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A Note On An Inverse Parabolic Problem

Shin-ichi Nakamura

1. Introduction.

Let us consider the following Cauchy problem :

$$\partial_t u(x,t) = \Delta u(x,t) + q(x)u(x,t) \quad \text{in } \mathbf{R}^n \times (0,\infty)\,(n \geq 2), \tag{1.1}$$

$$u(x,0) = f(x) \quad \text{on } \mathbf{R}^n, \tag{1.2}$$

where $q(x)$, $f(x)$ are bounded continuous functions and $\operatorname{supp} q \subset\subset \{x : |x| < R\}\,(R > 0)$. Without loss of generality, we may assume $0 \notin \operatorname{supp} q$. Various inverse problems are studied for determining $q(x)$ from the additional infomations (cf. [2], [5]).

In this paper, we study the following inverse problem:
Determine $q(x)$ from the knowledge of $\{u(f)(R\omega,t) : \omega \in \mathbf{S}^{n-1}\}$ (considred as the set of observed data) and $\{f(x)\}$ (considered as the set of input data).

For the wave equation $u_{tt} = \Delta u + q(x)u$, their high frequency beam solutions had used to derive the uniqueness of $q(x)$ from the Neumann to Dirichlet map (cf. [4], [7], [9]). The Neumann to Dirichlet map uniquely determines the X-ray transformation of $q(x)$. However the parabolic equations can not have the beam type solutions. For the parabolic equation $u_t = \Delta u + q(x)u$, Theorem 9.1.2 in [5] shows that the maximum principle and the enery estimates for the parabolic one derive the uniqueness of $q(x)$ from the Neumann to Dirichlet map. Therefore we need another idea to obtain the X-ray transformation of $q(x)$. In the case of parabolic equations, by combining the Feynman-Kac formula and the n-dimensional Brownian bridge process, we can represent their solutions directly and we shall see that we can get the X-ray transformation of $q(x)$. These considerations leads us to the proof of the following theorem:

Theorem. The quantities

$$\lim_{t\downarrow 0}\lim_{\epsilon\downarrow 0}\frac{1}{t}\log\left(\frac{u(\epsilon,\theta)(R\omega,t)}{p(R\omega,R(\theta-\omega),t)}\right)\ (\forall(\theta,\omega)\in \mathbf{S}^{n-1}\times\mathbf{S}^{n-1})$$

uniquely determine $q(x)$, where $u(\epsilon,\theta)(x,t)$ is the unique solution to the problem (1.1), (1.2) with $f(x) = \rho_\varepsilon(x-R\theta)$ ($\rho_\varepsilon(x)$ is the mollifier defined by $\rho_\varepsilon(x) = \epsilon^{-n/2}\rho(x/\epsilon)$, where $\rho(x)$ is a smooth positive function supported in the unit ball with $\int \rho(x)\,dx = 1$) and we set $p(x,y,t) = (2\pi t)^{-n/2}\exp(-\frac{1}{2t}|x-y|^2)$.

2. Proof of Theorem.

First we need the following Feynman-Kac formula:

Lemma 1. We can represent $u(x,t)$ in (1.1) and (1.2) by the Feynman-Kac formula:

$$u(x,t) = E[f(x+B_t)\exp\left(\int_0^t q(x+B_s)\,ds\right)], \tag{1.3}$$

where B_t is the n-dimensional Brownian motion starting at $0\in\mathbf{R}^n$.

The proof of Lemma 1 can be found in [3], [6].

