Nihonkai Math. J.
Vol.15(2004), 153-168

SOME LINEAR FUNCTIONAL AND FOURIER TRANSFORM OVER $\mathcal{K}'_{e,k}$

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ABSTRACT. We introduce the space $\mathcal{K}_{e,k}$ that is the vector space of all C^∞ - functions f such that $exp(e^{k|x|})\partial^\alpha f$ vanishes at infinity for all $\alpha \in N^n, k \in Z, k < 0$ and its dual $\mathcal{K}'_{e,k}$. For $f, g \in \mathcal{K}'_{e,k}$, we study the linear functional $f \circledast g$ on $\mathcal{K}_{e,k}$ defined by

$$< f \circledast g > = < f(x), < g(y), \phi(x+y) >>, \quad \phi \in \mathcal{K}_{e,k}.$$

Also, we show a representation theorem for the usual distributional Fourier transform over the spaces $\mathcal{K}'_{e,k}$, and an inversion formula which enables to prove that $\mathcal{K}'_{e,k}$ is a commutative algebra with unit element with respect to $\circledast$

1. Introduction

The Schwartz space $\mathcal{S}$ is the space of all infinitely differentiable function f on R^n such that $(1+|x|^2)^k \partial^\alpha f(x)$ vanishes at infinity for all $k \in Z$ and all $\alpha \in N^n$. The space $\mathcal{S}$ is equipped with the locally convex topology defined by the family $(q_{k,\alpha})$ of seminorms $(q_{k,\alpha}) = (1+|x|^2)^k |\partial^\alpha f(x)|$, where k runs through N and α through N^n. By $\mathcal{S}'$, we mean the space of continuous linear functionals on $\mathcal{S}$. Motivated by the Schwartz space $\mathcal{S}$, J. Horváth introduced the space $\mathcal{S}_k$, k is a fixed integer, that is defined as the vector space of all functions f on R^n such that $(1+|x|^2)^k \partial^\alpha f(x)$ vanishes at infinity for all $\alpha \in N^n$ in [3]. Horváth defined on $\mathcal{S}_k$ the seminorms $(\mu_{k,\alpha}) = (1+|x|^2)^k |\partial^\alpha f(x)|$ for a fixed k and every $\alpha \in N^n$. And B.J.Gonzalez and E.R.Negrin studied the convolution and Fourier transform over $\mathcal{S}_k, k \in Z, k < 0$, in [1] and [2], respectively.

In the meantime, the Schwartz space $\mathcal{S}$ is extended by G. Sampson and Z. Zielezny in [5]. They introduced the space $\mathcal{K}_p, p > 1$, of the space of all infinitely differentiable functions f on R^n such that $e^{k|x|^p} \partial^\alpha f(x)$ vanishes at infinity for all $k \in Z$ and all $\alpha \in N^n$. The space $\mathcal{K}_p, p > 1$, is equipped with the locally convex topology defined by the family of seminorms $(\gamma_{k,\alpha}) = e^{k|x|^p} |\partial^\alpha f(x)|$, where k runs

Key words and phrases. linear functional, Fourier transform.
2000 Mathematics Subject Classifications. 46F10, 46F05
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This work was supported by Inje Research and Scholarship Foundation Grant 2003-2004

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

through N and α through N^n. They also studied convolution over $\mathcal{K}'_p$, the dual of $\mathcal{K}_p$, in terms of their Fourier transform.

The extended Schwartz space $\mathcal{K}_p$, is extended to the spaces $\mathcal{K}_e$ by D. H. Pahk in [4]. D. H. Pahk denote $\mathcal{K}_e$ the space of all functions $\phi \in C^\infty(R^n)$ such that

$$\nu_k(\phi) = \sup_{x \in R^n, |\alpha| \leq k} exp(e^{k|x|})|D^\alpha \phi(x)| < \infty, k = 1, 2, ...$$

The space $\mathcal{K}_e$ with seminorms $\nu_k, k = 1, 2, ...$ is a Frechét space and the space of C^∞-functions with compact support $\mathcal{D}$ is a dense subset of $\mathcal{K}_e$. By $\mathcal{K}'_e$ we mean the space of continuous linear functionals on $\mathcal{K}_e$.

Motivated by the space $\mathcal{K}_e$, we introduce the spaces $\mathcal{K}_{e,k}(R^n), k \in Z, k < 0$ that is defined as the vector spaces of all functions f defined on R^n which possess continuous partial derivatives of all orders and satisfy the condition that if $\alpha \in N^n$ and $\epsilon > 0$, then there exists $C = C(f, \alpha, \epsilon) > 0$ such that

$$exp(e^{k|x|})|\partial^\alpha f(x)| \leq \epsilon,$$

for $|x| > C(f, \alpha, \epsilon)$.

