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POISSON SUMMATION FORMULA FOR THE SPACE OF FUNCTIONALS

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Abstract

In the preceding work, we formulated a Fourier transformation on the infinite-dimensional space of functionals. Here we first calculate the Fourier transformation of infinite-dimensional Gaussian distribution $\exp\left(-\pi\xi\int_{-\infty}^{\infty}\alpha^2(t)dt\right)$ for $\xi\in\mathbf{C}$ with $\mathrm{Re}(\xi)>0$, $\alpha\in L^2(\mathbf{R})$, using our formulated path integral. Secondly we develop the Poisson summation formula for the space of functionals, and define a functional Z_s, $s\in\mathbf{C}$, so that our path integral of the functional Z_s corresponds to Riemann's zeta function in the case that $\mathrm{Re}(s)>1$.

0. Introduction

In the preceding paper([N-O2]), we defined a delta functional δ and a Fourier transformation F on the space of functionals in the infinitesimal analysis as one of generalizations for Kinoshita's infinitesimal Fourier transformation in the space of functions. Historically, in 1962, Gaishi Takeuchi([T]) introduced a δ-function for the space of functions under nonstandard analysis. In 1988, 1990, Kinoshita([K1],[K2]) defined his Fourier transformation in the infinitesimal analysis for the space of functions. He called it "an infinitesimal Fourier transformation". Nitta and Okada([N-O1],[N-O2]) defined, for funtionals, an infinitesimal Fourier transformation, using a concept of double infinitesimals, and calculated the infinitesimal Fourier transform for two typical examples. The main idea is to use the concept of double infinitesimals and taking standard parts twice st(st(.)). In our theory, the infinitesimal Fourier transform of δ, δ^2, ... , and $\sqrt{\delta}$, ... can be calculated as constant functionals, 1, infinite, ... , and infinitesimal, ...

Now let H be an even infinite number in $^*\mathbf{R}$, and L be a lattice with infinitesimal spacing

$L:=\left\{\varepsilon z \;\middle|\; z\in {}^*\mathbf{Z},\, -\frac{H}{2}\le\varepsilon z<\frac{H}{2}\right\}$, where $\varepsilon=\frac{1}{H}$, and let H' be an even infinite number in $^*(^*\mathbf{R})$, and L' be a lattice with infinitesimal spacing

$L':=\left\{\varepsilon' z' \;\middle|\; z'\in {}^*({}^*\mathbf{Z}),\, -\frac{H'}{2}\le\varepsilon' z'<\frac{H'}{2}\right\}$, where $\varepsilon'=\frac{1}{H'}$. We hereafter call a lattice with infinitesimal spacing, for short, *an infinitesimal lattice.*

Then we calculate the Fourier transform of a nonstandard functional of Gaussian type. The functional of Gaussian type means that the standard part of the image for $\alpha\in L^2$ is $\exp\left(-\pi\xi\int_{-\infty}^{\infty}\alpha^2(t)dt\right)$, for $\xi\in\mathbf{C}$ with $\mathrm{Re}(\xi)>0$. We choose such a nonstandard functional and calculate the Fourier transform of it. Then the standard part of the Fourier transform satisfies that the image of $\alpha\in L^2$ is $C_\xi\exp\left(-\pi\xi^{-1}\int_{-\infty}^{\infty}\alpha^2(t)dt\right)$, in which C_ξ is a constant independent of b.

On the other hand, an infinitesimal lattice $\{\varepsilon' z' \mid z' \in {}^*({}^*\mathbf{Z})\}$ has an equivalence relation : for $\varepsilon' z_1', \varepsilon' z_2'$ in the lattice, $\varepsilon' z_1'$ is equivalent to $\varepsilon' z_2'$ if $\varepsilon' z_1' - \varepsilon' z_2'$ is divided by ${}^*HH'$. The lattice L' is identified with the set of all equivalence classes. The set has a natural group structure, hence it induces group structures in L' and $X := \{a \mid a$ is an internal function with double meamings, from ${}^*L$ to $L'\}$. Then we obtain a Poisson summation formula for the Fourier transformation. If Y is a subgroup of X, we define

$$Y^{\perp\varepsilon} := \{b \in X \mid \exp(2\pi i\, {}^*\varepsilon \sum_{k \in L} a(k)b(k)) = 1 \text{ for } \forall a \in X\}.$$

The Poisson summation formula is the following :

$$|Y|^{-\frac{1}{2}} \sum_{a \in Y} f(a) = |Y^{\perp\varepsilon}|^{-\frac{1}{2}} \sum_{b \in Y^{\perp\varepsilon}} (Ff)(b),$$

where $|Y|$ and $|Y^{\perp\varepsilon}|$ are orders of Y and $Y^{\perp\varepsilon}$.

Finally we define a functional that associates to Riemann's zeta function. Our path integral of the functional Z_s corresponds to Riemann's zeta function in the case that $\mathrm{Re}(s) > 1$. Using our Poisson summation formula for the functional, we study a relationship between the functional and Riemann's zeta function.

1. Preliminaries

1-1. Infinitesimal Fourier transformations by Kinoshita (cf. [Ki],[N-O1],[N-O2])

Let Λ be an infinite set. Let F be a nonprincipal ultrafilter on Λ. For each $\lambda \in \Lambda$, let S_λ be a set. We put an equivalence relation $\sim$ induced from F on $\prod_{\lambda \in \Lambda} S_\lambda$. For $\alpha = (\alpha_\lambda)$, $\beta = (\beta_\lambda)$ $(\lambda \in \Lambda)$,

$$\alpha \sim \beta \Longleftrightarrow \{\lambda \in \Lambda \mid \alpha_\lambda = \beta_\lambda\} \in F.$$

The set of equivalence classes is called *ultraproduct* of S_λ for F with respect to $\sim$. If $S_\lambda = S$ for $\lambda \in \Lambda$, then it is called *ultraproduct* of S for F and it is written as ${}^*S$. The set S is naturally embedded in ${}^*S$ by the following mapping :

$$s \ (\in S) \mapsto [(s_\lambda = s), \lambda \in \Lambda] \ (\in {}^*S),$$

where $[\ \ ]$ denotes the equivalence class with respect to the ultrafilter F. We write the mapping as $*$, and call it naturally elementary embedding. From now on, we identify the image $*(S)$ as S.

Let H $(\in {}^*\mathbf{Z})$ be an infinite even number. The infinite number H is even, when for $H = [(H_\lambda), \lambda \in \Lambda]$, $\{\lambda \in \Lambda \mid H_\lambda \text{ is even}\} \in F$. We denote $\frac{1}{H}$ by ε. We define an infinitesimal lattice space $\mathbf{L}$, an infinitesimal lattice subspace L and a space of functions $R(L)$ on L as follows :

$$\mathbf{L} := \varepsilon\, {}^*\mathbf{Z} = \{\varepsilon z \mid z \in {}^*\mathbf{Z}\},$$

$L := \{\varepsilon z \mid z \in {}^*\mathbf{Z},\ -\frac{H}{2} \leq \varepsilon z < \frac{H}{2}\}\ (\subset \mathbf{L}),$
$R(L) := \{\varphi \mid \varphi \text{ is an internal function from } L \text{ to } {}^*\mathbf{C}\}.$

We extend $R(L)$ to the space of periodic functions on $\mathbf{L}$ with period H. We write the same notation $R(L)$ for the space of periodic functions.

Gaishi Takeuchi([T]) introduced an infinitesimal δ function. Furthermore Moto-o Kinoshita ([Ki]) constructed an infinitesimal Fourier transformation theory on $R(L)$.

We explain it briefly. For $\varphi, \psi \in R(L)$, the infinitesimal δ function, the infinitesimal Fourier transformation $F\varphi$ $(\in R(L))$, the inverse infinitesimal Fourier transformation $\overline{F}\varphi$ $(\in R(L))$ and the convolution $\varphi * \psi$ $(\in R(L))$ are defined as follows :

$$\delta \in R(L), \quad \delta(x) := \begin{cases} H & (x = 0), \\ 0 & (x \neq 0), \end{cases}$$

$(F\varphi)(p) := \sum_{x \in L} \varepsilon \exp(-2\pi i p x)\, \varphi(x),$
$(\overline{F}\varphi)(p) := \sum_{x \in L} \varepsilon \exp(2\pi i p x)\, \varphi(x),$
$(\varphi * \psi)(x) := \sum_{y \in L} \varepsilon \varphi(x - y)\psi(y).$

1-2. Formulation of infinitesimal Fourier transformation on the space of functionals (cf. [N-O1],[N-O2])

To treat a *-unbounded functional f in the nonstandard analysis, we need a second nonstandardization. Let $F_2 := F$ be a nonprincipal ultrafilter on an infinite set $\Lambda_2 := \Lambda$ as above. Denote the ultraproduct of a set S with respect to F_2 by ${}^*S$ as above. Let F_1 be another nonprincipal ultrafilter on an infinite set Λ_1. Take the *-ultrafilter ${}^*F_1$ on ${}^*\Lambda_1$. For an internal set S in the sense of *-nonstandardization, let ${}^\star S$ be the *-ultraproduct of S with respect to ${}^*F_1$. Thus, we define a double ultraproduct ${}^\star({}^*\mathbf{R})$, ${}^\star({}^*\mathbf{Z})$, etc for the set $\mathbf{R}$, $\mathbf{Z}$, etc. It is shown easily that

$${}^\star({}^*\mathbf{S}) = S^{\Lambda_1 \times \Lambda_2}/F_1^{F_2},$$

where $F_1^{F_2}$ denotes the ultrafilter on $\Lambda_1 \times \Lambda_2$ such that for any $A \subset \Lambda_1 \times \Lambda_2$, $A \in F_1^{F_2}$ if and only if

$$\{\lambda \in \Lambda_1 \mid \{\mu \in \Lambda_2 \mid (\lambda, \mu) \in A\} \in F_2\} \in F_1.$$

We always work with this double nonstandardization. The natural imbedding ${}^\star S$ of an internal element S which is not considered as a set in *-nonstandardization is often denoted simply by S.

