Y)\{\eta(AX)AW$$
$$- g(AW, AX)\xi\} + \eta(Y)\phi(\nabla_W A)X - g((\nabla_W A)X, \phi Y)\xi$$
$$- g(AX, \eta(Y)AW - g(AW, Y)\xi)\xi - g(AX, \phi Y)\phi AW].$$

From this, taking skew-symmetric part and using the above formula of the curvature tensor, we get

$$
\begin{aligned}
(R(W,X)S)Y &= (Wh)\{g(\phi Y,X)\xi - \eta(Y)\phi X\} \\
&- (Xh)\{g(\phi Y,W)\xi - \eta(Y)\phi W\} \\
&+ h[-g(AW,Y)\eta(X)\xi + g(AX,Y)\eta(W)\xi + g(\phi Y,X)\phi AW \\
&- g(\phi Y,W)\phi AX - g(\phi AW,Y)\phi X + g(\phi AX,Y)\phi W \\
&- \eta(Y)\{\eta(X)AW - \eta(W)AX\}] - [-\eta(X)g(AW,Y)A\xi \\
&+ \eta(W)g(AX,Y)A\xi + g(\phi Y,X)(\nabla_W A)\xi - g(\phi Y,W)(\nabla_X A)\xi \\
&+ g(\phi Y,X)A\phi AW - g(\phi Y,W)A\phi AX - g((\nabla_W A)\xi,Y)\phi X \\
&+ g((\nabla_X A)\xi,Y)\phi W - g(A\phi AW,Y)\phi X + g(A\phi AX,Y)\phi W \\
&- g(A\xi,Y)(\eta(X)AW - \eta(W)AX)] \\
&- 2[g(\phi AW,Y)\phi AX - g(\phi AX,Y)\phi AW + \eta(Y)\{\eta(AX)AW - \eta(AW)AX\} \\
&+ \eta(Y)\phi((\nabla_W A)X - (\nabla_X A)W) - g((\nabla_W A)X - (\nabla_X A)W,\phi Y)\xi \\
&+ g(AW,Y)\eta(AX)\xi - g(AX,Y)\eta(AW)\xi - g(AX,\phi Y)\phi AW \\
&+ g(AW,\phi Y)\phi AX].
\end{aligned}
\tag{3.2}
$$

Now let us take $e_1, e_2, \ldots, e_{2n-1}$ be local fields of orthonormal vectors on M. Then from (3.2) and the assumption of constant mean curvature it follows that

$$
\begin{aligned}
\Sigma_{i=1}^{2n-1} g((R(e_i,X)S)\xi,\phi e_i) &= -(2n-1)g(A\phi AX,\xi) - (2n-3)g((\nabla_X A)\xi,\xi) \\
&- \eta(X)g((\nabla_\xi A)\xi,\xi),
\end{aligned}
$$

where we have used the equation of Codazzi (1.4). From this, using the equation of Gauss (1.3) to the left side, we can calculate the following

$$\begin{aligned}\Sigma_{i=1}^{2n-1} g((R(e_i, X)S)\xi, \phi e_i) &= \Sigma_{i=1}^{2n-1} g(R(e_i, X)(S\xi), \phi e_i) \\ &\quad - \Sigma_{i=1}^{2n-1} g(R(e_i, X)\xi, S\phi e_i) \\ &= 2ng(\phi X, S\xi) + g(\phi AX, AS\xi) + g(SAX, \phi A\xi),\end{aligned}$$

where we have used the fact $TrAS\phi = 0$. Thus from these equations it follows that

$$\begin{aligned}&2ng(\phi S\xi, X) + g(A\phi AS\xi, X) - g(AS\phi A\xi, X) \\ &= (2n-1)g(A\phi AX, \xi) + (2n-3)g((\nabla_X A)\xi, \xi) + \eta(X)g((\nabla_\xi A)\xi, \xi).\end{aligned} \tag{3.3}$$

Also from (3.2) we can calculate the following

$$\begin{aligned}\Sigma_{i=1}^{2n-1} g((R(e_i, \phi e_i)S)\xi, X) &= -2g((\nabla_X A)\xi, \xi) + 2\eta(X)g((\nabla_\xi A)\xi, \xi) \\ &\quad - 2g(A\phi AX, \xi) - 2\big[g(\xi, A\phi AX) - g(A\phi A\xi, X)\big].\end{aligned}$$