In what follows, we shall write $\mathcal{K}_{e,k}$ instead of $\mathcal{K}_{e,k}(R^n)$. For every $\alpha \in N^n$ and fixed $k \in Z, k < 0$, we define on $\mathcal{K}_{e,k}$ the seminorms

$$q_{k,\alpha}(f) = \max_{x \in R^n} exp(e^{k|x|})|\partial^\alpha f(x)|.$$

The space $\mathcal{K}_{e,k}$ equipped with the countable family of seminorms is a locally convex space. Then $\mathcal{D}$ is a dense subspace of $\mathcal{K}_{e,k}$. By $\mathcal{K}'_{e,k}$, we mean the space of continuous linear functionals on $\mathcal{K}_{e,k}$.

In this paper, we will study convolutional type of linear functional on $\mathcal{K}_{e,k}$ as in the case of $\mathcal{S}_k$ in [1]. We will prove that for $f, g \in \mathcal{K}'_{e,k}, k \in Z, k < 0$, the linear functional $f \circledast g$ defined by

$$< f \circledast g, \phi > = < f(x), < g(y), \phi(x+y) >>, \quad \phi \in \mathcal{K}_{e,k},$$

has sense as the application of the functional $f \in \mathcal{K}'_{e,k}$ to $< g(y), \phi(x+y) > \in \mathcal{K}_{e,k}$.

Lastly, as in the case on $\mathcal{S}_k$ in [2] we will show that we can derive a representation theorem for the usual distributional Fourier transform over the spaces $\mathcal{K}'_{e,k}, k \in Z, k < 0$, and an inversion formula which enables us to prove that $\mathcal{K}'_{e,k}$ is a commutative algebra with unit element with respect to $\circledast$.

Throughout this paper we will use the notations and terminologies of [3].

2. Convolutional type of linear functional over $\mathcal{K}'_{e,k}$

First, we will prove that for $f, g \in \mathcal{K}'_{e,k}, \phi \in \mathcal{K}_{e,k}, k \in Z, k < 0$, the linear functional $f \circledast g$ defined by

$$(1) \qquad < f \circledast g, \phi > = < f(x), < g(y), \phi(x+y) >>$$

has sense as the application of the functional $f \in \mathcal{K}'_{e,k}$ to $< g(y), \phi(x+y) > \in \mathcal{K}_{e,k}$. It is also obtained that $f \circledast g \in \mathcal{K}'_{e,k}$.

For the proof of the above results, we need the following several lemmas.

Lemma 2.1. *Let $x \in R^n$ be a fixed vector, $\phi \in \mathcal{K}'_{e,k}, k \in Z, k < 0$, then $\phi(x+y) \in \mathcal{K}_{e,k}$.*

Proof. Since $\phi \in \mathcal{K}'_{e,k}$, for all $\epsilon > 0$ and $\alpha \in N^n$, there exists $A(\phi, \alpha, \epsilon) > 0$ such that

$$exp(e^{k|z|})|\partial^\alpha \phi(z)| \le \epsilon,$$

for $|z| > A(\phi, \alpha, \epsilon)$. Then, since $k < 0$, if we take $B(\phi, \alpha, \epsilon, x) = A(\phi, \alpha, \epsilon) + |x|$, then for $|y| > B(\phi, \alpha, \epsilon, x)$,

$$\begin{aligned} exp(e^{k|y|})|\partial^\alpha \phi(x+y)| &= exp(e^{k|y|} - e^{k|x+y|} + e^{k|x+y|})|\partial^\alpha \phi(x+y)| \\ &\le \frac{exp(e^{k|y|})}{exp(e^{k|x+y|})}\epsilon \\ &\le \frac{exp(e^{k|x+y|} \cdot e^{-k|x|})}{exp(e^{k|x+y|})}\epsilon \\ &\le \frac{exp(\frac{1}{2}(e^{k|x+y|})^2) \cdot exp(\frac{1}{2}(e^{-k|x|})^2)}{exp(e^{k|x+y|})}\epsilon \\ (2) \qquad &\le C exp(\frac{1}{2}(e^{-k|x|})^2)\epsilon. \end{aligned}$$

Therfore, for each fixed vector $x \in R^n$, $\phi(x+y) \in \mathcal{K}_{e,k}$. □

Lemma 2.2. *If $g \in \mathcal{K}'_{e,k}$ and $\phi \in \mathcal{K}_{e,k}$ with $k \in Z, k < 0$, then, for all $m \in N^n$,*

$$(3) \qquad \partial^m < g(y), \phi(x+y) > = < g(y), \partial^m \phi(x+y) > .$$

Proof. We will prove (3) by induction on $|m|$. Assume $|m| = 1$. For each fixed $x \in R^n$ and each fixed $i = 1, 2, ...n$, set $h_i = (h_{i,1}, h_{i,2}, ..., h_{i,n}) \in R^n$ given by $h_{i,i} = \Delta x_i \ne 0$ and $h_{i,j} = 0$ for $j \ne i$. Now consider

$$\frac{1}{\Delta x_i}\{< g(y), \phi(x+y+h_i) > - < g(y), \phi(x+y) >\}$$

$$- < g(y), \frac{\partial}{\partial x_i}\phi(x+y) > = < g(y), \theta_{h_i,x}(y) >,$$

where

$$\theta_{h_i,x}(y) = \frac{1}{\triangle x_i}\{\phi(x+y+h_i) - \phi(x+y)\} - \frac{\partial}{\partial x_i}\phi(x+y).$$

