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Fourier Transform in Comonodiffric Functions

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1. Introduction

The theory of monodiffric and comnodiffric functions has been developed by Berzsenyi [1, 2], Kurowski [3] and Tu [4, 5]. In a recent paper [5], some properties of the comonodiffric Laplace transform by use of comonodiffric exponential functions are obtained. Our results can be applied to solve difference equations with constant coefficients. In the present paper, we shall investigate the Fouier transform via comonodiffric exponential functions. First, we deal with some basic properties of Fourier transforms. Then we deal with an inversion formula, a convolution theorem and an analog Parseval's identity. Finally, we give an application to the filtering problem.

2. Definition and Notation

Let $\mathbb{C}$ be the complex plane, $D = \{z \in \mathbb{C} \mid z = x+iy, x$ and y are integers$\}$ and $f\colon D \longrightarrow \mathbb{C}$.

Definition 1. The function $f\colon D \longrightarrow \mathbb{C}$ is said to be comonodiffric at z if

$$(i-1)f(z) + f(z-i) - if(z-1) = 0.$$

The function f is said to be comonodiffric in D, if it is comonodiffric at any point in D.

In the theory of comonodiffric function, a function

$$e(z, a) = (1-a)^{-x}(1-ai)^{-y} \quad \text{where} \quad z = x+iy,\ 1-a \neq 0,\ 1-ai \neq 0,$$

satisfies the equation $f'(z) - af(z) = 0$ with $f(0) = 1$, if we defined the comonodiffric derivative f' of f as

$$f'(z) = \frac{1}{2}[(1-i)f(z) - f(z-1) + if(z-i)].$$

The function $e(z, a)$ is called to be comonodiffric exponential function.

In this paper, we are maimly concerned with the case $y = 0$ and $a = \frac{2is}{1+is}$, $s \in \mathbb{R}$.

$$\text{i.e.,} \quad e(x, is) = \left(\frac{1+is}{1-is}\right)^x \quad \text{where } x \text{ is an integer.}$$

Definition 2. Let $f(x)$ and $g(x)$ be two complex valued function defined for an integer x and the double dot line integral of f and g are defined by

$$\int_{-\infty}^{\infty} f(x):\ g(x)dx = \sum_{x=-\infty}^{\infty} \left[\frac{f(x) + f(x-1)}{2}\right]\left[\frac{g(x) + g(x-1)}{2}\right].$$

Definition 3. Let I be a set of integers and $f: I \longrightarrow \mathbb{C}$. If

$$\mathfrak{F}(f)(s) = \int_{-\infty}^{\infty} f(x):\ e(x,\ is)dx \quad \text{exists for every real} \quad s,$$

then its function value $F(s)$ is called the Fourier transform of $f(x)$ and we denote it by

$$\mathfrak{F}(f)(s) = F(s).$$

For convenience, we denote $\mathfrak{F}(f)(s)$ by $\mathfrak{F}(f)$. It is easy to see that if $\sum_{x=-\infty}^{\infty} |f(x) + f(x-1)| < \infty$ then $\mathfrak{F}(f) = F(s)$ for all real s. i.e., the Fourier transform of $f(x)$ exists for all real s. We define

$$\mathfrak{F}:(I) = \{f|\ f: I \longrightarrow \mathbb{C} \text{ and } \sum_{x=-\infty}^{\infty} |f(x) + f(x-1)| < \infty\}.$$

3. Some Basic Properties of Fourier Transform

Let $\mathfrak{F}(f) = F(s)$ and $\mathfrak{F}(g) = G(s)$ with f and g are in $\mathfrak{F}:(I)$. Then we have the following basic properties:

(a) $\mathfrak{F}(af(x) + bg(x)) = a\mathfrak{F}(f) + b\mathfrak{F}(g) = aF(s) + bG(s)$, where a and b are real or complex numbers,

(b) $\mathfrak{F}(f(-x)) = F(-s)$,

(c) $\mathfrak{F}(\bar{f}(x)) = \bar{F}(-s)$,

(d) $\mathfrak{F}(f(x+k)) = e(-k, is)F(s)$ where k is an integer.

Properties (a) through (d) follow immediately from the definition of the Fourier transform.

4. Inversion Formula

We begin with following lemmas.