An infinite number in ${}^\star({}^*\mathbf{R})$ is defined to be greater than any element in ${}^*\mathbf{R}$. We remark that an infinite number in ${}^*\mathbf{R}$ is not infinite in ${}^\star({}^*\mathbf{R})$, that is, the word "an infinite number in ${}^\star({}^*\mathbf{R})$" has a double meaning. An infinitesimal number in ${}^\star({}^*\mathbf{R})$ is also defined to be nonzero and whose absolute value is less than each positive number in ${}^*\mathbf{R}$.

DEFINITION 1.1. Let $H(\in {}^*\mathbf{Z})$, $H'(\in {}^*({}^*\mathbf{Z}))$ be even positive numbers such that H' is larger than any element in ${}^*\mathbf{Z}$, and let $\varepsilon(\in {}^*\mathbf{R})$, $\varepsilon'(\in {}^*({}^*\mathbf{R}))$ be infinitesimals satifying $\varepsilon H = 1$, $\varepsilon' H' = 1$. We define as follows :

$$\mathbf{L} := \varepsilon\, {}^*\mathbf{Z} = \{\varepsilon z \mid z \in {}^*\mathbf{Z}\}, \ \ \mathbf{L}' := \varepsilon'\, {}^*({}^*\mathbf{Z}) = \{\varepsilon' z' \mid z' \in {}^*({}^*\mathbf{Z})\},$$

$$L := \left\{\varepsilon z \;\middle|\; z \in {}^*\mathbf{Z}, \ -\frac{H}{2} \leq \varepsilon z < \frac{H}{2}\right\} \ (\subset \mathbf{L})$$

$$L' := \left\{\varepsilon' z' \;\middle|\; z' \in {}^*({}^*\mathbf{Z}), \ -\frac{H'}{2} \leq \varepsilon' z' < \frac{H'}{2}\right\} \ (\subset \mathbf{L}').$$

Here L is an ultraproduct of lattices

$$L_\mu := \left\{\varepsilon_\mu z_\mu \;\middle|\; z_\mu \in \mathbf{Z}, \ -\frac{H_\mu}{2} \leq \varepsilon_\mu z_\mu < \frac{H_\mu}{2}\right\} \quad (\mu \in \Lambda_2)$$

in $\mathbf{R}$, and L' is also an ultraproduct of lattices

$$L'_\lambda := \left\{\varepsilon'_\lambda z'_\lambda \;\middle|\; z'_\lambda \in {}^*\mathbf{Z}, \ -\frac{H'_\lambda}{2} \leq \varepsilon'_\lambda z'_\lambda < \frac{H'_\lambda}{2}\right\} \quad (\lambda \in \Lambda_1)$$

in ${}^*\mathbf{R}$ that is an ultraproduct of

$$L'_{\lambda\mu} := \left\{\varepsilon'_{\lambda\mu} z'_{\lambda\mu} \;\middle|\; z'_{\lambda\mu} \in \mathbf{Z}, \ -\frac{H'_{\lambda\mu}}{2} \leq \varepsilon'_{\lambda\mu} z'_{\lambda\mu} < \frac{H'_{\lambda\mu}}{2}\right\} \quad (\mu \in \Lambda_2)$$

We define a latticed space of functions X as follows,

$X := \{a \mid a$ is an internal function with double meamings, from ${}^*L$ to $L'\}$

$= \{[(a_\lambda),\ \lambda \in \Lambda_1] \mid a_\lambda$ is an internal function from L to $L'_\lambda\}$,

where $a_\lambda : L \to L'_\lambda$ is $a_\lambda = [(a_{\lambda\mu}),\ \mu \in \Lambda_2]$, $a_{\lambda\mu} : L_\mu \to L'_{\lambda\mu}$.

We define three equivalence relations $\sim_H$, $\sim_{\star(H)}$ and $\sim_{H'}$ on $\mathbf{L}$, $\star(\mathbf{L})$ and $\mathbf{L}'$:

$x \sim_H y \Longleftrightarrow x - y \in H\, {}^*\mathbf{Z}$, $\ x \sim_{\star(H)} y \Longleftrightarrow x - y \in \star(H)\, {}^*({}^*\mathbf{Z})$,

$x \sim_{H'} y \Longleftrightarrow x - y \in H'\, {}^*({}^*\mathbf{Z})$.

Then we identify $\mathbf{L}/\sim_H$, ${}^*(\mathbf{L})/\sim_{\star(H)}$ and $\mathbf{L}'/\sim_{H'}$ as L, ${}^*(L)$ and L'. Since ${}^*(L)$ is identified with L, the set ${}^*(\mathbf{L})/\sim_{\star(H)}$ is identified with $\mathbf{L}/\sim_H$. Furthermore we represent X as the following internal set :

$\{a \mid a$ is an internal function with a double meaning, from ${}^*(\mathbf{L})/\sim_{\star(H)}$ to $\mathbf{L}'/\sim_{H'}$ $\}$.

We use the same notation as a function from ${}^*(L)$ to L' to represent a function in the above internal set. We define the space A of functionals as follows :

$A := \{f \mid f$ is an internal function with a double meaning, from X to ${}^*({}^*\mathbf{C})\}$.

We define an infinitesimal delta function $\delta(a)(\in A)$, an infinitesimal Fourier transform of $f(\in A)$, an inverse infinitesimal Fourier transform of f and a convolution of f, $g(\in A)$, by the following :

DEFINITION 1.2. We define

$$\delta(a) := \begin{cases} (H')^{({}^*H)^2} & (a=0) \\ 0 & (a \neq 0), \end{cases}$$

and, with $\varepsilon_0 := (H')^{-({}^*H)^2} \in {}^*({}^*\mathbf{R})$,
$(Ff)(b) := \sum_{a\in X} \varepsilon_0 \exp\left(-2\pi i \sum_{k\in L} a(k)b(k)\right) f(a)$,
$(\overline{F}f)(b) := \sum_{a\in X} \varepsilon_0 \exp\left(2\pi i \sum_{k\in L} a(k)b(k)\right) f(a)$,
$(f * g)(a) := \sum_{a'\in X} \varepsilon_0 f(a-a')g(a')$.

We define an inner product on A :

$(f,g) := \sum_{b\in X} \varepsilon_0 \overline{f(b)} g(b)$, where $\overline{f(b)}$ is the complex conjugate of $f(b)$.

In the section 7, we write down Riemann zeta function as a nonstandard functional in Definition 1.2. In general, $\sum_{k\in L} a^2(k)$ is infinite, and it is difficult to consider the meaining of F, $\overline{F}$ in Definition 1.2 as standard objects. They are defined only algebraically. In order to understand Definition 1.2 analytically for a standard one, we change the definition briefly, to Definition 1.3.

Replacing the definitions of L', δ, ε_0, F, $\overline{F}$ in Definition 1.1 and Definition 1.2 as the following, we shall define another type of infinitesimal Fourier transformation. The different point is only the definition of an inner product of the space of functions X. In Definition 1.2, the inner product of a, $b(\in X)$ is $\sum_{k\in L} a(k)b(k)$, and in the following definition, it is ${}^*\varepsilon \sum_{k\in L} a(k)b(k)$.

DEFINITION 1.3.

$$L' := \left\{ \varepsilon' z' \;\middle|\; z' \in {}^*({}^*\mathbf{Z}), -{}^*H\frac{H'}{2} \leq \varepsilon' z' < {}^*H\frac{H'}{2} \right\},$$

$$\delta(a) := \begin{cases} ({}^*H)^{\frac{1}{2}({}^*H)^2} H'^{({}^*H)^2} & (a=0), \\ 0 & (a \neq 0), \end{cases}$$

$\varepsilon_0 := ({}^*H)^{-\frac{1}{2}({}^*H)^2} H'^{-({}^*H)^2}$
$(Ff)(b) := \sum_{a\in X} \varepsilon_0 \exp\left(-2\pi i \, {}^*\varepsilon \sum_{k\in L} a(k)b(k)\right) f(a)$,
$(\overline{F}f)(b) := \sum_{a\in X} \varepsilon_0 \exp\left(2\pi i \, {}^*\varepsilon \sum_{k\in L} a(k)b(k)\right) f(a)$.

Then we obtain the following theorem :

THEOREM 1.4([N-O2]).
(1) $\delta = F1 = \overline{F}1$, (2) F is unitary, $F^4 = 1, \overline{F}F = F\overline{F} = 1$,
(3) $f * \delta = \delta * f = f$, (4) $f * g = g * f$,
(5) $F(f * g) = (Ff)(Fg)$, (6) $\overline{F}(f * g) = (\overline{F}f)(\overline{F}g)$,

(7) $F(fg) = (Ff) * (Fg)$, (8) $\overline{F}(fg) = (\overline{F}f) * (\overline{F}g)$.

The definition implies the following proposition :

PROPOSITION 1.5([N-O2]). If $l \in \mathbf{R}^+$, then $F\delta^l = (H')^{(l-1)({}^*H)^2}$.

If there exists α, $\beta \in L^2(\mathbf{R})$ so that $a = {}^*\alpha|_L$, $b = {}^*\beta|_L$, that is, $a(k) = \star({}^*\alpha(k))$, $b(k) = \star({}^*\beta(k))$, then $\mathrm{st}(\mathrm{st}({}^*\varepsilon \sum_{k\in L} a(k)b(k))) = \int_{-\infty}^{\infty} \alpha(x)b(x)dx$. Definition 1.3 is easier understanding than Definition 1.2 for a standard meaning in analysis. For the reason, we consider mainly Definition 1.3 about several examples. However Definition 1.2 is also treated algebraically, as algebraically defined functions are not always L^2-functions on $\mathbf{R}$. The two types of Fourier transforms are different in a standard meaning.

2. Examples of the infinitesimal Fourier transformation on the space of functions

We calculate the infinitesimal Fourier transforms of φ_ξ, $\varphi_{im} \in R(L)$:

1. $\varphi_\xi(x) = \exp(-\xi\pi x^2)$, where $\xi \in \mathbf{C}$, $\mathrm{Re}(\xi) > 0$,
2. $\varphi_{im}(x) = \exp(-im\pi x^2)$, where $m \in \mathbf{Z}$.