Similarly, if we use the equation of Gauss (1.3) to the left side of this equation, we get

$$\begin{aligned}\Sigma_{i=1}^{2n-1} g((R(e_i, \phi e_i)S)\xi, X) &= \Sigma_{i=1}^{2n-1} g(R(e_i, \phi e_i)S\xi, X) - \Sigma_{i=1}^{2n-1} g(R(e_i, \phi e_i)\xi, SX) \\ &= g(-4n\phi S\xi + 2(SA\phi A - A\phi AS)\xi, X).\end{aligned}$$

From these equations we also get the following

$$\begin{aligned}&g(-2n\phi S\xi + (SA\phi A - A\phi AS)\xi, X) \\ &= -g((\nabla_X A)\xi, \xi) - 3g(A\phi AX, \xi) + \eta(X)g((\nabla_\xi A)\xi, \xi).\end{aligned} \tag{3.4}$$

Summing up (3.3) and (3.4) and noticing the fact that $SA\phi A\xi = AS\phi A\xi$, we have

$$(3.5) \qquad (n-2)\{g(A\phi AX,\xi)+g((\nabla_X A)\xi,\xi)\}+\eta(X)g((\nabla_\xi A)\xi,\xi)=0.$$

From this, replacing X by ξ, then we get $g((\nabla_\xi A)\xi,\xi)=0$ for a case where $n\geq 2$. Thus from the assumption $n\geq 3$ (3.5) reduces to

$$(3.6) \qquad g(A\phi AX,\xi)+g((\nabla_X A)\xi,\xi)=0.$$

Since we have assumed that $\eta(A\xi)$ is constant, we know that

$$g((\nabla_X A)\xi,\xi)=-2g(A\phi AX,\xi),$$

from which together with (3.6), it follows

$$(3.7) \qquad A\phi A\xi=0.$$

On the other hand, (3.2) can be contracted as the following

$$\begin{aligned}
&\Sigma_{i=1}^{2n-1}g((R(e_i,X)S)Y,e_i)=(\xi h)g(\phi Y,X)-(\phi Xh)\eta(Y)\\
&+h[-g(A\xi,Y)\eta(X)+g(AX,Y)-g(\phi AX,\phi Y)-g(\phi A\phi X,Y)\\
&-\eta(Y)\eta(X)TrA+\eta(Y)g(AX,\xi)]\\
&-[-\eta(X)g(A^2\xi,Y)+g(AX,Y)g(A\xi,\xi)+g(\phi Y,X)Tr(\nabla_\xi A)\\
(3.8)\quad &-g((\nabla_X A)\xi,\phi Y)-g(A\phi AX,\phi Y)-g((\nabla_{\phi X}A)\xi,Y)-g(A\phi A\phi X,Y)\\
&-g(A\xi,Y)\eta(X)TrA+g(A\xi,Y)g(AX,\xi)]\\
&-2[g(\phi A\phi AX,Y)+\eta(Y)\eta(AX)TrA-\eta(Y)g(AX,A\xi)\\
&+2(n-1)\eta(Y)\eta(X)-g(\phi X,\phi Y)+g(A\xi,Y)\eta(AX)-\eta(A\xi)g(AX,Y)\\
&+g(\phi AX,A\phi Y)],
\end{aligned}$$

where we have used the fact that $\Sigma_{i=1}^{2n-1} g(\phi e_i, e_i) = 0, \Sigma_{i=1}^{2n-1} g(\phi A e_i, e_i) = 0,$ and $\Sigma_{i=1}^{2n-1} g(A\phi A e_i, e_i) = 0.$

Now let us note that the left hand side of (3.8) is symmetric with respect to X and Y, because

$$\Sigma_{i=1}^{2n-1} g((R(e_i, X)S)Y, e_i) = \Sigma_{i=1}^{2n-1} g(R(e_i, X)(SY), e_i) - \Sigma_{i=1}^{2n-1} g(R(e_i, X)Y, Se_i)$$
$$= g(SX, SY) - \Sigma_{i=1}^{2n-1} g(R(e_i, X)Y, Se_i),$$

and if we use the first Bianchi identity to the second term, we get

$$-\Sigma_{i=1}^{2n-1} g(R(e_i, X)Y, Se_i) = \Sigma_{i=1}^{2n-1} g(R(X, Y)e_i, Se_i) + \Sigma_{i=1}^{2n-1} g(R(Y, e_i)X, Se_i)$$
$$= trS{\cdot}R(X, Y) - \Sigma_{i=1}^{2n-1} g(R(e_i, Y)X, Se_i)$$
$$= -\Sigma_{i=1}^{2n-1} g(R(e_i, Y)X, Se_i).$$