In (1.3), putting $f(x)$ as $\rho_\varepsilon(x-R\theta)$, where $\rho_\varepsilon(x)$ is the mollifier and $\theta\in\mathbf{S}^{n-1}$ and using the Brownian bridge process (cf. [6], [10]), we obtain

$$\begin{aligned}u(\epsilon,\theta)(x,t) &= E[\rho_\varepsilon(x-R\theta+B_t)\exp\left(\int_0^t q(x+B_s)\,ds\right)]\\ &= \int \rho_\varepsilon(x-R\theta+y)p(x,y,t)\times E_{0,x}^{t,y}[\exp\left(\int_0^t q(x+B_s)\,ds\right)]\,dy, \quad (1.4)\end{aligned}$$

where $p(x,y,t) = (2\pi t)^{-n/2}\exp(-\frac{1}{2t}|x-y|^2)$ and the expectation $E_{0,x}^{t,y}$ is with respect to a Brownian bridge starting at time 0 from x and ending at time t in y.
In (1.4), we set $x = R\omega$ and letting $\varepsilon\downarrow 0$. It is easily seen that

$$\lim_{\epsilon\downarrow 0} u(\epsilon,\theta)(R\omega,t) = p(R\omega,R(\theta-\omega),t)\times E_{0,R\omega}^{t,R(\theta-\omega)}[\exp\left(\int_0^t q(x+B_s)\,ds\right)]. \tag{1.5}$$

To continue the proof of the theorem we need the key lemma:

Lemma 2. The following equality holds

$$E_{0,x}^{t,y}[\exp\left(\int_0^t q(x+B_s)\,ds\right)] = \exp\left(t\int_0^1 q(sy+(1-s)x)\,ds + o(t)\right). \quad (1.6)$$

The proof of Lemma 2 can be found in [1].

Combining (1.5) and (1.6), we see that

$$\lim_{\epsilon\downarrow 0}\frac{u(\epsilon,\theta)(R\omega,t)}{p(R\omega,R(\theta-\omega),t)} = \exp\left(t\int_0^1 q(sR(\theta-\omega)+(1-s)R\omega)\,ds + o(t)\right).$$

Hence we have

$$\lim_{t\downarrow 0}\lim_{\epsilon\downarrow 0}\frac{1}{t}\log\left(\frac{u(\epsilon,\theta)(R\omega,t)}{p(R\omega,R(\theta-\omega),t)}\right) = \int_0^1 q(sR(\theta-\omega)+(1-s)R\omega)\,ds.$$

By the assumption that $\operatorname{supp} q \subset\subset \{x : |x| < R\}$, we conclude that for any $(\theta,\omega) \in \mathbf{S}^{n-1}\times\mathbf{S}^{n-1}$

$$\lim_{t\downarrow 0}\lim_{\epsilon\downarrow 0}\frac{1}{t}\log\left(\frac{u(\epsilon,\theta)(R\omega,t)}{p(R\omega,R(\theta-\omega),t)}\right) = \int_{-\infty}^{+\infty} q(R\omega+sR(\theta-2\omega))\,ds.$$

Therefore $\lim_{t\downarrow 0}\lim_{\epsilon\downarrow 0}\frac{1}{t}\log\left(\frac{u(\epsilon,\theta)(R\omega,t)}{p(R\omega,R(\theta-\omega),t)}\right)$, $\forall(\theta,\omega)\in\mathbf{S}^{n-1}\times\mathbf{S}^{n-1}$ uniquely determine the X-ray transfomation of $q(x)$ defined by $\int_{-\infty}^{+\infty} q(y+s\eta)\,ds$ for any $y\in\mathbf{R}^n$ and $\eta\in\mathbf{S}^{n-1}$. We know that the X-ray transformation of $q(x)$ uniquely determines the Fourier transfomation of $q(x)$ (cf. [8]) and hence $\lim_{t\downarrow 0}\lim_{\epsilon\downarrow 0}\frac{1}{t}\log\left(\frac{u(\epsilon,\theta)(R\omega,t)}{p(R\omega,R(\theta-\omega),t)}\right)$, $\forall(\theta,\omega)\in\mathbf{S}^{n-1}\times\mathbf{S}^{n-1}$ uniquely determine $q(x)$.
The proof is completed.

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Department of General Education
Sasebo National College of Technology
1-1, Okishin-cho
Sasebo, Nagasaki 857-1193
Japan

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