For φ_ξ, we obtain :

Proposition 2.1.

$(F\varphi_\xi)(p) = c_\xi(p)\varphi_\xi(\frac{p}{\xi})$, where $c_\xi(p) = \sum_{x\in L} \varepsilon \exp(-\xi\pi(x + \frac{i}{\xi}p)^2)$.

If p is finite, then $\mathrm{st}(c_\xi(p)) = \frac{1}{\sqrt{\xi}}$.

Proof. The infinitesimal Fourier transforms of φ_ξ is :

$(F\varphi_\xi)(p) = \sum_{x\in L} \varepsilon \exp(-2\pi ipx)\exp(-\xi\pi x^2)$
$= (\sum_{x\in L} \varepsilon \exp(-\xi\pi(x + \frac{i}{\xi}p)^2))\exp(-\pi\frac{1}{\xi}p^2) = c_\xi(p)\varphi_\xi(\frac{p}{\xi})$,

where $c_\xi(p) = \sum_{x\in L} \varepsilon \exp(-\xi\pi(x + \frac{i}{\xi}p)^2)$. If p is finite, then $\mathrm{st}(c_\xi(p))$
$= \int_{-\infty}^{\infty} \exp\left(-\xi\pi\left(t + \frac{i}{\xi}\mathrm{st}(p)\right)^2\right)dt = \frac{1}{\sqrt{\xi}}$.

Using Theorem 1.4(8), we obtain for c_ξ :

Proposition 2.2. $\varphi_\xi(x') = \left(\overline{F}c_\xi(p) * \left(c_{\frac{1}{\xi}}(-x)\varphi_\xi(x)\right)\right)(x')$.

Proof. We obtain : $(F\varphi_\xi)(p) = c_\xi(p)\varphi_\xi(\frac{p}{\xi})$, and put $\overline{F}$ to the above :

$(\overline{F}(F\varphi_\xi))(x) = (\overline{F}(c_\xi(p)\varphi_\xi(\frac{p}{\xi})))(x)$
$= (\overline{F}c_\xi(p) * \overline{F}\varphi_\xi(\frac{p}{\xi}))(x)$, that is, $\varphi_\xi(x) = (\overline{F}c_\xi(p) * \overline{F}\varphi_\xi(\frac{p}{\xi}))(x)$.

Now $(\overline{F}\varphi_\xi(\frac{p}{\xi}))(x) = \sum_{p\in L} \varepsilon \exp(-2\pi ipx)\exp(-\xi(\frac{p}{\xi})^2\pi)$
$= \sum_{p\in L} \varepsilon \exp(-\pi\frac{1}{\xi}(p^2 - 2\pi i\xi px)) = \left(\sum_{p\in L} \varepsilon \exp(-\frac{\pi}{\xi}(p - i\xi x)^2)\right)\varphi_\xi(x)$.

By the definition : $c_\xi(p) = \sum_{x\in L} \varepsilon \exp(-\pi\xi(x + i\frac{1}{\xi}p)^2)$, the sum $\sum_{p\in L} \varepsilon \exp(-\frac{\pi}{\xi}(p-i\xi x)^2)$ is $c_{\frac{1}{\xi}}(-x)$. Hence $\varphi_\xi(x') = \left(\overline{F}c_\xi(p) * \left(c_{\frac{1}{\xi}}(-x)\varphi_\xi(x)\right)\right)(x')$.

For the following proposition 2.3, we recall the Gauss sum(cf.[R]) :

For $z \in \mathbf{N}$, Gauss sum $\sum_{l=0}^{z-1} \exp(-i\frac{2\pi}{z}l^2)$ is equal to $\sqrt{z}\frac{1+(-i)^z}{1-i}$.

Proposition 2.3. If $m|2H^2$ and $m|\frac{p}{\varepsilon}$, then $(F\varphi_{im})(p) = c_{im}(p)\exp(i\pi\frac{1}{m}p^2)$, where $c_{im}(p) = \sqrt{\frac{m}{2}}\frac{1+i^{\frac{2H^2}{m}}}{1+i}$ for positive m and $c_{im}(p) = \sqrt{\frac{-m}{2}}\frac{1+(-i)^{\frac{2H^2}{-m}}}{1-i}$ for negative m.

Proof. $(F\varphi_{im})(p) = \sum_{x\in L} \varepsilon \exp(-im\pi x^2)\exp(-2\pi ixp)$
$= c_{im}(p)\exp(i\pi\frac{1}{m}p^2)$, where $c_{im}(p) = \sum_{x\in L} \varepsilon \exp(-im\pi(x + \frac{p}{m})^2)$.
Since $m|\frac{p}{\varepsilon}$, the element $\frac{p}{m}$ is in L. We remark that $\exp(-i\pi mx^2) = \exp(-i\pi m(x + H)^2)$. For positive m,

$$c_{im}(p) = \sum_{x\in L} \varepsilon \exp(-im\pi x^2) = \frac{m}{2}\overline{\left(\varepsilon\sqrt{\frac{2H^2}{m}}\frac{1+(-i)^{\frac{2H^2}{m}}}{1-i}\right)}$$

by the above Gauss sum. Hence $c_{im}(p) = \sqrt{\frac{m}{2}}\frac{1+i^{\frac{2H^2}{m}}}{1+i}$. For negative m, the proof is as same as the above.

3. Examples of the infinitesimal Fourier transformation for the space of functionals

We define an equivalence relation $\sim_{{}^\star HH'}$ in $\mathbf{L}'$ by $x \sim_{{}^\star HH'} y \Leftrightarrow x - y \in {}^\star HH'{}^\star({}^\star\mathbf{Z})$. We identify $\mathbf{L}'/\sim_{{}^\star HH'}$ with L'. Let

$X_{H,{}^\star HH'} := \{a' \mid a'$ is an internal function with a double meaning, from ${}^\star\mathbf{L}/\sim_{\star(H)}$ to $\mathbf{L}'/\sim_{{}^\star HH'}\}$,

and let $\mathbf{e}$ be a mapping from X to $X_{H,{}^\star HH'}$, defined by $(\mathbf{e}(a))([k]) = [a(\hat{k})]$, where [ ] on the left-hand side represents the equivalence class for the equivalence relation $\sim_{\star(H)}$ in ${}^\star\mathbf{L}$, $\hat{k}$ is a representative in ${}^\star(L)$ satisfying $k \sim_{\star(H)} \hat{k}$, and [ ] on the right-hand side represents the equivalence class for the equivalence relation $\sim_{{}^\star HH'}$ in $\mathbf{L}'$. Furthermore let $\mathbf{e}^\sharp(f)(a')$ be defined by $f(\mathbf{e}^{-1}(a'))$.

3-1. The infinitesimal Fourier transform of $g_\xi(a) = \exp\left(-\pi\,{}^\star\varepsilon\xi\sum_{k\in L} a^2(k)\right)$ with $\xi \in \mathbf{C}$, $\mathbf{Re}(\xi) > 0$

We calculate the infinitesimal Fourier transform of

$g_\xi(a) = \exp\left(-\pi\,{}^\star\varepsilon\xi\sum_{k\in L} a^2(k)\right)$, where $\xi \in \mathbf{C}$, $\mathrm{Re}(\xi) > 0$, in the space A of functionals, for Definition 1.3. We identify $^*(^*\xi) \in \mathbf{C}$ with $\xi \in \mathbf{C}$.

Theorem 3.1. $(F(\mathbf{e}^\sharp(g_\xi)))(b) = C_\xi(b)g_\xi(\frac{b}{\xi})$, where $b \in X$ and
$C_\xi(b) = \sum_{a\in X} \varepsilon_0 \exp\left(-\pi\, ^*\varepsilon\xi \sum_{k\in L}(a(k) + i\frac{1}{\xi}b(k))^2\right)$.

Proof. We do the infinitesimal Fourier transform of $\mathbf{e}^\sharp(g_\xi)(a)$.
$(F(\mathbf{e}^\sharp(g_\xi)))(b) = F\left(\exp\left(-\pi\, ^*\varepsilon\xi \sum_{k\in L} a^2(k)\right)\right)(b)$
$= \sum_{a\in X} \varepsilon_0 \exp\left(-2i\pi\, ^*\varepsilon \sum_{k\in L} a(k)b(k)\right) \exp\left(-\pi\, ^*\varepsilon\xi \sum_{k\in L} a^2(k)\right)$
$= C_\xi(b)g_\xi(\frac{b}{\xi})$.

Let $\star \circ * : \mathbf{R} \to {}^*(^*\mathbf{R})$ be the natural elementary embedding and let $\mathbf{st}(c)$ for $c \in {}^*(^*\mathbf{R})$ be the standard part of c with respect to the natural elementary embedding $\star \circ *$. Let $\mathrm{st}_2(c)$ be the standard part of c with respect to the natural elementary embedding $\star$.

Theorem 3.2. If the image of b $(\in X)$ is bounded by a finite value of $^*\mathbf{R}$, that is, $\exists b_0 \in {}^*\mathbf{R}$ s.t. $k \in L \Rightarrow |b(k)| \leq \star(b_0)$, then

$$\mathrm{st}_2(C_\xi(b)) = \left(*\left(\tfrac{1}{\sqrt{\xi}}\right)\right)^{H^2} \ (\in {}^*\mathbf{R}) \text{ and } \mathbf{st}\left(\frac{C_\xi(b)}{\star\left(\left(*\left(\frac{1}{\sqrt{\xi}}\right)\right)^{H^2}\right)}\right) = 1.$$

Proof. $\mathrm{st}_2(C_\xi(b)) = \mathrm{st}_2(\sum_{a\in X} \prod_{k\in L} \sqrt{\varepsilon}\varepsilon' \exp\left(-\pi\xi\{\sqrt{\varepsilon}(a(k)) + i\sqrt{\varepsilon}\frac{1}{\xi}(b(k))\}^2\right))$
$= \prod_{k\in L} \int_{-^*\infty}^{^*\infty} \exp\left(-\pi\xi\{x + i\sqrt{\varepsilon}\frac{1}{\xi}\mathrm{st}_2(b(k))\}^2\right) dx$
$= \prod_{k\in L} \int_{-^*\infty}^{^*\infty} \exp(-\pi\xi x^2)\, dx.$

The argument is same about the infinitesimal Fourier transform of $g'_\xi(a) = \exp(-\pi\xi \sum_{k\in L} a^2(k))$, for Definition 1.2, as the above.