We will prove that $\theta_{h_i,x} \to 0$, in $\mathcal{K}_{e,k}$ for $|h_i| \to 0$, which assures that

$$\frac{\partial}{\partial x_i} < g(y), \phi(x+y) > = < g(y), \frac{\partial}{\partial x_i}\phi(x+y) > .$$

First, we will check that $\theta_{h_i,x}(y) \in \mathcal{K}_{e,k}$. For all $\alpha \in N^n$ and $y \in R^n$,

$$\partial^\alpha \phi(x+y+h_i) = \partial^\alpha \phi(x+y) + \triangle x_i \frac{\partial}{\partial x_i}\partial^\alpha \phi(x+y)$$

$$+ \int_0^{\triangle x_i} (\triangle x_i - \xi)\frac{\partial^2}{\partial x_i^2}\partial^\alpha \phi(x+y+t_{i,\xi})d\xi,$$

where $t_{i,\xi} = (t_{i,1,\xi}, t_{i,2,\xi}, ..., t_{i,n,\xi})$ with $t_{i,j,\xi} = \xi$ for $j = i$ and $t_{i,j,\xi} = 0$ for $j \neq i$. Therefore,

$$\partial^\alpha \theta_{h_i,x}(y) = \int_0^{\triangle x_i} (\triangle x_i - \xi)\frac{\partial^2}{\partial x_i^2}\partial^\alpha \phi(x+y+t_{i,\xi})d\xi.$$

Since $\phi \in \mathcal{K}_{e,k}$, given $\epsilon > 0$ and $\alpha \in N^n$, there exist $A(\phi,\alpha,\epsilon) > 0$ such that if $|z| > A(\phi,\alpha,\epsilon)$, then

$$exp(e^{k|z|})|\frac{\partial^2}{\partial z_i^2}\partial^\alpha \phi(z)| < \epsilon.$$

Now, for $|t| \leq |h_i| < 1$,

$$exp(e^{k|y|})|\frac{\partial^2}{\partial y_i^2}\partial^\alpha \phi(x+y+t)|$$

$$= exp(e^{k|y|} - e^{k|x+y+t|} + e^{k|x+y+t|})|\frac{\partial^2}{\partial y_i^2}\partial^\alpha \phi(x+y+t)|. \tag{4}$$

Since $\phi \in \mathcal{K}_{e,k}$, we have that for $|t| \leq 1$ and $|x+y+t| > A(\phi,\alpha,\epsilon)$,

$$exp(e^{k|x+y+t|})|\frac{\partial^2}{\partial y_i^2}\partial^\alpha \phi(x+y+t)| < \epsilon.$$

If we let $B(\phi,\alpha,\epsilon,x) = A(\phi,\alpha,\epsilon) + |x| + 1$, since $k < 0$, we have that for $|y| > B(\phi,\alpha,\epsilon,x)$ and $|t| \leq 1$, (4) is less than or qual to

$$
\begin{aligned}
exp(e^{k|y|}-e^{k|x+y+t|})\epsilon &\le \frac{exp(e^{k|y|})}{exp(e^{k|x+y+t|})}\epsilon \\
&\le \frac{exp(e^{k|x+y+t|}\cdot e^{-k|x+t|})}{exp(e^{k|x+y+t|})}\epsilon \\
&\le \frac{exp(\frac{1}{2}(e^{k|x+y+t|})^2)\cdot exp(\frac{1}{2}(e^{-k|x+t|})^2)}{exp(e^{k|x+y+t|})}\epsilon \\
&\le Cexp(\frac{1}{2}(e^{-k|x+t|})^2)\epsilon \\
&\le Cexp(\frac{1}{2}(e^{-k|x|})^2\cdot(e^{-k|t|})^2)\epsilon \\
&\le Cexp(\frac{1}{2}(e^{-k})^2\cdot(e^{-k|x|})^2)\epsilon.
\end{aligned}
$$