Hence taking the skew-symmetric part of (3.8) and account of the symmetry of the left hand side of (3.8), we have

$$\begin{aligned}
0 &= 2(\xi h)g(\phi Y, X) - (\phi X h)\eta(Y) + (\phi Y h)\eta(X) \\
&- h\{g(\phi AX, \phi Y) - g(\phi AY, \phi X)\} \\
&- [-\eta(X)\eta(A^2Y) + \eta(Y)\eta(A^2X) + 2g(\phi Y, X)Tr(\nabla_\xi A) \\
&- g((\nabla_X A)\xi, \phi Y) + g((\nabla_Y A)\xi, \phi X) - g(A\phi AX, \phi Y) + g(A\phi AY, \phi X) \\
&- g(\nabla_{\phi X} A)\xi, Y) + g((\nabla_{\phi Y} A)\xi, X) - g(A\phi A\phi X, Y) + g(A\phi A\phi Y, X) \\
&- h\eta(AY)\eta(X) + h\eta(AX)\eta(Y)] \\
&+ 2\{\eta(Y)\eta(A^2X) - \eta(X)\eta(A^2Y)\}.
\end{aligned} \tag{3.9}$$

On the other hand, by using the equation of Codazzi (1.4) we have

$$g((\nabla_X A)\xi, \phi Y) - g((\nabla_Y A)\xi, \phi X) + g((\nabla_{\phi X} A)\xi, Y) - g((\nabla_{\phi Y} A)\xi, X)$$

$$= g((\nabla_\xi A)X - \phi X, \phi Y) - g((\nabla_\xi A)Y - \phi Y, \phi X)$$

$$+ g((\nabla_\xi A)\phi X - \phi^2 X, Y) - g((\nabla_\xi A)\phi Y - \phi^2 Y, X)$$

$$= 0.$$

From this and the fact $g(\phi AX, \phi Y) = g(AX, Y) - \eta(Y)\eta(AX)$,(3.9) reduces to the following

$$0 = 2(\xi h)g(\phi Y, X) - (\phi X h)\eta(Y) + (\phi Y h)\eta(X) + \eta(Y)\eta(A^2 X)$$

$$- \eta(X)\eta(A^2 Y) + 2g(\phi X, Y)Tr(\nabla_\xi A) + 2g(A\phi AX, \phi Y) - 2g(A\phi AY, \phi X).$$

Put $Y = \xi$ in this equation, then

$$0 = \phi X h = \eta(A^2 X) - \eta(X)\eta(A^2\xi) - 2g(A\phi A\xi, \phi X), \tag{3.10}$$

because the mean curvature h is constant on M. Substituting (3.7) into (3.10), we get

$$\eta(A^2 X) = \eta(X)\eta(A^2\xi), \quad i.e., A^2\xi = \eta(A^2\xi)\xi. \tag{3.11}$$

Now we put $A\xi = \alpha\xi + \beta U$, where U is a unit vector field orthogonal to ξ. From (3.7) and this expression it follows

$$A\phi A\xi = \beta A\phi U = 0. \tag{3.12}$$

Then in order to prove that ξ is principal it suffices to show that $\beta = 0$ on M. Now let us put

$$AU = \beta\xi + \gamma U + \delta V,$$

where U,V and ξ are mutually orthogonal. Then from this and (3.11) it follows that

$$\begin{aligned} A^2\xi &= (\alpha^2 + \beta^2)\xi \\ &= \alpha^2\xi + \alpha\beta U + \beta^2\xi + \beta\gamma U + \beta\delta V. \end{aligned}$$

Since we have assumed that the mean curvature is constant, (3.1) and (1.6) gives that

$$\begin{aligned} &-3(\nabla_X\eta)(Y)\xi - 3\eta(Y)\phi AX + h(\nabla_X A)Y - (\nabla_X A^2)Y \\ &= h\{g(\phi Y, X)\xi - \eta(Y)\phi X\} - \{g(\phi Y, X)A\xi - g(A\xi, Y)\phi X\} \\ &- 2\{\eta(Y)\phi AX - g(AX, \phi Y)\xi\}. \end{aligned} \tag{3.13}$$

Now let us denote $\alpha^2 + \beta^2$ by λ^2. Then differentiating $A^2\xi = \lambda^2\xi$ gives

$$(\nabla_X A^2)\xi = (X\lambda^2)\xi + \lambda^2\nabla_X\xi - A^2\phi AX.$$

Substituting $Y = \xi$ in (3.13) and using this fact, we get

$$h(\nabla_X A)\xi = \phi AX + \{(X\lambda^2)\xi + \lambda^2\phi AX - A^2\phi AX\} - h\phi X + g(A\xi, \xi)\phi X.$$