Theorem 3.3. $(F(\mathbf{e}^\sharp(g'_\xi)))(b) = B_\xi(b)g'_\xi(\frac{b}{\xi})$, where $b \in X$ and
$B_\xi(b) = \sum_{a\in X} \varepsilon_0 \exp\left(-\pi\xi \sum_{k\in L}(a(k) + i\frac{1}{\xi}b(k))^2\right)$. Furthermore, if the image of b $(\in X)$ is bounded by a finite value of $^*\mathbf{R}$, that is, $\exists b_0 \in {}^*\mathbf{R}$ s.t. $k \in L \Rightarrow |b(k)| \leq \star(b_0)$, then

$$\mathrm{st}(B_\xi(b)) = \left(*\left(\tfrac{1}{\sqrt{\xi}}\right)\right)^{H^2} \ (\in {}^*\mathbf{R}) \text{ and } \mathbf{st}\left(\frac{B_\xi(b)}{\star\left(\left(*\left(\frac{1}{\sqrt{\xi}}\right)\right)^{H^2}\right)}\right) = 1.$$

3-2. The infinitesimal Fourier transform of $g_{im} = \exp(-i\pi m\, ^*\varepsilon \sum_{k\in L} a^2(k))$ with $m \in \mathbf{Z}$

We calculate the infinitesimal Fourier transform of

$g_{im}(a) = \exp(-i\pi m\, ^*\varepsilon \sum_{k\in L} a^2(k))$, where $m \in \mathbf{Z}$,

for Definition 1.3.

Proposition 3.4. $(F(\mathbf{e}^\sharp(g_{im})))(b)$ is written as $C_{im}(b)g_{\frac{1}{im}}(b)$.

If $m|2{}^\star HH'^2$ and $m|\frac{b(k)}{\varepsilon'}$ for an arbitrary k in L, then $(F(\mathbf{e}^\sharp(g_{im})))(b) = C_{im}(b)g_{\frac{1}{im}}(b)$, where $C_{im}(b) = \left(\sqrt{\frac{m}{2}}\frac{1+i^{\frac{2{}^\star HH'^2}{m}}}{1+i}\right)^{({}^\star H)^2}$ for a positive m and

$$C_{im}(b) = \left(\sqrt{\frac{-m}{2}}\frac{1+(-i)^{\frac{2{}^\star HH'^2}{-m}}}{1-i}\right)^{({}^\star H)^2} \quad \text{for a negative } m.$$

Proof. $(F(\mathbf{e}^\sharp(g_{im})))(b) = C_{im}(b)g_{\frac{1}{im}}(b)$,
where

$$C_{im}(b) = \sum_{a\in X}\varepsilon_0 \exp(-i\pi m{}^\star\varepsilon\sum_{k\in L}(a(k)+\frac{1}{m}b(k))^2).$$

When we denote $a(k)$, $b(k)$ by $\varepsilon'n'$, $\varepsilon'l'$,

$\sum_{-{}^\star H\frac{H'^2}{2}\le a(k)<{}^\star H\frac{H'^2}{2}} \exp(-i\pi m{}^\star\varepsilon\sum_{k\in L}(a(k)+\frac{1}{m}b(k))^2)$
$= \sum_{-{}^\star H\frac{H'^2}{2}\le \varepsilon'n'<{}^\star H\frac{H'^2}{2}} \exp(-i\pi m{}^\star\varepsilon\sum_{k\in L}(\varepsilon'n'+\varepsilon'\frac{n'}{m})^2)$.

Since $m|\frac{b(k)}{\varepsilon'}$, for a positive m, it is equal to

$$\sum_{-{}^\star H\frac{H'^2}{2}\le \varepsilon'n'<{}^\star H\frac{H'^2}{2}} \exp(-i\pi m{}^\star\varepsilon\varepsilon'^2n'^2) = \frac{m}{2}\sqrt{\frac{2{}^\star HH'^2}{m}}\frac{1+i^{\frac{2{}^\star HH'^2}{m}}}{1+i},$$

by Proposition 2.3. Hence $C_{im} = \left(\sqrt{\frac{m}{2}}\frac{1+i^{\frac{2{}^\star HH'^2}{m}}}{1+i}\right)^{({}^\star H)^2}$ for a positive m. For a negative m, the proof is as same as the above.

The argument for the infinitesimal Fourier transform of
$g'_{im}(a) = \exp(-i\pi m\sum_{k\in L}a^2(k))$,
for Definition 1.2, is as same as the above one of g_{im} for Definition 1.3.

Proposition 3.5. If $m|2{}^\star HH'^2$ and $m|\frac{b(k)}{\varepsilon'}$ for an arbitrary k in L, then $(F(\mathbf{e}^\sharp(g'_{im})))(b) = B_{im}(b)g'_{\frac{1}{im}}(b)$, where $B_{im}(b) = \left(\sqrt{\frac{m}{2}}\frac{1+i^{\frac{2H'^2}{m}}}{1+i}\right)^{({}^\star H)^2}$ for a positive m and $B_{im}(b) = \left(\sqrt{\frac{-m}{2}}\frac{1+(-i)^{\frac{2H'^2}{-m}}}{1-i}\right)^{({}^\star H)^2}$ for a negative m.

4. Poisson summation formula for Kinoshita's infinitesimal Fourier transformation

We extend the Poisson summation formula of finite group to Kinoshita's infinitesimal Fourier transformation.

4-1. Formulation

Theorem 4.1. Let S be an internal subgroup of L. Then we obtain, for $\varphi \in R(L)$,

$|S^{\perp}|^{-\frac{1}{2}} \sum_{p \in S^{\perp}} (F\varphi)(p) = |S|^{-\frac{1}{2}} \sum_{x \in S} \varphi(x)$,

where $S^{\perp} := \{p \in L \mid \exp(2\pi i p x) = 1 \text{ for } \forall x \in S\}$.

Since L is an internal cyclic group, S is also an internal cyclic group. The generator of L is ε. The generator of S is written as εs ($s \in {}^*\mathbf{Z}$). Since the order of L is H^2, so s is a factor of H^2.

We prepare the following lemma for the proof of Theorem 4.1.

Lemma 4.2. $S^{\perp} = < \varepsilon \frac{H^2}{s} >$.

Proof of Lemma 4.2. For $p \in S^{\perp}$, we write $p = \varepsilon t$. Then we obtain the following :

$\exp(2\pi i p \varepsilon s) = 1 \Longleftrightarrow \exp(2\pi i \varepsilon t \varepsilon s) = 1 \Longleftrightarrow \exp(2\pi i t \frac{s}{H^2}) = 1 \Longleftrightarrow t\frac{s}{H^2} \in {}^*\mathbf{Z}$.

Hence the generater of $S^{\perp}$ is $\varepsilon\frac{H^2}{s}$.

Proof of Theorem 4.1. By Lemma 4.2, $|S| = \frac{H^2}{s}$ and $|S^{\perp}| = s$. If $x \notin S$, then $\varepsilon\frac{H^2}{s} x s = \varepsilon H^2 x \in {}^*\mathbf{Z}$, and $\left(\exp\left(2\pi i \varepsilon \frac{H^2}{s} x\right)\right)^s = 1$. For $x \in L$,

$$\sum_{p \in S^{\perp}} \exp(2\pi i p x) = \begin{cases} \dfrac{(\exp(2\pi i(-\frac{H}{2})x)(1-(\exp(2\pi i \varepsilon \frac{H^2}{s} x)^s)))}{(1-\exp(2\pi i \varepsilon \frac{H^2}{s} x))} & (x \notin S) \\ \sum_{p \in S^{\perp}} 1 & (x \in S) \end{cases}$$
$$= \begin{cases} 0 & (x \notin S) \\ s & (x \in S) \end{cases}.$$

Hence

$$\sum_{p \in S^{\perp}} (F\varphi)(p) = \sum_{p \in S^{\perp}} \varepsilon (\sum_{x \in L} \varphi(x) \exp(2\pi i p x))$$
$$= \varepsilon \sum_{x \in L} \varphi(x) (\sum_{p \in S^{\perp}} \exp(2\pi i p x)) = \frac{s}{H} \sum_{x \in S} \varphi(x).$$

Thus

$\frac{1}{\sqrt{s}} \sum_{p \in S^{\perp}} (F\varphi)(p) = \sqrt{\frac{s}{H^2}} \sum_{x \in S} \varphi(x) \quad \cdots (\sharp_1)$,

hence $|S^{\perp}|^{-\frac{1}{2}} \sum_{p \in S^{\perp}} (F\varphi)(p) = \frac{1}{|S|^{\frac{1}{2}}} \sum_{x \in S} \varphi(x)$.

Proposition 4.3 Especially if s is equal to H, then $(\sharp_1)$ implies that

$\sum_{p \in S^{\perp}} (F\varphi)(p) = \sum_{x \in S} \varphi(x)$.

The standard part of the above is

$\mathrm{st}(\sum_{p \in S^{\perp}} (F\varphi)(p)) = \mathrm{st}(\sum_{x \in S} \varphi(x))$.