So, for $|y| > B(\phi,\alpha,\epsilon,x)$,

$$
\begin{aligned}
exp(e^{k|y|})|\partial^\alpha\theta_{h_i,x}(y)| &\le C\frac{exp(\frac{1}{2}(e^{-k})^2\cdot(e^{-k|x|})^2)\epsilon}{|\Delta x_i|}\int_0^{\Delta x_i}(\Delta x_i-\xi)d\xi \\
&= \frac{|\Delta x_i|}{2}exp(\frac{1}{2}(e^{-k})^2\cdot(e^{-k|x|})^2)\epsilon, \qquad (5)
\end{aligned}
$$

and thus $\theta_{h_i,x}(y)\in\mathcal{K}_{e,k}$. On the other hand, for $|y|\le B(\phi,\alpha,\epsilon,x)$ and $|y|\le 1$,

$$
exp(e^{k|y|})|\frac{\partial^2}{\partial y_i^2}\partial^\alpha\phi(x+y+t)|\le M_1,
$$

for some constant M_1. Setting $M_2=\max\{M_1, exp(\frac{1}{2}(e^{-k})^2\cdot(e^{-k|x|})^2)\epsilon\}$ and taking into account (5), for all $y\in R^n$,

$$
\begin{aligned}
exp(e^{k|y|})|\partial^\alpha\theta_{h_i,x}(y)| &\le \frac{M_2}{\Delta x_i}\int_0^{\Delta x_i}(\Delta x_i-\xi)d\xi \\
&= \frac{|\Delta x_i|}{2}M_2,
\end{aligned}
$$

which tends to 0 as $|h_i|\to 0$. This proves the conclusion for $|m|=1$. Now, the result of this lemma follows by induction on $|m|$. □

Lemma 2.3. *If $g\in\mathcal{K}'_{e,k}, \phi\in\mathcal{K}_{e,k}, k\in Z, k<0$, then $< g(y),\phi(x+y) >\in\mathcal{K}_{e,k}$.*

Proof. From Lemma 2.2, one has that $< g(y),\phi(x+y) >$ is smooth. It remains to prove that for any $m\in N^n$ and any $\epsilon>0$, there exist $B>0$ such that if $|x|>B$, then $exp(e^{k|x|})|\partial^m < g(y),\phi(x+y) > |\le\epsilon$. In fact, from Lemma 2.2

and [3, remark of Proposition 2, p.97] there exists a positive constant C and a nonnegative integer r such that

$$(6) \qquad |< g, \phi >| \leq C \max_{0 \leq s \leq r} q_{k,\alpha_s}(\phi),$$

for $\phi \in \mathcal{K}_{e,k}$.

Here C and r depend on g but not on ϕ. First, we will show that this lemma holds for $\phi \in \mathcal{D}(R^n)$. Since $\mathcal{D} \subset \mathcal{K}_{e,k}$, by (6), for any $m \in N^n$ and $\phi \in \mathcal{D}$,

$$\begin{aligned} exp(e^{k|x|})|\partial_x^m < g(y), \phi(x+y) > | &= exp(e^{k|x|})| < g(y), \partial_x^m \phi(x+y) > | \\ &\leq C \max_{0 \leq s \leq r} \max_{y \in R^n} exp(e^{k|x|}) exp(e^{k|y|}) \\ &\times |\partial_x^m \partial_y^{\alpha_s} \phi(x+y)| \\ &\leq C \max_{0 \leq s \leq r} exp(e^{k|x|}) M_{m,\alpha_s}, \end{aligned}$$

where $M_{m,\alpha_s} = \max_{z \in R^n} |\partial^{m+\alpha_s}\phi(z)|$. Since $k < 0$, this lemma holds for $\phi \in \mathcal{D}$. Next, since $\mathcal{D}$ is a dense subset of $\mathcal{K}_{e,k}$, for $\phi \in \mathcal{K}_{e,k}$, there exists a sequence $\{\phi_j\} \subset \mathcal{D}$ with $\phi_j \to \phi$ in $\mathcal{K}_{e,k}$ as $j \to \infty$. Hence for any $\epsilon > 0$ and any $\alpha \in N^n$, there exist $j_0^* = j_0^*(\epsilon, \alpha) \in N$ such that

$$\max_{z \in R^n} exp(e^{k|z|})|\partial^\alpha \{\phi_j(z) - \phi(z)\}| \leq \frac{\epsilon}{2C},$$

for $j \geq j_0^*$. So, for any $\epsilon > 0$ and any $\alpha \in N^n$, if $j \geq j_0 = \max\{j_0^*(\epsilon, m+\alpha_s)\}, s = 0, 1, ...r$,

$$\begin{aligned} &exp(e^{k|x|})\partial_x^m |\{< g(y), \phi_j(x+y) > - < g(y), \phi(x+y) >\}| \\ &\quad \leq C \max_{0 \leq s \leq r} \max_{y \in R^n} exp(e^{k|x|}) exp(e^{k|y|}) |\partial_y^{\alpha_s} \partial_x^m \{\phi_j(x+y) - \phi(x+y)\}| \\ &\quad = C \max_{0 \leq s \leq r} \max_{y \in R^n} exp(e^{k|x|}) exp(e^{k|y|}) \\ &\qquad \times exp(-e^{k|x+y|} + e^{k|x+y|}) |\partial^{m+\alpha_s} \{\phi_j(x+y) - \phi(x+y)\}| \\ &\quad = C \max_{0 \leq s \leq r} \max_{y \in R^n} exp(e^{k|x|} + e^{k|y|} - e^{k|x+y|}) \\ &\qquad \times exp(e^{k|x+y|}) |\partial^{m+\alpha_s} \{\phi_j(x+y) - \phi(x+y)\}| \\ &\quad \leq C \max_{0 \leq s \leq r} \max_{z \in R^n} exp(e^{k|z|}) |\partial^{m+\alpha_s} \{\phi_j(z) - \phi(z)\}| \\ (7) &\quad < \frac{\epsilon}{2}. \end{aligned}$$