Lemma 1. Let $\mathfrak{F}(f) = F(s)$ with f is in $\mathfrak{F}:(I)$, and $P(\rho) = \sum_{k=-\infty}^{\infty} \rho^k |f(x)+f(x-1)|$ where $\rho \equiv e(1, is) = \frac{1+is}{1-is}$ then $F(s) = \frac{1+\rho}{4\rho}P(\rho)$.

Proof. $F(s) = \int_{-\infty}^{\infty} f(x){:}\ e(x,is)dx = \sum_{x=-\infty}^{\infty} \left[\frac{f(x)+f(x-1)}{2}\right]\left[\frac{\rho^x+\rho^{x-1}}{2}\right] = \frac{1+\rho}{4\rho} P(\rho)$.

Lemma 2. For $f \in \mathfrak{F}{:}\ (I)$, we have

(1) $\frac{1}{2\pi i} \int_C \frac{P(\rho)}{\rho^{x+1}} d\rho = f(x) + f(x-1)$,

(2) $\frac{P(\rho)}{\rho^{x+1}}$ is continuous relative to the unit circle C, where C is the unit circle taken in the counterclockwise sense and x is an integer.

Proof. Since $f \in \mathfrak{F}{:}(I)$

$$|P(\rho)| = |\sum_{x=-\infty}^{\infty} \rho^k[f(k) + f(k-1)]| \leq \sum_{x=-\infty}^{\infty} |f(k) + f(k-1)| < \infty,$$

$P(\rho)$ exists. And

$$(3.1) \qquad \frac{1}{2\pi i}\int_C \frac{P(\rho)}{\rho^{x+1}}\, d\rho = \frac{1}{2\pi i}\int_C \sum_{k=-\infty}^{\infty} \rho^{k-x-1}[f(k) + f(k-1)]d\rho$$

$$= \frac{1}{2\pi i}\sum_{k=-\infty}^{\infty} [f(k) + f(k-1)]\int_C \rho^{k-x-1}\, d\rho.$$

Now,

$$\int_C \rho^{k-x-1}\, d\rho = \int_0^{2\pi} (e^{i\theta})^{k-x-1} de^{i\theta} = \int_0^{2\pi} ie^{i\theta(k-x)} d\theta.$$

If $k = x$ then $\int_C \rho^{k-x-1} d\rho = 2\pi i$, and if $k \neq x$ then $\int_C \rho^{k-x-1} d\rho = 0$. (3.1) becomes $\frac{1}{2\pi i}\int \frac{P(\rho)}{\rho^{x+1}}\, d\rho = f(x) + f(x-1)$.

Proof of (2). Let $g_k(\rho) = \rho^{k-x-1}[f(k) + f(k-1)]$ and $M_k = |f(k) + f(k-1)|$ then

$$|g_k(\rho)| = M_k$$

and $g_k(\rho)$ is continuous throughout C for each k. For $f \in \mathfrak{F}:(I)$, we have

$$\sum_{k=-\infty}^{\infty} M_k = \sum_{k=-\infty}^{\infty} |[f(k) + f(k-1)]| < \infty.$$

By Weierstrass M–test we obtain that $\sum_{k=-\infty}^{\infty} g_k(\rho)$ is uniformly convergent throughout C and $\sum_{k=-\infty}^{\infty} g_k(\rho)$ converges to $\frac{P(\rho)}{\rho^{x+1}}$ which is continuous throughout C.

Lemma 3. For $f \in \mathfrak{F}:(I)$ with $\mathfrak{F}(f) = F(s)$, we have

$$f(x) + f(x-1) = \frac{1}{\pi} \int_{-\infty}^{\infty} F(s)[e(x, -is) + e(x-1, -is)]ds$$

for all integers x.