If there exists a standard function $\varphi' : \mathbf{R} \to \mathbf{C}$ so that $\varphi = {}^*\varphi'|_L$, then the right hand side is equal to $\sum_{-\infty<x<\infty} \varphi'(x)$, that is, $\sum_{-\infty<x<\infty} \mathrm{st}(\varphi)(x)$. Furthermore if εs is infinitesimal and φ' is integrable on $\mathbf{R}$, then

$$\mathrm{st}(\varepsilon s \sum_{x\in S} \varphi(x)) = \int_{-\infty}^{\infty} \varphi'(x)dx.$$

Since $(\sharp_1)$ implies that

$$\sum_{p\in S^\perp} (F\varphi)(p) = \varepsilon s \sum_{x\in S} \varphi(x),$$

we obtain $\mathrm{st}(\sum_{p\in S^\perp} (F\varphi)(p)) = \int_{-\infty}^{\infty} \varphi'(x)dx$, that is, $\int_{-\infty}^{\infty} \mathrm{st}(\varphi)(x)dx$.

We decompose H to prime factors $H = p_1^{l_1} p_2^{l_2} \cdots p_m^{l_m}$, where $p_1 = 2$, $p_1 < p_2 < \cdots < p_m$, each p_i is a prime number, $0 < l_i$. Since S is a subgroup of L, the number s is a factor of H^2. When we write s as $p_1^{k_1} p_2^{k_2} \cdots p_m^{k_m}$, the order of S is equal to $p_1^{2l_1-k_1} p_2^{2l_2-k_2} \cdots p_m^{2l_m-k_m}$ and the order of $S^\perp$ is $p_1^{k_1} p_2^{k_2} \cdots p_m^{k_m}$. Hence $(\sharp_1)$ is

$$(p_1^{k_1} p_2^{k_2} \cdots p_m^{k_m})^{-\frac{1}{2}} \sum (_{p\in S^\perp} (F\varphi)(p)) = (p_1^{2l_1-k_1} p_2^{2l_2-k_2} \cdots p_m^{2l_m-k_m})^{-\frac{1}{2}} \sum_{x\in S} \varphi(x).$$

4-2. Examples

We apply Theorem 4.1 to the following two functions :

1. $\varphi_i(x) = \exp(-i\pi x^2)$,
2. $\varphi_\xi(x) = \exp(-\xi\pi x^2)$,

whose infinitesimal Fourier transforms are :

1. $(F\varphi_i)(p) = \exp(-i\frac{\pi}{4})\overline{\varphi_i(p)} \cdots (\sharp_2)$,
2. $(F\varphi_\xi)(p) = c_\xi(p)\varphi_\xi(\frac{p}{\xi})$,

where $\mathrm{st}(c_\xi(p)) = \frac{1}{\sqrt{\xi}}$, if p is finite. Hence we obtain :

1. $|S^\perp|^{-\frac{1}{2}} \exp(-i\frac{\pi}{4}) \sum_{p\in S^\perp} \overline{\varphi_i(p)} = |S|^{-\frac{1}{2}} \sum_{x\in S} \varphi_i(x)$,
2. $|S^\perp|^{-\frac{1}{2}} \sum_{p\in S^\perp} c_\xi(p)\varphi_\xi(\frac{p}{\xi}) = |S|^{-\frac{1}{2}} \sum_{x\in S} \varphi_\xi(x)$.

When the generator of S is εs, we write this as the following, explicitly :

1. $H \exp(-i\frac{\pi}{4}) \sum_{p\in S^\perp} \exp(i\pi p^2) = s \sum_{x\in S} \exp(-i\pi x^2)$,
2. $H \sum_{p\in S^\perp} c_\xi(p)\exp(-\frac{1}{\xi}\pi p^2) = s \sum_{x\in S} \exp(-\xi\pi x^2)$.

We obtain the following proposition :

Proposition 4.4
(i) If $s = H$, then the generator of S is 1 and $S = S^\perp = L \cap {}^*\mathbf{Z}$. Hence

1. $\exp(-i\frac{\pi}{4}) \sum_{p\in L\cap {}^*\mathbf{Z}} \exp(i\pi p^2) = \sum_{x\in L\cap {}^*\mathbf{Z}} \exp(-i\pi x^2)$,

the first equation is a trivial one, and the second is the following :

2. $\sum_{p\in L\cap {}^*\mathbf{Z}} c_\xi(p)\exp(-\frac{1}{\xi}\pi p^2) = \sum_{x\in L\cap {}^*\mathbf{Z}} \exp(-\xi\pi x^2)$.

Taking their standard parts, we obtain :

2. $\mathrm{st}(\sum_{p\in L\cap {}^*\mathbf{Z}} c_\xi(p)\exp(-\frac{1}{\xi}\pi p^2)) = \mathrm{st}(\sum_{x\in L\cap {}^*\mathbf{Z}} \exp(-\xi\pi x^2))$

$= \sum_{-\infty<n<\infty} \exp(-\xi\pi n^2) = \theta(i\xi)$,
where $\theta(z)$ is a θ-function, defined by $\theta(z) = \sum_{-\infty<n<\infty} \exp(i\pi z n^2)$.

(ii) If εs is infinitesimal, then the equation : 2. $H\sum_{p\in S^{\perp}} c_{\xi}(p)\exp(-\frac{1}{\xi}\pi p^2) = s\sum_{x\in S}\exp(-\xi\pi x^2)$ implies the following :

2. $\mathrm{st}(\sum_{p\in S^{\perp}} c_{\xi}(p)\exp(-\frac{1}{\xi}\pi p^2)) = \mathrm{st}(\varepsilon s\sum_{x\in S}\exp(-\xi\pi x^2))$
$= \int_{-\infty}^{\infty}\exp(-\xi\pi x^2)dx = \frac{1}{\sqrt{\xi}}$.

It is known that $\mathrm{st}(c_{\xi}(p)) = \frac{1}{\sqrt{\xi}}$, and $\sum_{-\infty<x<\infty}\exp(-\xi\pi x^2)$ in 2 of (i) is equal to $\frac{1}{\sqrt{\xi}}\sum_{-\infty<p<\infty}\exp(-\frac{1}{\xi}\pi p^2)$ by the standard Poisson summation formula. Hence, by 2 of (i), $\mathrm{st}(\sum_{p\in S^{\perp}} c_{\xi}(p)\exp(-\frac{1}{\xi}\pi p^2)) = \sum_{-\infty<p<\infty}\mathrm{st}(c_{\xi}(p)\exp(-\frac{1}{\xi}\pi p^2))$.

We extend the formula ($\sharp_2$) for $\varphi_i(x)$ to $\varphi_{im}(x) = \exp(-im\pi x^2)$, for an integer m so that $m|2H^2$. If $m|\frac{p}{\varepsilon}$, we recall

$(F\varphi_{im})(p) = c_{im}(p)\exp(i\pi\frac{1}{m}p^2)$,

where $c_{im}(p) = \sqrt{\frac{m}{2}}\frac{1+i^{\frac{2H^2}{m}}}{1+i}$ for a positive m and $c_{im}(p) = \sqrt{\frac{-m}{2}}\frac{1+(-i)^{\frac{2H^2}{-m}}}{1-i}$ for a negative m.

Hence $|S^{\perp}|^{-\frac{1}{2}}\sum_{p\in S^{\perp}} c_{im}(p)\varphi_{\frac{1}{im}}(p) = |S|^{-\frac{1}{2}}\sum_{x\in S}\varphi_{im}(x)$. When the generator $\varepsilon s'$ of $S^{\perp}$ satifies $m|s'$, that is, the generator εs of S satifies $m|\frac{H^2}{s}$, it reduces to the following :

$$H\sqrt{\frac{m}{2}}\frac{1+i^{\frac{2H^2}{m}}}{1+i}\sum_{p\in S^{\perp}}\exp(i\pi\frac{1}{m}p^2) = s\sum_{x\in S}\exp(-im\pi x^2)$$

for a positive m,

$$H\sqrt{\frac{-m}{2}}\frac{1+(-i)^{\frac{2H^2}{-m}}}{1-i}\sum_{p\in S^{\perp}}\exp(i\pi\frac{1}{m}p^2) = s\sum_{x\in S}\exp(-im\pi x^2)$$

for a negative m.

5. Poisson summation formula for Definition 1.2 on the space of functionals

We extend Poisson summation formula of finite group to our infinitesimal Fourier transformation, Definition 1.2, on the space of functionals.

5-1. Formulation

Theorem 5.1. Let Y be an internal subgroup of X. Then we obtain, for $f \in A$, $|Y^{\perp}|^{-\frac{1}{2}}\sum_{b\in Y^{\perp}}(Ff)(b) = |Y|^{-\frac{1}{2}}\sum_{a\in Y} f(a)$, where $Y^\perp := \{b \in X \mid \exp(2\pi i < a,b >) = 1 \text{ for } \forall a \in X\}$ and $< a,b >:= \sum_{k\in L} a(k)b(k)$.

Lemma 5.2. $|Y^\perp| = \frac{|X|}{|Y|}$.

Proof of Lemma 5.2. For $k \in L$, we denote $Y_k := \{a(k) \in L' \mid a \in Y\}$.

$b \in Y^\perp \iff \forall a \in Y,\ \exp(2\pi i \sum_{k\in L} a(k)b(k)) = 1$
$\iff \forall k \in L,\ b(k) \in Y_k^\perp$
$\iff b : L \to L',\ \forall k \in L,\ b(k) \in Y_k^\perp$.

Hence $|Y^\perp| = \prod_{k\in L} |Y_k^\perp|$. Lemma 4.2 implies $|Y_k^\perp| = \dfrac{H'^2}{|Y_k|}$. Thus

$$|Y^\perp| = \prod_{k\in L}\left(\frac{H'^2}{|Y_k|}\right) = \frac{H'^{2\,{}^*H^2}}{\prod_{k\in L}|Y_k|} = \frac{|X|}{|Y|}.$$

Proof of Theorem 5.1.

$|Y^\perp|^{-\frac{1}{2}} \sum_{b\in Y^\perp} (Ff)(b)$
$= |Y^\perp|^{-\frac{1}{2}} \sum_{a\in X} \varepsilon_0 (\sum_{b\in Y^\perp} \exp(-2\pi i < a,b >)) f(a)$.