Also, since $< g(y), \phi_{j_0}(x+y) > \in \mathcal{K}_{e,k}$, for any $\epsilon > 0$ and $m \in N^n$, there exist $A(\epsilon, m, \phi_{j_0})$ such that

$$exp(e^{k|x|})|\partial_x^m < g(y), \phi_{jo}(x+y) > | < \frac{\epsilon}{2},$$

for $|x| > A(\epsilon, m, \phi_{jo})$. Hence taking $B = A(\epsilon, m, \phi_{jo})$, for $|x| > B$, then, by (7) and above fact,

$$\begin{aligned}
&exp(e^{k|x|})|\partial_x^m < g(y), \phi(x+y) > | \\
&\quad \leq exp(e^{k|x|})|\partial_x^m < g(y), \phi_{jo}(x+y) > | \\
&\quad\quad + exp(e^{k|x|})|\{\partial_x^m < g(y), \phi(x+y) > -\partial_x^m < g(y), \phi_{jo}(x+y) >\}| \\
&\quad < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

Thus the result follows. □

Lemma 2.4. *Assume that $k \in Z, k < 0, g \in \mathcal{K}'_{e,k}$ and $\phi_j \to 0$ in $\mathcal{K}_{e,k}$ for $j \to \infty$, then $< g(y), \phi_j(x+y) > \to 0$ in $\mathcal{K}_{e,k}$ as $j \to \infty$.*

Proof. By (6) in the proof of Lemma 2.3 above,

$$exp(e^{k|x|})|\partial^m < g(y), \phi_j(x+y) > | \leq C \max_{0 \leq s \leq r} q_{k,m+\alpha_s}(\phi_j).$$

From the above fact the result of this lemma follows immediately. □

Now, we conclude that

Theorem 2.5. *If $f, g \in \mathcal{K}'_{e,k}, k \in Z, k < 0$, then $f \circledast g \in \mathcal{K}'_{e,k}$.*

Proof. Let $\{\phi_j\} \subset \mathcal{K}_{e,k}$ such that $\phi_j \to 0$ in $\mathcal{K}_{e,k}$ as $j \to \infty$. By Lemma 2.1 and Lemma 2.3

$$< f \circledast g, \phi_j > = < f(x), < g(y), \phi_j(x+y) >>$$

has sense, and by Lemma 2.4 and $f \in \mathcal{K}'_{e,k}$, $< f \circledast g, \phi_j >$ tends to zero as $j \to \infty$. □

Remark. Since the weight function $exp(e^{k|x|})$ is not infinitely differentiable, we can not have the structure theorem for $f \in \mathcal{K}'_{e,k}$, i.e., $f \in \mathcal{K}'_{e,k}$ can be identified with the space of all distributions of the form $exp(e^{k|x|}) \sum \partial^\alpha \mu_\alpha$, where (μ_α) is some finite family of measure belonging to the dual of Banach space of all continuous functions on R^n vanishing at infinity. To apply the above form of the distribution in the structure theorem to a test function, one would have to differentiate the product of the weight function $exp(e^{k|x|})$ by a test function, and this differentiation produces terms which are not bounded relative to the weight function. Although the weight function $exp(e^{k|x|})$ can be replaced by an equivalent infinitely differentiable weight function $exp(e^{k(1+|x|^2)^{\frac{1}{2}}})$, there is some difficulties to provide the reasonable notion of integrable distribution on open sets different from R^n. Therefore, we can not present that the product defined in (1) is the same as the general convolution $f * g$ in the sense of Laurent Schwartz, or Horváth in [3].

3. Fourier Transform over $\mathcal{K}'_{e,k}$

In this section, we will state a representation theorem for the usual distributional Fourier transform over the space $\mathcal{K}_{e,k}, k \in Z, k < 0$. It's inversion formula is also obtained, which enables us to prove that $\mathcal{K}'_{e,k}$ is commutative convolution algebra with unit element with respect to the linear functional $\circledast$ defined in (1).