Proof. Let $\rho = \frac{1+is}{1-is} = e^{i\theta}$, $-\pi \leq \theta \leq \pi$, we obtain $s = \tan \frac{1}{2} \theta$. From Lemma 2,

$$f(x) + f(x-1) = \frac{1}{2\pi i} \int_C \frac{P(\rho)}{\rho^{x+1}} d\rho$$

$$= \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{P(e(1,is))}{e(x+1,is)} d(e(1, is))$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{P(e(1,is))}{4e(x+1,is)} [1 + e(1, is)]^2 ds$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1+e(1,is)}{4e(1,is)} P(e(1, is))\frac{1+e(1,is)}{e(x,is)}ds$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} F(s)[e(x,-is) + e(x-1, -is)]ds.$$

Thus Lemma 3 is proved. Now, we give the Inversion Formula as follows:

Theorem 1 (The Inversion Formula). If $f \in \mathfrak{F}:(I)$, then the Fourier transform $F(s)$ of $f(x)$ exists for all real s and

$$f(x) = (-1)^x[f(0) - \frac{1}{\pi} \oint_{-\infty}^{\infty} F(s)ds] + \frac{1}{\pi} \oint_{-\infty}^{\infty} F(s)e(x, -is)ds$$

where $\frac{1}{\pi} \oint_{-\infty}^{\infty} F(s)ds = \lim_{T\to\infty} \int_{-T}^{T} F(s)ds$ expresses Cauchy principle value.

Proof. First, for all s, we want to show that

$$\int_{-\infty}^{\infty} F(s)[(-1)^{x+1} + e(x, -is)]ds \quad \text{exists.}$$

Since $f \in \mathfrak{F}:(I)$, i.e. $F(s)$ wxists for all real s. By Lemma 3,

$$f(x) + f(x-1) = \frac{1}{\pi} \int_{-\infty}^{\infty} F(s)[e(x,-is) + e(x-1, -is)]ds$$

Multiplying both sides by $(-1)^{x-1}$ and summing for $x = 1, 2, \ldots, n$,

$$f(1) + f(0) = \frac{1}{\pi}\int_{-\infty}^{\infty} F(s)[e(x, -is) + e(0, -is)]ds$$

$$-f(2) - f(1) = \frac{1}{\pi}\int_{-\infty}^{\infty} F(s)[-e(2, -is) - e(1, -is)]ds$$

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$$(-1)^{n-1}[f(n) - f(n-1)] = \frac{1}{\pi}\int_{-\infty}^{\infty} F(s)[(-1)^{n-1}e(n,-is) + (-1)^{n-1}e(n-1,-is)]ds$$

gives

$$f(n) = (-1)^n f(0) + \frac{1}{\pi}\int_{-\infty}^{\infty} F(s)[(-1)^{n+1} + e(n, -is)]ds.$$

Thus $\int_{-\infty}^{\infty} F(s)[(-1)^{n+1} + e(n, -is)]ds$ exists for all real s. Next, we shall show that $\int_{-T}^{T}[e(x, -is) + e(x-1, is)]ds$ is uniformly bounded in T and x.

Let $T > 0$, then for any integer x

$$\int_{-T}^{T}[e(x, -is) + e(x-1, is)]ds = 2\int_{-T}^{T} \frac{e(x, is)}{1+is}ds. \tag{3.2}$$

Take $w = 2\tan^{-1}s$ and let $w_0 = 2\tan^{-1}T$ then $e(x, is) = e^{iwx}$ and

$$2\int_{-T}^{T} \frac{e(x, is)}{1+is}ds = \int_{-w_0}^{w_0} \frac{\exp[i(x-\frac{1}{2})w]}{\cos w/2}\, dw$$

$$= \int_{-w_0}^{w_0} \frac{\cos(x-\frac{1}{2})w}{\cos w/2}\, dw + i\int_{-w_0}^{w_0} \frac{\sin(x-\frac{1}{2})w}{\cos w/2}\, dw$$

$$= \int_{-2\tan^{-1}T}^{2\tan^{-1}T} \frac{\cos(x-\frac{1}{2})w}{\cos w/2}\, dw.$$

Since $\frac{\cos(x-\frac{1}{2})w}{\cos w/2} = \cos xw + (\sin xw)\cdot\tan\frac{w}{2}$ is continuous on $(-\pi, \pi)$, (3.2) becomes

$$\lim_{T\to\infty} \int_{-T}^{T} [e(x, is) + e(x-1, is)]ds$$

$$= \int_{-\pi}^{\pi} \frac{\cos(x-\frac{1}{2})w}{\cos w/2}\, dw$$

$$= 2\int_{0}^{\pi} \frac{(-1)^{x+1}\sin[(x-\frac{1}{2})(w+\pi)]}{\sin\frac{w+\pi}{2}}\, dw$$