From this it follows that

$$\begin{aligned} hg((\nabla_X A)Y, \xi) =& g(\phi AX, Y) + (X\lambda^2)\eta(Y) + \lambda^2 g(\phi AX, Y) \\ & - g(A^2\phi AX, Y) - (h - \alpha)g(\phi X, Y), \end{aligned}$$

where $\eta(A\xi) = g(A\xi, \xi)$ is denoted by α. Then taking skew-symmetric part and using the equation of Codazzi, we have

$$
\begin{aligned}
-2g(\phi X, Y)h =& g((\phi A + A\phi)X, Y) + (X\lambda^2)\eta(Y) - (Y\lambda^2)\eta(X) \\
&+ \lambda^2 g((\phi A + A\phi)X, Y) - g((A^2\phi A + A\phi A^2)X, Y) \\
&- 2(h - \alpha)g(\phi X, Y),
\end{aligned} \tag{3.14}
$$

If we put $Y = \xi$ in (3.14), then

$$
X\lambda^2 = (\xi\lambda^2)\eta(X) + g(\phi A\xi, X) + \lambda^2 g(\phi A\xi, X) + g((A^2\phi A + A\phi A^2)X, \xi). \tag{3.15}
$$

Substituting this into (3.14), we have

$$
\begin{aligned}
-2g(\phi X, Y)h =& g((\phi A + A\phi)X, Y) + g(\phi A\xi, X)\eta(Y) \\
&- g(\phi A\xi, Y)\eta(X) + \lambda^2\{g(\phi A\xi, X)\eta(Y) \\
&- g(\phi A\xi, Y)\eta(X)\} + \{g((A^2\phi A + A\phi A^2)X, \xi)\eta(Y) \\
&- g((A^2\phi A + A\phi A^2)Y, \xi)\eta(X)\} + \lambda^2 g((\phi A + A\phi)X, Y) \\
&- g((A^2\phi A + A\phi A^2)X, Y) - 2(h - \alpha)g(\phi X, Y).
\end{aligned} \tag{3.16}
$$

Thus if we put $X = \phi A\xi$ in (3.16) and use (3.12), we have

$$
g(A\phi^2 A\xi, Y) + \|\phi A\xi\|^2\eta(Y) + \lambda^2\|\phi A\xi\|^2\eta(Y)
$$

$$
+\lambda^2 g((\phi A + A\phi)\phi A\xi, Y) + 2\alpha g(\phi^2 A\xi, Y) = 0.
$$

From this and (1.1) it follows

$$(3.17)\quad \begin{aligned}&(\alpha^2+\beta^2+1)\{-g(A^2\xi,Y)+\eta(A\xi)g(A\xi,Y)\}\\&+(\alpha^2+\beta^2+1)\|\phi A\xi\|^2\eta(Y)-2\alpha g(A\xi,Y)+2\alpha^2\eta(Y)=0.\end{aligned}$$

Then substituting $Y=U$ in (3.17), we have

$$(\alpha^2+\beta^2-1)\alpha\beta=0.$$

Now let us consider an open set $\mathcal{U}=\{p\in M|\beta(p)\neq 0\}$. Then $\alpha^2+\beta^2=1$, or $\alpha=0$ on $\mathcal{U}$. For a case where $\alpha^2+\beta^2=1$ (3.11),(3.12) and (3.15) imply that $g(\phi A\xi,X)=0$ for any tangent vector field X on $\mathcal{U}$. Thus $\phi A\xi=0$. From this it follows that ξ is a principal vector and $\beta=0$, which makes a contradiction. Thus this case can not occur on $\mathcal{U}$.

Next we consider for a case where $\alpha=0$. Then by virtue of (3.1) and the expression of $A\xi$ and AU we know $g(A\xi,AU)=\beta\gamma=0$. Since $\beta\neq 0$, $\gamma=0$ on $\mathcal{U}$. This means that

$$AU = \beta\xi+\delta V,$$

where V is orthogonal unit vector to ξ. From this expression and (3.11) we also know that

$$\beta^2\xi=A^2\xi=A(A\xi)=\beta AU=\beta^2\xi+\beta\delta V.$$

Thus $\beta\delta V=0$ implies $\delta=0$ on $\mathcal{U}$. This gives $AU=\beta\xi$ and $\phi AU=0$ on $\mathcal{U}$.

Unless otherwise stated, hereafter let us discuss our statement on the above open set $\mathcal{U}$.