For $k < 0$ and each $y \in R^n$, the function $x \mapsto e^{ixy}$ is a member of $\mathcal{K}_{e,k}$. Hence, the application of the functional $f \in \mathcal{K}'_{e,k}$ to e^{ixy} yields the following complex-valued function of y,

$$\begin{aligned} F(y) &= (\mathcal{F}f)(y) \\ &= < f(x), e^{ixy} > \end{aligned} \tag{8}$$

Next, if we only replace $(1 + |x|^2)^k$ and $\mathcal{S}_k$ by $exp(e^{k|x|})$ and $\mathcal{K}_{e,k}$, respectively, we can show exactly like Theorem 2.1 in [2] that the function (8) represents the usual distributional Fourier transform when it acts over members $f \in \mathcal{K}'_{e,k}, k \in Z, k < 0$ and finctions in $\mathcal{K}_e$, i.e.,

Theorem 3.1. *Let $f \in \mathcal{K}'_{e,k}, k \in Z, k < 0$. Then for all $\phi \in \mathcal{K}_e$, the Parseval eqality*

$$< f, \mathcal{F}\phi > = < \mathbb{T}_{<f(x),e^{ixy}>}, \phi(y) >,$$

follows, where $\mathbb{T}_{<f(x),e^{ixy}>}$ is the member of $\mathcal{K}'_e$ given by

$$< \mathbb{T}_{<f(x),e^{ixy}>}, \phi(y) > = \int_{R^n} < f(x), e^{ixy} > \phi(y) dy,$$

and $\mathcal{F}\phi$ denotes the classical Fourier transform of ϕ, namely,

$$(\mathcal{F}\phi)(t) = \int_{R^n} \phi(y) e^{ixy} dy, \quad t \in R^n.$$

Now, in order to obtain an inversion formula for the Fourier transform over the space $\mathcal{K}_{e,k}$, we need the following lemma. The techniques employed in the next lemma are extracted from B. J. Gonzalez and E. R. Negrin [2, Lemma 3.1] and A. H. Zemmanian [6, Lemma 3.5-2].

Lemma 3.2. *Let $\phi_1, ..., \phi_n \in \mathcal{D}(R), x = (x_1, ..., x_n) \in R, t = (t_1, ..., t_n) \in R$, then, for any $k \in Z, k < 0$, one has*

$$\frac{1}{\pi^n} \int_{R^n} \prod_{j=1}^{n} \phi_j(x_j + t_j) \frac{\sin Y t_j}{t_j} dt \longrightarrow \phi_1(x_1) \cdots \phi_n(x_n)$$

in $\mathcal{K}_{e,k}$ as $Y \to +\infty$.

Proof. First, we will show that for $\phi \in \mathcal{D}(R^n)$, and $p \in N, \alpha \in R, \alpha < 0, Y > 0$, then

$$\Psi_Y(x) = \frac{1}{\pi} \int_{\infty}^{\infty} \phi(t + x) \frac{\sin Yt}{t} dt \in \mathcal{K}_{e,k}, \tag{9}$$

and

$$\max_{x \in R} exp(e^{\alpha|x|}) |D^p\{\Psi_Y(x) - \phi(x)\}| \longrightarrow 0, \tag{10}$$

for $Y \to +\infty$.

In order to show (9) and (10), we need the following;

$$D_x^p \int_{-\infty}^{\infty} \phi(t + x) \frac{\sin Yt}{t} dt = \int_{-\infty}^{\infty} D_x^p \phi(t + x) \frac{\sin Yt}{t} dt, \; p \in N. \tag{11}$$

For $p = 0$, (11) is trivial. For $p = 1$, since $\phi \in \mathcal{D}(R)$, $supp\phi \subset [a, b]$ for some $a, b \in R$ with $a < b$,

$$\begin{aligned} D_x \int_{-\infty}^{\infty} \phi(t + x) \frac{\sin Yt}{t} dt &= D_x \int_{a-x}^{b-x} \phi(t + x) \frac{\sin Yt}{t} dt \\ &= -\phi(b) \frac{\sin Y(b - x)}{b - x} + \phi(a) \frac{\sin Y(a - x)}{a - x} + \int_{a-x}^{b-x} D_x \phi(t + x) \frac{\sin Yt}{t} dt \\ &= \int_{a-x}^{b-x} D_x \phi(t + x) \frac{\sin Yt}{t} dt = \int_{-\infty}^{\infty} D_x \phi(t + x) \frac{\sin Yt}{t} dt. \end{aligned}$$

Since $D^p\phi \in \mathcal{D}(R)$, by induction on $p \in N$,we can prove (11) for any $p \in N$.

Now, let $\alpha \in R$ with $\alpha < 0$ and $\epsilon > 0$. Then

$$
\begin{aligned}
&| \, exp(e^{\alpha|x|}) D_x \int_{-\infty}^{\infty} \phi(t+x) \frac{\sin Yt}{t} dt \, | \\
&\leq exp(e^{\alpha|x|}) \cdot \int_{-\infty}^{\infty} | \, D_x^p \phi(t+x) \frac{\sin Yt}{t} \, | \, dt \\
&\leq C(Y) \cdot exp(e^{\alpha|x|}) \cdot \int_{a-x}^{b-x} | \, D_x^p \phi(t+x) \frac{\sin Yt}{t} \, | \, dt \\
&\leq C(Y) \cdot exp(e^{\alpha|x|}) \cdot \sup_{a \leq t \leq b} D^p \phi(t) \cdot (b-a) \\
&< \epsilon,
\end{aligned}
$$

for $|x| > B$, where $B > 0$ is suitable constant. Hence (9) holds.