Since $\sum_{b\in Y^\perp} \exp(-2\pi i < a,b >) = \begin{cases} 0 & (a \notin Y) \\ |Y^\perp| & (a \in Y) \end{cases}$, the above is equal to

$|Y^\perp|^{-\frac{1}{2}} \varepsilon_0 |Y^\perp| \sum_{a\in Y} f(a) = |Y^\perp|^{\frac{1}{2}} H'^{-\,{}^*H^2} \sum_{a\in Y} f(a) = |Y|^{-\frac{1}{2}} \sum_{a\in Y} f(a)$.

In the special case where $f(a) = \prod_{k\in L} f_k(a(k))$,

$(Ff)(b) = \sum_{a\in X} \varepsilon_0 \exp(-2\pi i \sum_{k\in L} a(k)b(k)) \prod_{k\in L} f_k(a(k))$
$= \prod_{k\in L} (\sum_{a(k)\in L'} \varepsilon' \exp(-2\pi i a(k)b(k)) f_k(a(k)))$.

Namely, the Fourier transform in functional space is the product of those in function space.

Corollary 5.3

(i) If each generator of Y_k is equal to 1, f is written as $\prod_{k\in L} f_k$, $f_k = {}^*(\mathrm{st}(f_k))|_{L'}$, and $\sum_{-\infty<n<\infty} \mathrm{st}(f_k)(n)$ converges, then

$\mathrm{st}(\sum_{b\in Y^\perp}(Ff)(b)) = \prod_{k\in L}(\sum_{-\infty<n<\infty} \mathrm{st}(f_k)(n))$.

(ii) If each generator of Y_k is infinitesimal, f is written as $\prod_{k\in L} f_k$, $f_k = {}^*(\mathrm{st}(f_k))|_{L'}$ and $\mathrm{st}(f_k)$ is L_1-integrable on $\mathbf{R}$, then

$\mathrm{st}(\sum_{b\in Y^\perp}(Ff)(b)) = \prod_{k\in L} \int_{-\infty<t<\infty} \mathrm{st}(f_k)(t)dt$.

5-2. Examples

From now on the infinitesimal Fourier transform $F(\mathrm{e}^\sharp(f))$ for a functional $f \in A$ is often denoted simply Ff. We apply Theorem 5.1 to the following two functionals :

1. $f_i(a) = \exp(-i\pi \sum_{k\in L} a(k)^2)$,
2. $f_\xi(a) = \exp(-\xi\pi \sum_{k\in L} a(k)^2)$, where $\xi \in \mathbf{C}$, $\mathrm{Re}(\xi) > 0$.

The infinitesimal Fourier transforms of the functionals are :

1. $(Ff_i)(b) = (-1)^{\frac{H}{2}} \overline{f_i(b)} \cdots (\natural_3)$,
2. $(Ff_\xi)(b) = B_\xi(b) f_\xi(\frac{b}{\xi})$,

hence we obtain :

1. $|Y^\perp|^{-\frac{1}{2}}(-1)^{\frac{H}{2}} \sum_{b\in Y^\perp} \overline{f_i(b)} = |Y|^{-\frac{1}{2}} \sum_{a\in Y} f_i(a)$,
2. $|Y^\perp|^{-\frac{1}{2}} \sum_{b\in Y^\perp} B_\xi(b) f_\xi(\frac{b}{\xi}) = |Y|^{-\frac{1}{2}} \sum_{a\in Y} f_\xi(a)$.

We write this as the following, explicitly :

1. $|Y^\perp|^{-\frac{1}{2}}(-1)^{\frac{H}{2}} \sum_{b\in Y^\perp} \exp(-i\pi \sum_{k\in L} b(k)^2) = |Y|^{-\frac{1}{2}} \sum_{a\in Y} \exp(-i\pi \sum_{k\in L} a(k)^2)$,
2. $|Y^\perp|^{-\frac{1}{2}} \sum_{b\in Y^\perp} B_\xi(b) \exp(-\frac{1}{\xi}\pi \sum_{k\in L} b(k)^2) = |Y|^{-\frac{1}{2}} \sum_{a\in Y} \exp(-\xi\pi \sum_{k\in L} a(k)^2)$.

Corollaly 5.3 implies the following proposition 5.4.

Proposition 5.4

(i) If each generator of Y_k is equal to 1, then

1. $(-1)^{\frac{H}{2}} \mathrm{st}(\sum_{b\in Y^\perp} \exp(-i\pi \prod_{k\in L} b(k)^2)) = (\sum_{-\infty<n<\infty} \exp(-i\pi n^2))^{H^2}$,
2. $\mathrm{st}(\sum_{b\in Y^\perp} B_\xi(b) \exp(-\frac{1}{\xi}\pi \sum_{k\in L} b(k)^2)) = (\sum_{-\infty<n<\infty} \exp(-\xi\pi n^2))^{H^2}$ $\left(= (\theta(i\xi))^{H^2}\right)$,

(ii) If each generator of Y_k is equal to a natural number m_k, then

1. $(-1)^{\frac{H}{2}} \mathrm{st}(\sum_{b\in Y^\perp} \exp(-i\pi \prod_{k\in L} b(k)^2)) = \prod_{k\in L}(m_k \sum_{-\infty<n<\infty} \exp(-i\pi m_k^2 n^2))$,
2. $\mathrm{st}(\sum_{b\in Y^\perp} B_\xi(b) \exp(-\frac{1}{\xi}\pi \sum_{k\in L} b(k)^2)) = \prod_{k\in L}(m_k \sum_{-\infty<n<\infty} \exp(-\xi\pi m_k^2 n^2))$ $(= \prod_{k\in L}(m_k \theta(i m_k^2 \xi)))$,

(iii) If each generator of Y_k is infinitesimal, then

2. $\mathrm{st}(\sum_{b\in Y^\perp} B_\xi(b) \exp(-\frac{1}{\xi}\pi \sum_{k\in L} b(k)^2)) = (\int_{-\infty}^{\infty} \exp(-\xi\pi t^2) dt)^{H^2}$ $\left(= \left(*\left(\frac{1}{\sqrt{\xi}}\right)\right)^{H^2}\right)$.

We extend the above formula $(\natural_3)$ for $f_i(a)$ to $f_{im}(a) = \exp(-im\pi \sum_{k\in L} a^2(k))$, for an integer m so that $m|2H'^2$. If $m|\frac{b(k)}{\varepsilon'}$, we recall

$(Ff_{im})(b) = B_{im}(b) f_{\frac{1}{im}}(b)$, where $B_{im}(b) = \left(\sqrt{\frac{m}{2}} \frac{1+i^{\frac{2H'^2}{m}}}{1+i}\right)^{(^*H)^2}$ for a positive m and $B_{im}(b) = \left(\sqrt{\frac{-m}{2}} \frac{1+(-i)^{\frac{2H'^2}{-m}}}{1-i}\right)^{(^*H)^2}$ for a negative m.

Hence $|Y^\perp|^{-\frac{1}{2}} \sum_{b\in Y^\perp} B_{im}(b) f_{\frac{1}{im}}(b) = |Y|^{-\frac{1}{2}} \sum_{a\in Y} f_{im}(a)$. When each generator $\varepsilon' s'_k$ of $Y_k^\perp$ satisfies $m|s'_k$, that is, each generator $\varepsilon' s_k$ of Y_k satisfies $m|\frac{H'^2}{s_k}$, it reduces to the following :

$$H'^{({}^*H)^2}\left(\sqrt{\frac{m}{2}}\frac{1+i^{\frac{2H'^2}{m}}}{1+i}\right)^{({}^*H)^2}\sum_{b\in Y^\perp}\exp(i\pi\frac{1}{m}\sum_{k\in L}b(k)^2)$$

$= \prod_{k\in L} s_k \sum_{a\in Y}\exp(-im\pi\sum_{k\in L}a(k)^2)$ for a positive m, and

$$H'^{({}^*H)^2}\left(\sqrt{\frac{-m}{2}}\frac{1+(-i)^{\frac{2H'^2}{-m}}}{1-i}\right)^{({}^*H)^2}\sum_{b\in Y^\perp}\exp(i\pi\frac{1}{m}\sum_{k\in L}b(k)^2)$$

$= \prod_{k\in L} s_k \sum_{a\in Y}\exp(-im\pi\sum_{k\in L}a(k)^2)$ for a negative m.

If $s_k = H'$ and $m|H'$, then

$$\left(\sqrt{\frac{m}{2}}\frac{1+i^{\frac{2H'^2}{m}}}{1+i}\right)^{({}^*H)^2}\sum_{b\in Y^\perp}\exp(i\pi\frac{1}{m}\sum_{k\in L}b(k)^2)$$

$= \sum_{a\in Y}\exp(-im\pi\sum_{k\in L}a(k)^2)$ for a positive m, and

$$\left(\sqrt{\frac{-m}{2}}\frac{1+(-i)^{\frac{2H'^2}{-m}}}{1-i}\right)^{({}^*H)^2}\sum_{b\in Y^\perp}\exp(i\pi\frac{1}{m}\sum_{k\in L}b(k)^2)$$

$= \sum_{a\in Y}\exp(-im\pi\sum_{k\in L}a(k)^2)$ for a negative m, that is,

$$\left(\sqrt{m}\exp(-i\frac{\pi}{4})\right)^{({}^*H)^2}\sum_{b\in Y^\perp}\exp(i\pi\frac{1}{m}\sum_{k\in L}b(k)^2)$$

$= \sum_{a\in Y}\exp(-im\pi\sum_{k\in L}a(k)^2)$ for a positive m, and
$(\sqrt{-m}\exp(i\frac{\pi}{4}))^{({}^*H)^2}\sum_{b\in Y^\perp}\exp(i\pi\frac{1}{m}\sum_{k\in L}b(k)^2)$
$= \sum_{a\in Y}\exp(-im\pi\sum_{k\in L}a(k)^2)$ for a negative m.

6. Poisson summation formula for Definition 1.3 on the space of functionals

We extend Poisson summation formula of finite group to our infinitesimal Fourier transformation, Definition 1.3, on the space of functionals originally defined in [N-O1].

6-1. Formulation

We obtain the following theorem for Definition 1.3 as the argument in the section 5.