$$= 2(-1)^{x+1}\int_{\pi}^{2\pi} \frac{\sin xt\cos t/2 - \cos xt\sin t/2}{\sin t/2}\, dt$$

$$= 2(-1)^{x+1}\int_{0}^{\pi} (\sin xt\cot\frac{t}{2} - \cos xt)dt.$$

From Zygmund [6, p.57] we can show

$$\int_0^{\zeta} \sin nt \cot \frac{t}{2}\, dt \quad \text{is uniformly bounded,}$$

therefore, $\lim_{T\to\infty} \int_{-T}^{T} [e(x,\ is) + e(x-1,\ is)]ds$ is uniformly bounded. Now, it remains to prove that $\oint_{-\infty}^{\infty} F(s)ds$ exists. Since

$$\int_{-T}^{T} F(s)ds = \int_{-T}^{T} \sum_{x=-\infty}^{\infty} \left[\frac{f(x) + f(x-1)}{2}\right]\left[\frac{e(x,\ is) + e(x-1,\ is)}{2}\right]ds$$

$$= \sum_{x=-\infty}^{\infty} \frac{1}{4}[f(x) + f(x-1)] \cdot \int_{-T}^{T} [e(x,\ is) + e(x-1,\ is)]ds,$$

we have

$$\left|\int_{-T}^{T} F(s)ds\right| \le \frac{1}{4} \sum_{x=-\infty}^{\infty} |f(x) + f(x-1)| \int_{-T}^{T} |e(x,\ is) + e(x-1,\ is)|ds.$$

Thus,

$$\left|\oint_{-\infty}^{\infty} F(s)ds\right|$$

$$\le \frac{1}{4} \sum_{x=-\infty}^{\infty} |f(x) + f(x-1)| \oint_{-\infty}^{\infty} |e(x,\ is) + e(x-1,\ is)|ds < \infty. \qquad \text{Q.E.D.}$$

From Theorem 1, we immediately get the following.

Corollary. If f and g $\in \mathfrak{F}$:(I) with $\mathfrak{F}(F) = \mathfrak{F}(g)$ then $f(x) - g(x) = (-1)^x K$ for all integer x and some constant K.

Remark. In the inversion formula, it is of interest to ask "Can we change the Cauchy principle value $\oint_{-\infty}^{\infty} F(s)ds$ into the improper integral in the ordinary sense, that is $\oint_{-\infty}^{\infty} F(s)ds = \lim_{\substack{T\to\infty \\ a\to-\infty}} \int_a^T F(s)ds$. The answer is negative. Let $f(n) = (-1)^n$ for $n \geq 0$ and zero otherwise. Clearly, $f \in \mathfrak{F}$:(I). And

$$F(s) = \int_{-\infty}^{\infty} f(s)\text{: } e(x, is)ds = \frac{1}{2(1+s^2)} - \frac{is}{2(1+s^2)} .$$

Here $\int_{-\infty}^{\infty} \frac{1}{2(1+s^2)} ds$ exists; however, the integral

$$\int_{-\infty}^{\infty} \frac{is}{2(1+s^2)} ds = \lim_{\substack{T\to\infty \\ a\to-\infty}} \int_a^T \frac{is}{2(1+s^2)} ds$$

does not exist in the ordinary sense. But its Cauchy principle value is zero.

4. The Convolution Theorem and Parseval's Identity

Definition 4. For f and g $\in \mathfrak{F}$:(I), the convolution product of f and g is defined by

$$(f\star g)(x) = \int_{-\infty}^{\infty} f(x-t):\ g(t)dt.$$

Theorem 2. Let f and g $\in \mathfrak{F}$:(I) with $\mathfrak{F}(f) = F(s)$ and $\mathfrak{F}(g) = G(s)$, then $\mathfrak{F}(f\star g) = F(s)G(s)$.