For (10), assuming $Y > 0$ and recalling $\int_{-\infty}^{\infty} \frac{\sin Yt}{t} dt = \pi$, we will prove that for $p \in N$,

$$
\begin{aligned}
&\max_{x \in R} | \, exp(e^{\alpha|x|}) \frac{1}{\pi} D_x^p \int_{-\infty}^{\infty} [\phi(t+x) - \phi(x)] \frac{\sin Yt}{t} dt \, | \\
&\to 0
\end{aligned}
\tag{12}
$$

as $Y \to +\infty$.

For any $\delta > 0$, (12) is less than or equal to

$$
\max_{x \in R} \frac{1}{\pi} \Big(\int_{-\infty}^{-\delta} + \int_{-\delta}^{\delta} + \int_{\delta}^{\infty} \Big) \, | \, exp(e^{\alpha|x|}) D_x^p [\phi(t+x) - \phi(x)] \frac{\sin Yt}{t} \, | \, dt.
$$

Let

$$
I_1 = \max_{x \in R} \frac{1}{\pi} \int_{-\infty}^{-\delta} | \, exp(e^{\alpha|x|}) D_x^p [\phi(t+x) - \phi(x)] \frac{\sin Yt}{t} \, | \, dt,
$$

$$
I_2 = \max_{x \in R} \frac{1}{\pi} \int_{-\delta}^{\delta} | \, exp(e^{\alpha|x|}) D_x^p [\phi(t+x) - \phi(x)] \frac{\sin Yt}{t} \, | \, dt,
$$

and

$$
I_3 = \max_{x \in R} \frac{1}{\pi} \int_{\delta}^{\infty} | \, exp(e^{\alpha|x|}) D_x^p [\phi(t+x) - \phi(x)] \frac{\sin Yt}{t} \, | \, dt.
$$

In order to estimate I_2, let

$$
H_{p,\phi}(x,t) = exp(e^{\alpha|x|}) \frac{1}{t} D_x^p [\phi(t+x) - \phi(x)]
$$

for $(x,t) \in R^2$ with $t \neq 0$, and let

$$H_{p,\phi}(x,t) = exp(e^{\alpha|x|})D_x^{p+1}\phi(x)$$

for $(x,t) \in R^2$ with $t = 0$.

Since $\lim_{t\to 0} \frac{1}{t} D_x^p[\phi(t+x) - \phi(x)] = D_x^{p+1}\phi(x)$, $H_{p,\phi}(x,t)$ is continuous on R^2. Moreover, since $supp\phi \subset [a,b]$, there exist $L = L(p,\phi)$ such that $|H_{p,\phi}(x,t)| \leq L$ for all $(x,t) \in R \times [-1,1]$. Now, given $\epsilon > 0$ choose δ with $0 \leq \delta \leq 1$ such that

$$I_2 = \max_{x\in R} \mid \frac{1}{\pi} \int_{-\delta}^{\delta} H_{p,\phi}(x,t) \sin Yt dt \mid \leq \frac{2L\delta}{\pi} < \epsilon.$$

In order to estimate I_1, consider that

$$I_1 \leq I_{11} + I_{12},$$

where

$$I_{11} = \max_{x\in R} \frac{1}{\pi} \int_{-\infty}^{-\delta} \mid exp(e^{\alpha|x|}) D_x^p \phi(t+x) \frac{\sin Yt}{t} \mid dt,$$

and

$$I_{12} = \max_{x\in R} \frac{1}{\pi} \int_{-\infty}^{-\delta} \mid exp(e^{\alpha|x|}) D_x^p \phi(x) \frac{\sin Yt}{t} \mid dt.$$

But, since $D^p\phi \in \mathcal{D}$,

$$I_{12} \leq \frac{1}{\pi} \cdot \sup_{a\leq x\leq b} |D^p\phi(x)| \cdot \mid \int_{-\infty}^{-Y\delta} \frac{\sin z}{z} dz \mid$$

tend to 0 as $Y \to +\infty$.

On the other hand, by an integration by parts,

$$I_{11} \leq I_{111} + I_{112},$$

where

$$I_{111} = \max_{x\in R} \mid \frac{\cos Y\delta}{\pi Y\delta} exp(e^{\alpha|x|}) D_x^p \phi(x-\delta) \mid$$

and

$$I_{112} = \max_{x\in R} \mid \frac{1}{\pi} \int_{-\infty}^{-\delta} exp(e^{\alpha|x|}) \cos(Yt) D_t[\frac{1}{t} D_x^p \phi(t+x)] dt \mid .$$

Since $exp(e^{\alpha|x|}) \leq 1$ for $\alpha < 0$,

$$I_{111} \leq \frac{1}{\pi Y \delta} \sup_{a \leq x \leq b} |D^p \phi|$$

tend to 0 as $Y \to +\infty$.