Theorem 6.1. Let Y be an internal subgroup of X. Then we obtain, for $f \in A$, $|Y^{\perp\varepsilon}|^{-\frac{1}{2}}\sum_{b\in Y^{\perp\varepsilon}}(Ff)(b) = |Y|^{-\frac{1}{2}}\sum_{a\in Y}f(a)$, where $Y^{\perp\varepsilon} := \{b \in X \mid \exp(2\pi i < a,b >_\varepsilon) = 1 \text{ for } \forall a \in Y\}$ and $< a,b >_\varepsilon := {}^*\varepsilon \sum_{k\in L} a(k)b(k)$.

Lemma 6.2. $|Y^{\perp\varepsilon}| = \frac{|X|}{|Y|}$.

Proof of Lemma 6.2. For $k \in L$, we denote $Y_k := \{a(k) \in L' \mid a \in Y\}$.

$b \in Y^{\perp\varepsilon} \Longleftrightarrow \forall a \in Y,\ \exp(2\pi i\, {}^*\varepsilon \sum_{k\in L} a(k)b(k)) = 1$
$\Longleftrightarrow \forall k \in L,\ {}^*\varepsilon b(k) \in Y_k^{\perp}$.

For $k \in L$, we write m, n as generators defined by :

$Y_k =< \varepsilon' m >,\ \{b(k) \in L' \mid {}^*\varepsilon b(k) \in Y_k^{\perp}\} =< \varepsilon' n >$.

Now

$\exp(2\pi i\, {}^*\varepsilon\varepsilon' m \varepsilon' n) = 1 \Longleftrightarrow {}^*\varepsilon\varepsilon' m\varepsilon' n = 1$.

We write $Y_k^{\perp\varepsilon} := \{b(k) \in L' \mid {}^*\varepsilon b(k) \in Y_k^{\perp}\}$. Then $|Y_k^{\perp\varepsilon}| = m$. This is equal to $\dfrac{{}^*HH'^2}{{}^*HH'^2/m} = \dfrac{|L'|}{|Y_k|}$. Hence

$|Y^{\perp\varepsilon}| = \prod_{k\in L} |Y_k^{\perp\varepsilon}| = \frac{|X|}{|Y|}$.

Proof of Theorem 6.1.

$|Y^{\perp\varepsilon}|^{-\frac{1}{2}} \sum_{b\in Y^{\perp\varepsilon}} (Ff)(b)$
$= |Y^{\perp\varepsilon}|^{-\frac{1}{2}} \sum_{a\in X} \varepsilon_0 (\sum_{b\in Y^{\perp\varepsilon}} \exp(-2\pi i < a,b >_\varepsilon)) f(a)$.

Since $\sum_{b\in Y^{\perp\varepsilon}} \exp(-2\pi i < a,b >_\varepsilon) = \begin{cases} 0 & (a \notin Y) \\ |Y^{\perp\varepsilon}| & (a \in Y) \end{cases}$, the above is equal to

$|Y^{\perp\varepsilon}|^{-\frac{1}{2}} \varepsilon_0 |Y^{\perp\varepsilon}| \sum_{a\in Y} f(a) = |Y|^{-\frac{1}{2}} \sum_{a\in Y} f(a)$.

We obtain the following :

Corollary 6.3

(i) If each generator of Y_k is equal to 1, f is written as $\prod_{k\in L} f_k$, $f_k = {}^*(\mathrm{st}(f_k))|_{L'}$, and $\sum_{-\infty<n<\infty} \mathrm{st}(f_k)(n)$ converges, then

$H^{\frac{H^2}{2}} \mathrm{st}(\sum_{b\in Y^\perp} (Ff)(b)) = \prod_{k\in L} (\sum_{-\infty<n<\infty} \mathrm{st}(f_k)(n))$.

(ii) If each generator of Y_k is infinitesimal, f is written as $\prod_{k\in L} f_k$, $f_k = {}^*(\mathrm{st}(f_k))|_{L'}$, and $\mathrm{st}(f_k)$ is L_1-integrable on $\mathbf{R}$, then

$H^{\frac{H^2}{2}} \mathrm{st}(\sum_{b\in Y^\perp} (Ff)(b)) = \prod_{k\in L} \int_{-\infty}^{\infty} \mathrm{st}(f_k)(t)dt$.

6-2. Examples

We apply Theorem 3.3 to the following two functionals :

1. $g_i(a) = \exp(-i\pi\, {}^*\varepsilon \sum_{k\in L} a(k)^2)$,
2. $g_\xi(a) = \exp(-\xi\pi\, {}^*\varepsilon \sum_{k\in L} a(k)^2)$,

whose infinitesimal Fourier transforms are :

1. $(Fg_i)(b) = (-1)^{\frac{H}{2}}\overline{g_i(b)} \cdots (\natural_4)$,
2. $(Fg_\xi)(b) = C_\xi(b) g_\xi(\frac{b}{\xi})$,

hence we obtain :

1. $|Y^{\perp\varepsilon}|^{-\frac{1}{2}}(-1)^{\frac{H}{2}}\sum_{b\in Y^{\perp\varepsilon}}\overline{g_i(b)} = |Y|^{-\frac{1}{2}}\sum_{a\in Y} g_i(a)$,
2. $|Y^{\perp\varepsilon}|^{-\frac{1}{2}}\sum_{b\in Y^{\perp\varepsilon}} C_\xi(b) g_\xi(\frac{b}{\xi}) = |Y|^{-\frac{1}{2}}\sum_{a\in Y} g_\xi(a)$.

We write this as the following, explicitly :

1. $|Y^{\perp\varepsilon}|^{-\frac{1}{2}}(-1)^{\frac{H}{2}}\sum_{b\in Y^{\perp\varepsilon}}\exp(-i\pi\,{}^\star\varepsilon\sum_{k\in L} b(k)^2)$
$= |Y|^{-\frac{1}{2}}\sum_{a\in Y}\exp(-i\pi\,{}^\star\varepsilon\sum_{k\in L} a(k)^2)$,
2. $|Y^{\perp\varepsilon}|^{-\frac{1}{2}}\sum_{b\in Y^{\perp\varepsilon}} C_\xi(b)\exp(-\frac{1}{\xi}\pi\,{}^\star\varepsilon\sum_{k\in L} a(k)^2)$
$= |Y|^{-\frac{1}{2}}\sum_{a\in Y}\exp(-\xi\pi\,{}^\star\varepsilon\sum_{k\in L} a(k)^2)$.

Corollaly 5.3 implies the following proposition 5.8.

Proposition 6.4

(i) If each generator of Y_k is equal to 1, then the standard parts are :

1. $H^{\frac{H^2}{2}}(-1)^{\frac{H}{2}}\mathrm{st}(\sum_{b\in Y_\varepsilon^\perp}\exp(-i\pi\varepsilon\sum_{k\in L} b(k)^2)) = (\sum_{-\infty<n<\infty}\exp(-i\pi\varepsilon n^2))^{H^2}$,
2. $H^{\frac{H^2}{2}}\mathrm{st}(\sum_{b\in Y_\varepsilon^\perp} C_\xi(b)\exp(-\frac{1}{\xi}\pi\varepsilon\sum_{k\in L} b(k)^2)) = (\sum_{-\infty<n<\infty}\exp(-\xi\pi\varepsilon n^2))^{H^2}$
$\left(= (\theta(i\xi))^{H^2}\right)$,

(ii) If each generator of Y_k is equal to a natural number m_k, then

1. $H^{\frac{H^2}{2}}(-1)^{\frac{H}{2}}\mathrm{st}(\sum_{b\in Y_\varepsilon^\perp}\exp(-i\pi\varepsilon\sum_{k\in L} b(k)^2))$
$= \prod_{k\in L}(m_k\sum_{-\infty<n<\infty}\exp(-i\pi\varepsilon m_k^2 n^2))$,
2. $H^{\frac{H^2}{2}}\mathrm{st}(\sum_{b\in Y_\varepsilon^\perp} C_\xi(b)\exp(-\frac{1}{\xi}\pi\varepsilon\sum_{k\in L} b(k)^2))$
$= \prod_{k\in L}(m_k\sum_{-\infty<n<\infty}\exp(-\xi\pi\varepsilon m_k^2 n^2))$
$(= \prod_{k\in L}(m_k\theta(i m_k^2\xi)))$,

(iii) If each generator of Y_k is infinitesimal, then

2. $\mathrm{st}(\sum_{b\in Y_\varepsilon^\perp} C_\xi(b)\exp(-\frac{1}{\xi}\pi\varepsilon\sum_{k\in L} b(k)^2)) = (\int_{-\infty}^{\infty}\exp(-\xi\pi t^2)dt)^{H^2}$
$\left(= \left(*\left(\frac{1}{\sqrt{\xi}}\right)\right)^{H^2}\right)$.

We extend the above formulation $(\natural_4)$ of $g_i(a)$ to $g_{im}(a) = \exp(-im\pi\,{}^\star\varepsilon\sum_{k\in L} a^2(k))$, for an integer m so that $m|2\,{}^\star HH'^2$. If $m|\frac{b(k)}{\varepsilon'}$ for an arbitrary $k \in L$, we recall

$(Fg_{im})(b) = C_{im}(b) g_{\frac{1}{im}}(b)$, where $C_{im}(b) = \left(\sqrt{\frac{m}{2}}\frac{1+i^{\frac{2^\star HH'^2}{m}}}{1+i}\right)^{\star H^2}$ for a positive

m and $C_{im}(b) = \left(\sqrt{\frac{-m}{2}}\frac{1+(-i)^{\frac{2^\star HH'^2}{-m}}}{1-i}\right)^{\star H^2}$ for a negative m.