Proof. From Lemma 1, $F(s) = \frac{1+\rho}{4\rho} \sum_{x=-\infty}^{\infty} \rho^x[f(x) + f(x-1)]$. Similary,

$$G(s) = \frac{1+\rho}{4\rho} \sum_{x=-\infty}^{\infty} \rho^x[g(x) + g(x-1)].$$

$$F(s)G(s) = \frac{1}{16}(\frac{1+\rho}{\rho})^2 \sum_{x=-\infty}^{\infty} \rho^x(\sum_{k=-\infty}^{\infty} [f(x-k) + f(x-1-k)][g(k) + g(k-1)])$$

$$= \frac{1}{16}[(1+\rho^{-1})+(\rho^{-1}+\rho^{-2})] \sum_{x=-\infty}^{\infty} \rho^x(\sum_{k=-\infty}^{\infty} [f(x-k) + f(x-1-k)][g(k) + g(k-1)])$$

$$= \frac{1}{4} \sum_{x=-\infty}^{\infty} (\rho^x+\rho^{x-1})\int_{-\infty}^{\infty} f(x-1-t):\ g(t)dt + \frac{1}{4} \sum_{x=-\infty}^{\infty} (\rho^{x-1}+\rho^{x-2})\int_{-\infty}^{\infty} f(x-1-k):g(t)dt$$

$$= \frac{1}{4} \sum_{x=-\infty}^{\infty} (\rho^x+\rho^{x-1})\int_{-\infty}^{\infty} f(x-1-t):\ g(t)dt + \frac{1}{4} \sum_{x=-\infty}^{\infty} (\rho^x+\rho^{x-1})\int_{-\infty}^{\infty} f(x-t):g(t)dt$$

$$= \frac{1+\rho}{4\rho} \sum_{x=-\infty}^{\infty} \rho^x[\int_{-\infty}^{\infty} f(x-1-t):\ g(t)dt + \int_{-\infty}^{\infty} f(x-t):g(t)dt]$$

$$= \int_{-\infty}^{\infty}[\int_{-\infty}^{\infty} f(x-t): g(t)dt]: e(x, is)dx = \mathfrak{F}(f\star g).$$

This proves Theorem 2.

Theorem 3 (Parseval's Identity). For f and g are in $\mathfrak{F}$:(I) with $\mathfrak{F}(f) = F(s)$ and $\mathfrak{F}(g) = G(s)$

$$\int_{-\infty}^{\infty} f(x): \bar{g}(x)dx = \frac{1}{\pi}\int_{-\infty}^{\infty} F(s)\overline{G}(s)ds.$$

In particular,

$$\int_{-\infty}^{\infty} f(x): \bar{f}(x)dx = \frac{1}{\pi}\int_{-\infty}^{\infty} |F(s)|^2 ds.$$

Proof. By Lemma 3,

$$f(x) + f(x-1) = \frac{1}{\pi}\int_{-\infty}^{\infty} F(s)[e(x, -is) + e(x-1, -is)]ds$$

we have

$$\int_{-\infty}^{\infty} f(x): \bar{g}(x)dx = \sum_{x=-\infty}^{\infty} \frac{1}{4}[f(x) + f(x-1)]\bar{g}(x) + \bar{g}(x-1)]$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} F(s)[\int_{-\infty}^{\infty} \overline{g}(s):\ e(x,\ -is)dx]ds$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} F(s)\overline{G}(s)ds. \qquad \text{Q.E.D.}$$

Finally, we give an example of the filtering problem. For given two functions f and g in $\mathfrak{F}:(I)$, we want to find h such that $\int_{-\infty}^{\infty} f(x):\ h(t-x)dx = g(t)$. We can solve this problem by Theorems 1 and 2. Let

$$f(n) = \begin{cases} (-1)^n & n \geq 0 \\ 0 & n < 0 \end{cases}$$

and

$$g(n) = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0. \end{cases}$$

The problem is to find $h(x)$ such that

$$\int_{-\infty}^{\infty} f(x):\ h(t-x)dx = g(t).$$

Here $$f(n) + f(n-1) = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$$

$$F(s) = \frac{1}{4}[e(0,\ is) + e(1,\ -is)]$$
$$G(s) = \frac{1}{4}[e(0,\ is) + e(1,\ -is)][e(0,\ is) + e(1,\ is)].$$

Thus,

$$\mathfrak{F}(h(x)) = \frac{G(s)}{F(s)} = e(1, is)[e(0, -is) + e(1, -is)] = 4\ \mathfrak{F}(f(x-1)).$$

On solution $h(x)$ of this problem is $h(x) = 4f(x-1)$.

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