Now taking covariant derivative to $A\xi = \beta U$, then we get

$$(\nabla_X A)\xi + A\nabla_X\xi = (X\beta)U + \beta\nabla_X U.$$

From this it follows that

$$g((\nabla_X A)\xi, Y) + g(A\phi AX, Y) = (X\beta)g(U,Y) + \beta g(\nabla_X U, Y). \tag{3.18}$$

Thus using the equation of Codazzi (1.4), we find

$$\begin{aligned}-2g(\phi X, Y) + 2g(A\phi AX, Y) =& (X\beta)g(U,Y) - (Y\beta)g(U,X)\\ &+ \beta\{g(\nabla_X U, Y) - g(\nabla_Y U, X)\}.\end{aligned}$$

Now we put $Y = U$ in this equation, and note that $A\phi U = 0$, and $\phi AU = 0$, then

$$X\beta = (U\beta)g(U,X) + \beta g(\nabla_U U, X) - 2g(\phi X, U).$$

From this we also get

$$\begin{aligned}\phi A\xi\beta &= 2g(\phi U, \phi A\xi) + \beta g(\nabla_U U, \phi A\xi)\\ &= 2\beta\|\phi U\|^2 + \beta^2 g(\nabla_U U, \phi U).\end{aligned} \tag{3.19}$$

On the other hand, replacing X and Y in (3.18) by U and ϕU respectively and also using $A\phi U = 0$, we have

$$\begin{aligned}\beta g(\nabla_U U, \phi U) = g((\nabla_U A)\xi, \phi U) &= g((\nabla_\xi A)U, \phi U) - g(\phi U, \phi U)\\ &= \beta g(\phi A\xi, \phi U) - g(\phi U, \phi U) = \beta^2 - 1.\end{aligned}$$

Thus

$$g(\nabla_U U, \phi U) = \frac{\beta^2 - 1}{\beta}.$$

From this together with (3.19) it follows

$$(\phi A\xi)\beta = 2\beta\|\phi U\|^2 + \beta(\beta^2 - 1) = \beta^3 + \beta. \tag{3.20}$$

On the other hand, (3.15) together with $\alpha = 0$ implies

$$(\phi A\xi)\beta^2 = (\beta^2 + 1)\|\phi A\xi\|^2 = \beta^2(\beta^2 + 1).$$

Thus applying (3.20) to the left side $(\phi A\xi)\beta^2 = 2\beta(\phi A\xi)\beta$, we have

$$\beta^2(\beta^2 + 1) = 0.$$

Thus $\beta = 0$. This case can not occur on the open set $\mathcal{U}$. Hence there does not exist such an open set $\mathcal{U}$. From this fact we complete the proof of Lemma 3.1.

§4. Characterization of type A_1 and A_2 in $P_n(C)$.

Let M be a real hypersurface in $P_n(C)$. The Ricci tensor S is said to be *η-parallel* if $g((\nabla_X S)Y, Z) = 0$ for any X,Y and Z in $\xi^{\perp}$, where $\xi^{\perp}$ denotes the orthogonal complement of the structure vector ξ. Then to characterize of type A_1 and A_2 in $P_n(C)$ let us introduce known theorems as follows

Theorem 4.1.[12] *Let M be a real hypersurface in $P_n(C)$. Then Then the Ricci tensor is η-parallel and the structure vector field ξ is principal if and only if M is of type A_1,A_2 and B.*

Theorem 4.2.[4] *Let M be a connected complete real hypersurface in $P_n(C)$ and assume that ξ is principal vector field on M. If M satisfies $S\phi + \phi S = k\phi$ for some constant $k \neq 0$, then M is of type A_1,B or M is locally congruent to one of a certain hypersurface of type A_2.*

By using the above theorems we can prove the following

Theorem 4.3. *Let M be a real hypersurface in $P_n(C)$ ($n \geq 3$) with constant mean curvature. Then M satisfies*

$$(*) \qquad \begin{aligned}(\nabla_X S)Y =& h\{g(\phi Y, X)\xi - \eta(Y)\phi X\} - \{g(\phi Y, X)A\xi \\ & - g(A\xi, Y)\phi X\} - 2\{\eta(Y)\phi AX - g(AX, \phi Y)\xi\}\end{aligned}$$

if and only if M is of type A_1 and A_2 provided that $\eta(A\xi)$ is constant.