Hence $|Y^{\perp_\varepsilon}|^{-\frac{1}{2}} \sum_{b\in Y^\perp} C_{im}(b) g_{\frac{1}{im}}(b) = |Y|^{-\frac{1}{2}} \sum_{a\in Y} g_{im}(a)$. When each generator $\varepsilon' s'_k$ of $Y_k^{\perp_\varepsilon}$ satisfies $m|s'_k$, that is, each generator $\varepsilon' s_k$ of Y_k satisfies $m|\frac{{}^*HH'^2}{s_k}$, it reduces to the following :

$$H^{\frac{H^2}{2}} H'^{({}^*H)^2} \left(\sqrt{\frac{m}{2}} \frac{1+i^{\frac{2\,{}^*HH'^2}{m}}}{1+i}\right)^{({}^*H)^2} \sum_{b\in Y^{\perp_\varepsilon}} \exp(i\pi \frac{1}{m} {}^*\varepsilon \sum_{k\in L} b(k)^2)$$

$= \prod_{k\in L} s_k \sum_{a\in Y} \exp(-im\pi\, {}^*\varepsilon \sum_{k\in L} a(k)^2)$ for a positive m, and

$$H^{\frac{H^2}{2}} H'^{({}^*H)^2} \left(\sqrt{\frac{-m}{2}} \frac{1+(-i)^{\frac{2\,{}^*HH'^2}{-m}}}{1-i}\right)^{({}^*H)^2} \sum_{b\in Y^{\perp_\varepsilon}} \exp(i\pi \frac{1}{m} {}^*\varepsilon \sum_{k\in L} b(k)^2)$$

$= \prod_{k\in L} s_k \sum_{a\in Y} \exp(-im\pi\, {}^*\varepsilon \sum_{k\in L} a(k)^2)$ for a negative m.

If $s_k = H'$ and $m|H'$, then

$$H^{\frac{H^2}{2}} \left(\sqrt{\frac{m}{2}} \frac{1+i^{\frac{2\,{}^*HH'^2}{m}}}{1+i}\right)^{({}^*H)^2} \sum_{b\in Y^{\perp_\varepsilon}} \exp(i\pi \frac{1}{m} \sum_{k\in L} b(k)^2)$$

$= \sum_{a\in Y} \exp(-im\pi\, {}^*\varepsilon \sum_{k\in L} a(k)^2)$ for a positive m, and

$$H^{\frac{H^2}{2}} \left(\sqrt{\frac{-m}{2}} \frac{1+(-i)^{\frac{2\,{}^*HH'^2}{-m}}}{1-i}\right)^{({}^*H)^2} \sum_{b\in Y^{\perp_\varepsilon}} \exp(i\pi \frac{1}{m} \sum_{k\in L} b(k)^2)$$

$= \sum_{a\in Y} \exp(-im\pi\, {}^*\varepsilon \sum_{k\in L} a(k)^2)$ for a negative m, that is,

$$H^{\frac{H^2}{2}} \left(\sqrt{m} \exp(-i\frac{\pi}{4})\right)^{({}^*H)^2} \sum_{b\in Y^{\perp_\varepsilon}} \exp(i\pi \frac{1}{m} \sum_{k\in L} b(k)^2)$$

$= \sum_{a\in Y} \exp(-im\pi\, {}^*\varepsilon \sum_{k\in L} a(k)^2)$ for a positive m, and

$$H^{\frac{H^2}{2}} \left(\sqrt{-m} \exp(i\frac{\pi}{4})\right)^{({}^*H)^2} \sum_{b\in Y^{\perp_\varepsilon}} \exp(i\pi \frac{1}{m} \sum_{k\in L} b(k)^2)$$

$= \sum_{a\in Y} \exp(-im\pi\, {}^*\varepsilon \sum_{k\in L} a(k)^2)$ for a negative m.

7. The infinitesimal Fourier transform of a functional $Z_s(a)$

In this section, we define a functional on X, and study a relationship between the functional and Riemann's zeta function. We order all prime numbers as $p(1) = 2$, $p(2) = 3, \dots, p(n) < p(n+1), \dots$, that is, p is a mapping from **N** to the set {prime number}, $p : \mathbf{N} \to$ {prime number}. The nonstandard extension ${}^*p : {}^*\mathbf{N} \to {}^*$\{prime number\} is written as ${}^*p([l_\mu]) = [p(l_\mu)]$, and we define a mapping

$\tilde{p} : {}^*\mathbf{N} \to {}^*({}^*\{\text{prime number}\})$ as $\tilde{p}([l_\mu]) = {}^*[p(l_\mu)]$. For $s \in \mathbf{C}$, we define $Z_s (\in A)$ as the following :

$$Z_s(a) := \prod_{k\in L} \tilde{p}(H(k+\frac{H}{2})+1)^{(-s(a(k)+\frac{H'}{2}))}$$

Now $H(k+\frac{H}{2})+1$ is an element of ${}^*\mathbf{N}$ and $a(k)+H'/2$ is an element of ${}^*({}^*\mathbf{N})$. Then $Z_s(a)$ is calculated as $\exp(-s\sum_{k\in L}\log(\tilde{p}(H(k+\frac{H}{2})+1))a(k))\prod_{k\in L}\tilde{p}(H(k+\frac{H}{2})+1)^{-s\frac{H'}{2}}$. We obtain the following theorem for the Fourier transform of $\mathbf{e}^\sharp(Z_s)$ for Definition 1.2 :

Theorem 7.1. $(F(\mathbf{e}^\sharp(Z_s)))(b) = \left(\prod_{k\in L}\tilde{p}(H(k+\frac{H}{2})+1)\right)^{-s\frac{H'}{2}}$

$$\cdot\prod_{k\in L}\varepsilon'\frac{\sinh((2\pi i b(k)+s\log\tilde{p}(H(k+\frac{H}{2})+1))\frac{H'}{2})}{\exp(-\frac{\varepsilon'}{2}(2\pi i b(k)+s\log\tilde{p}(H(k+\frac{H}{2})+1))\sinh(\frac{\varepsilon'}{2}(2\pi i b(k)+s\log\tilde{p}(H(k+\frac{H}{2})+1))}.$$

Proof. $(F(\mathbf{e}^\sharp(Z_s)))(b) = \left(\prod_{k\in L}\tilde{p}(H(k+\frac{H}{2})+1)\right)^{-s\frac{H'}{2}}$
$\cdot\sum_{a\in X}\varepsilon_0\exp(-s\sum_{k\in L}\log\tilde{p}(H(k+\frac{H}{2})+1)a(k))\exp(-2\pi i\sum_{k\in L}a(k)b(k))$
$= \left(\prod_{k\in L}\tilde{p}(H(k+\frac{H}{2})+1)\right)^{-s\frac{H'}{2}}$
$\cdot\sum_{a\in X}\varepsilon_0\exp(-(2\pi i\, b(k)+s\log\tilde{p}(H(k+\frac{H}{2})+1))a(k))$

$$= \left(\prod_{k\in L}\tilde{p}(H(k+\frac{H}{2})+1)\right)^{-s\frac{H'}{2}}$$

$$\cdot\prod_{k\in L}\varepsilon'\frac{\sinh((2\pi i\, b(k)+s\log\tilde{p}(H(k+\frac{H}{2})+1))\frac{H'}{2})}{\exp(-\frac{\varepsilon'}{2}(2\pi i\, b(k)+s\log\tilde{p}(H(k+\frac{H}{2})+1))\sinh(\frac{\varepsilon'}{2}(2\pi i\, b(k)+s\log\tilde{p}(H(k+\frac{H}{2})+1))}.$$

We denote Riemann's zeta function by $\zeta(s)$, defined by $\zeta(s) = \prod_{l=1}^\infty \frac{1}{1-p(l)^{-s}}$ for $\mathrm{Re}(s) > 1$. Let $Y_\mathbf{Z}$ be a subgroup of X so that each generator of $(Y_\mathbf{Z})_k$ is equal to 1. Then we obtain the following theorem :

Theorem 7.2. If $\mathrm{Re}(s) > 1$, then $\mathrm{st}(\mathrm{st}(\sum_{a\in Y_\mathbf{Z}}\mathbf{e}^\sharp(Z_s))(a))) = \zeta(s)$.

Proof. $\mathrm{st}(\mathrm{st}(\sum_{a\in Y_\mathbf{Z}}\mathbf{e}^\sharp(Z_s))(a)))$

$$=\mathrm{st}\left(\mathrm{st}\left(\left(\prod_{k\in L}\tilde{p}(H(k+\frac{H}{2})+1)\right)^{(-s(a(k)+\frac{H'}{2}))}\right)\right)$$

$$= \mathrm{st}\left(\mathrm{st}\left(\prod_{k\in L}\frac{1-\tilde{p}(H(k+\frac{H}{2})+1)^{-sH'}}{1-\tilde{p}(H(k+\frac{H}{2})+1)^{-s}}\right)\right)$$

$$=\mathrm{st}\left(\prod_{k\in L}\frac{1}{1-\tilde{p}(H(k+\frac{H}{2})+1)^{-s}}\right) = \zeta(s).$$

Furthermore, Corollary 5.3.(1) and Theorem 7.2 imply the following :

Corollary 7.3. $\mathrm{st}(\sum_{b\in Y_{\mathbf{Z}}^{\perp}}(F(\mathrm{e}^{\sharp}(Z_s)))(b)) =$

$$\mathrm{st}\Big(\prod_{k\in L}\frac{1-\tilde{p}(H(k+\frac{H}{2})+1)^{-sH'}}{1-\tilde{p}(H(k+\frac{H}{2})+1)^{-s}}\Big).$$

Hence we obtain : $\mathrm{st}(\mathrm{st}(\sum_{b\in Y_{\mathbf{Z}}^{\perp}}(F(\mathrm{e}^{\sharp}(Z_s)))(b)))) = \zeta(s)$ for $\mathrm{Re}(s) > 1$.

FURTHER PROBLEMS. Now the following two points are not clear for us.

(i) In order to define a standard Fourier transformation for the space of functionals, how to apply the nonstandard Fourier transformation to the standard space of functionals. There is a canonical method to apply a nonstandard one to a standard one, but does it define a standard Fourier transformation or not?

(ii) For which class of standard functionals are the nonstandard Fourier transformation applicable? Furthermore, in which class of standard functionals is the image of the nonstandard transform realized?

These are remained for quite important problems , as it is shown that there exists no parallelizable standard Borel measure on the standard space of functions. Each of them is a big theme for our later study.

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