Proof. Now let us suppose M is of type A_1 and A_2. then for the characterization of type A_1 and A_2 Y.Maeda[10] and M.Okumura [11] asserted that

$$(4.1) \qquad (\nabla_X A)Y = -\eta(Y)\phi X - g(\phi X, Y)\xi, \textit{and}$$

$$A\phi X = \phi AX \tag{4.2}$$

for any X,Y in M respectively. Obviously (4.2) gives that ξ is a principal vector field, that is, $A\xi = \alpha\xi$. From this and (1.6) it follows that

$$\begin{aligned}(\nabla_X S)Y =& -3\{g(\phi AX, Y)\xi + \eta(Y)\phi AX\} + (hI - A)\{-\eta(Y)\phi X \\ & - g(\phi X, Y)\xi\} + \eta(AY)\phi X + g(\phi X, AY)\xi \\ =& h\{g(\phi Y, X)\xi - \eta(Y)\phi X\} + \eta(Y)A\phi X + g(\phi X, Y)A\xi \\ & + \eta(AY)\phi X + g(A\phi X, Y)\xi - 3g(\phi AX, Y)\xi - 3\eta(Y)\phi AX \\ =& h\{g(\phi Y, X)\xi - \eta(Y)\phi X\} - \{g(\phi Y, X)A\xi - g(A\xi, Y)\phi X\} \\ & - 2\{\eta(Y)A\phi X + g(\phi AX, Y)\xi\}.\end{aligned}$$

Thus M is of type A_1 and A_2 satisfies (*).

Conversely let us suppose that M satisfies (*) and the mean curvature is constant on M. Since (*) implies that ξ is a principal vector by Lemma 3.1, (*) also gives $g((\nabla_X S)Y, Z) = 0$ for any X,Y and Z in $\xi^{\perp}$. That is, the Ricci tensor S is η-parallel on M. Thus by Theorem 4.1 we have that M is of type A_1, A_2 and B. Now assume that M is of type B. Then from Theorem 4.2 it follows that for a non-zero constant k

$$S\phi + \phi S = k\phi.$$

From this, differentiating and using (1.2), we have

$$\begin{aligned}&(\nabla_X S)\phi Y + \phi(\nabla_X S)Y + \eta(Y)SAX - g(AX, Y)S\xi \\ &+ \eta(SY)AX - g(AX, SY)\xi = k\{\eta(Y)AX - g(AX, Y)\xi\}.\end{aligned} \tag{4.3}$$

From (*) the first and the second term of the above equation can be given as follows respectively

$$(\nabla_X S)\phi Y = (h-\alpha)\{-g(Y,X)\xi + \eta(Y)\eta(X)\xi\} - 2\{g(AX,Y)\xi - \eta(Y)\eta(AX)\xi\},$$

$$\phi(\nabla_X S)Y = (h-\alpha)\{\eta(Y)X - \eta(X)\eta(Y)\xi\} + 2\{\eta(Y)AX - \eta(AX)\eta(Y)\xi\}.$$

Substituting these equations into (4.3) and using (1.2), we get

$$\begin{aligned}
&(h-\alpha)\{\eta(Y)\eta(X) - g(X,Y)\}\xi - 2\{g(AX,Y) - \eta(Y)\eta(AX)\}\xi \\
&+ (h-\alpha)\{\eta(Y)X - \eta(X)\eta(Y)\xi\} + 2\eta(Y)\{AX - \eta(AX)\xi\} \\
&+ \eta(Y)SAX - g(AX,Y)S\xi + \eta(SY)AX - g(AX,SY)\xi \\
&= k\{\eta(Y)AX - g(AX,Y)\xi\}.
\end{aligned}$$

From this, replacing Y by ξ, and taking a vector field X orthogonal to ξ, we get

$$(h-\alpha)X + 2AX + SAX + \eta(S\xi)AX = kAX, \tag{4.4}$$

where $\eta(S\xi) = 2(n-1) + \alpha(h-\alpha)$.

Since we have assumed M is of type B, our real hypersurface M has three distinct constant principal curvatures $\frac{(1+t)}{(1-t)}$, $\frac{(t-1)}{(t+1)}$ with multiplicities $n-1$ respectively, and $\alpha = 2cot2r$, where $t = 2cotr$, $0 < r < \frac{\pi}{4}$. Now let us denote $\frac{1+t}{1-t}$ by λ, and denote by X the principal vector orthogonal to ξ with the principal curvature λ such that $AX = \lambda X$. Then

$$S(AX) \quad = \quad \{(2n+1) + \lambda h - \lambda^2\}\lambda X.$$

From this, substituting into (4.4), we get

$$(4.5) \qquad (h-\alpha)+2\lambda+\{(2n+1)+\lambda h-\lambda^2\}\lambda+\{2(n-1)+\alpha(h-\alpha)\}\lambda=k\lambda.$$

Similarly, also using (4.4) for the principal vector Y orthogonal to ξ with the principal curvature $-\frac{1}{\lambda}$ such that $AY = -\frac{1}{\lambda}Y$, we have

$$(h-\alpha)-\frac{2}{\lambda}+\{(2n+1)-\frac{h}{\lambda}-\frac{1}{\lambda^2}\}(\frac{-1}{\lambda})+\{2(n-1)+\alpha(h-\alpha)\}(\frac{-1}{\lambda})=k(\frac{-1}{\lambda}).$$

Thus, multiplicating $-\lambda^2$ to both sides, we get

$$(4.6) \quad -(h-\alpha)\lambda^2+2\lambda+\{(2n+1)-\frac{h}{\lambda}-\frac{1}{\lambda^2}\}\lambda+\{2(n-1)+\alpha(h-\alpha)\}\lambda=k\lambda.$$

Substracting (4.6) from (4.5) and using the fact that $\alpha=\lambda-\frac{1}{\lambda}$ we find that

$$(h-\alpha)(1+\lambda^2)+(1+\lambda^2)h-\lambda^3+\frac{1}{\lambda}=0.$$

Since $(1+\lambda^2)\neq 0$ we have

$$(4.7) \qquad 2h-\alpha-(\lambda-\frac{1}{\lambda})=0.$$

On the other hand, $\lambda-\frac{1}{\lambda}=cot(r-\frac{\pi}{4})-tan(r-\frac{\pi}{4})=-\frac{4}{\alpha}$, (4.7) reduces to $\alpha^2-4(2n-3)=0$.

To complete our proof of the above Theorem let us introduce the following theorem.

Theorem 4.4.[8] *Let M be a real hypersurface with constant mean curvature in $P_n(C)$. Suppose that ξ is a principal curvature vector and the corresponding principal curvature is non-zero. If $\nabla_\xi S = 0$, then M is a tube of radius r over one of the following Kaehler submanifolds;*

(A_1) *hyperplane $P_n(C)$, where $0 < r < \frac{\pi}{2}$ and $r \neq \frac{\pi}{4}$,*

(A_2) *totally geodesic $P_k(C)$ $(1 \leq k \leq n-2)$, where $0 < r < \frac{\pi}{2}$ and $r \neq \frac{\pi}{4}$,*

(B) complex quadric Q_{n-1}, where $0 < r < \frac{\pi}{4}$ and $cot^2 2r = n-2$.

(C) $P_1(C) \times P_{\frac{(n-1)}{2}}(C)$, where $0 < r < \frac{\pi}{4}$, $cot^2 2r = \frac{1}{(n-2)}$ and $n (\geq 5)$ is odd,

(D) complex Grassmann $G_{2,5}(C)$, where $0 < r < \frac{\pi}{4}$, $cot^2 2r = \frac{3}{5}$ and $n = 9$,

(E) Hermitian symmetric space $SO(10)/U(5)$, where $0 < r < \frac{\pi}{4}$, $cot^2 2r = \frac{5}{9}$ and $n = 15$.

Since we have assumed M is of type B, we know that ξ is a principal vector with *non-zero* principal curvature α. Moreover (*) gives $\nabla_\xi S = 0$. By virtue of these facts we can use the above Theorem 4.4 . Thus for a case where M is of type B M is a tube of radius r over the complex quadric Q_{n-1}, where $0 < r < \frac{\pi}{4}$ and $cot^2 2r = n-2$. Combining this fact with (4.7), we have a contradiction, because (4.7) implies $cot^2 2r = 2n-3$. Hence M is of type B can not occur. Now we complete the proof of Theorem 4.3.

On the other hand, if M is of type A, Theorem 4.3 together with the fact that ξ is principal implies

$$(**) \quad \begin{aligned} (\nabla_X S)Y = &- (h-\alpha)\{g(\phi X, Y)\xi + \eta(Y)\phi X\} \\ &- 2\{\eta(Y)\phi AX + g(\phi AX + g(\phi AX, Y)\xi\}. \end{aligned}$$

As a converse problem of this fact we also get the following by using the same method in the proof of Theorem 4.3

Corollary 4.5. *Let M be a real hypersurface in $P_n(C)$ with constant mean curvature and assume that ξ is principal vector field on M. Then M satisfies (**) if and only if M is of type A_1,A_2.*

Remark 4.1 In the paper [10] it is known that if ξ is princiapl vector field on M, then its principal curvatire $\alpha = \eta(A\xi)$ is constant on M.

§5. The proof of Theorem B.

Motivated by Theorem 4.3 we will prove the main result in this section. Also we will discuss our statement under the condition such that the mean curvature is constant on M and the structure vector field ξ is principal. Then Lemma 3.1 and (*) implies

$$(5.1)\quad (\nabla_X S)Y = -(h-\alpha)\{g(\phi X, Y)\xi + \eta(Y)\phi X\} - 2\{\eta(Y)\phi AX + g(\phi AX, Y)\xi\}.$$

Now let us define a new tensor field T on M as follows

$$(5.2)\quad \begin{aligned} T(X,Y) =&(\nabla_X S)Y + (h-\alpha)\{g(\phi X, Y)\xi + \eta(Y)\phi X\} \\ &+ 2\{\eta(Y)\phi AX + g(\phi AX, Y)\xi\}. \end{aligned}$$

Thus by Corollary 4.5 $T = 0$ holds on M if and only if M is of type A_1, A_2.

In order to estimate the norm of the covariant derivative of the Ricci tensor, we calculate the norm of (5.2) as follows

$$(5.3)\quad \begin{aligned} \|T\|^2 =&\Sigma_{i=1}^{2n-1} g(T(e_i,e_j), T(e_i,e_j)) = \|\nabla S\|^2 \\ &+ 4(h-\alpha)\Sigma_{i=1}^{2n-1} g((\nabla_{e_i} S)\xi, \phi e_i) + 8\Sigma_{i=1}^{2n-1} g(\phi Ae_i, (\nabla_{e_i} S)\xi) \\ &+ 4(n-1)(h-\alpha)^2 + 8\Sigma_{i=1}^{2n-1} g(\phi Ae_i, \phi Ae_i) \\ &+ 8(h-\alpha)\Sigma_{i=1}^{2n-1} g(\phi Ae_i, \phi e_i), \end{aligned}$$

where we have put $\{e_1, ..., e_{2n-1}\}$ be the orthonormal basis of $T_x(M)$ for any $x \in M$.

On the other hand, we calculate the following from (1.6) under the condition that ξ is principal

$$\Sigma_{i=1}^{2n-1} g((\nabla_{e_i} S)\xi, \phi e_i) = -3\Sigma_{i=1}^{2n-1} g(\phi Ae_i, \phi e_i) + \alpha\Sigma_{i=1}^{2n-1} g((hI - A)\phi Ae_i, \phi e_i)$$

$$-\Sigma_{i=1}^{2n-1} g((hI - A)A\phi Ae_i, \phi e_i) - \Sigma_{i=1}^{2n-1} g(\alpha^2\phi Ae_i, \phi e_i) + \alpha\Sigma_{i=1}^{2n-1} g(A\phi Ae_i, \phi e_i),$$

Thus

$$\Sigma_{i=1}^{2n-1} g((\nabla_{e_i} S)\xi, \phi e_i) = \alpha(h-\alpha)^2 - 3(h-\alpha) + hTr\phi A\phi A - Tr\phi A^2\phi A.$$

Similarly, we also calculate

$$\Sigma_{i=1}^{2n-1} g((\nabla_{e_i} S)\xi, \phi Ae_i) = \{-3+\alpha(h-\alpha)\}(TrA^2-\alpha^2)+hTrA\phi A\phi A - TrA\phi A^2\phi A.$$

From these facts and (5.3) it follows that

$$\|T\|^2 = \|\nabla S\|^2 + 4\alpha(h-\alpha)^3 + 4(n-2)(h-\alpha)^2 + 8\{\alpha(h-\alpha)-2\}(TrA^2-\alpha^2)$$
$$+ 4(h-\alpha)\{hTr\phi A\phi A - Tr\phi A^2\phi A\} + 8\{hTrA\phi A\phi A - TrA\phi A^2\phi A\}.$$

Thus we have proved Theorem B in §0.

Remark 5.1 Substituting (2.5) into (2.8), we know that the Ricci tensor S of M in $P_n(C)$ commutes with the almost contact structure tensor ϕ of M. Kimura[6] classified real hypersurfaces M which satisfy $S\phi = \phi S$ in $P_n(C)$. By virtue of his classification we know that M satisfying (2.8) is of type A_1,A_2 or is locally congruent to one of a certain hypersurfaces of type B, C, D or E.

For a case where M is in a complex hyperbolic space $H_n(C)(n\geq 3)$ the present authors [4] proved that $S\phi = \phi S$ if and only if M is a horosphere or a tube over $H_k(C)$ for a k=0,1,...,n-1.

Remark 5.2 (2.10) means that the trajectories of the structure vector field ξ is geodesics.

Remark 5.3 For the formula (2.9) we will discuss in a forthcoming paper.

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