Now, since $D_t[\frac{1}{t} D_x^p \phi(t+x)] = (-\frac{1}{t^2}) D_x^p \phi(t+x) + \frac{1}{t} D_x^{p+1} \phi(t+x)$,

$$I_{112} \leq I_{1121} + I_{1122},$$

where

$$I_{1121} = \max_{x \in R} | \frac{1}{\pi} \int_{-\infty}^{-\delta} exp(e^{\alpha |x|}) \cos(Yt)(-\frac{1}{t^2}) D_x^p \phi(t+x) dt |,$$

and

$$I_{1122} = \max_{x \in R} | \frac{1}{\pi} \int_{-\infty}^{-\delta} exp(e^{\alpha |x|}) \cos(Yt) \frac{1}{t} D_x^{p+1} \phi(t+x) dt | .$$

Since $exp(e^{\alpha |x|}) \leq 1$ for $\alpha < 0$,

$$I_{1121} \leq \frac{1}{\pi Y \delta^2} \sup_{a \leq x \leq b} |D^p \phi(x)|$$

tend to 0 as $y \to +\infty$ and also,

$$I_{1122} \leq \frac{1}{\pi Y \delta} \sup_{a \leq x \leq b} |D^{p+1} \phi(x)|$$

tend to 0 as $y \to +\infty$. Hence $I_{11} \to 0$ for $p = 0, 1, 2, \cdots$ as $Y \to +\infty$, and analogously $I_3 \to 0$ as $Y \to +\infty$. Thus, (10) holds.

Now, note that, since $k < 0$, for any $(x_1, ..., x_n) \in R$,

$$\begin{aligned} exp(e^{k|x|}) &= exp(e^{k({x_1}^2 + {x_2}^2 + \cdots + {x_n}^2)^{\frac{1}{2}}}) \\ &\leq exp(e^{\frac{k}{\sqrt{n}}(|x_1| + |x_2| + \cdots + |x_n|)}) \\ &= exp(e^{\frac{k}{\sqrt{n}}|x_1|} \cdot e^{\frac{k}{\sqrt{n}}|x_2|} \cdots e^{\frac{k}{\sqrt{n}}|x_n|}) \\ &\leq exp(\frac{e^{\frac{k}{\sqrt{n}}|x_1|} + e^{\frac{k}{\sqrt{n}}|x_2|} + \cdots + e^{\frac{k}{\sqrt{n}}|x_n|}}{n}) \\ &\leq exp(\frac{1}{n} e^{\frac{k}{\sqrt{n}}|x_1|}) \cdot exp(\frac{1}{n} e^{\frac{k}{\sqrt{n}}|x_2|}) \cdots exp(\frac{1}{n} e^{\frac{k}{\sqrt{n}}|x_n|}) \end{aligned}$$

Consider

$$(13)\qquad \Big|\frac{exp(e^{k|x|})}{\pi^n}\partial^p\Big(\int_{R^n}\prod_{j=1}^{n}\phi_j(x_j+t_j)\frac{\sin Yt_j}{t_j}dt-\phi_1(x_1)\cdots\phi_n(x_n)\Big)\Big|$$

for $x=(x_1,x_2,...,x_n)\in R^n$ and $p=(p_1,p_2,...,p_n)\in N^n$. Writing, for $j=1,2,...,n$,

$$\Psi_{j,Y}(x_j)=\frac{1}{\pi}\int_{-\infty}^{\infty}\phi_j(x_j+t_j)\frac{\sin Yt_j}{t_j}dt,$$

it follows that (13) can be written as

$$\begin{aligned}
&exp(e^{k|x|})|[\partial_1^{p_1}\Psi_{1,Y}(x_1)\partial_2^{p_2}\Psi_{2,Y}(x_2)\cdots\partial_n^{p_n}\Psi_{n,Y}(x_n)\\
&\quad-\partial_1^{p_1}\phi_1(x_1)\partial_2^{p_2}\phi_2(x_2)\cdots\partial_n^{p_n}\phi_n(x^n)]|\\
&=exp(e^{k|x|})|[\partial_1^{p_1}\Psi_{1,Y}(x_1)\partial_2^{p_2}\Psi_{2,Y}(x_2)\cdots\partial_n^{p_n}\Psi_{n,Y}(x_n)\\
&\quad-\partial_1^{p_1}\phi_1(x_1)\partial_2^{p_2}\Psi_{2,Y}(x_2)\cdots\partial_n^{p_n}\Psi_{n,Y}(x_n)\\
&\quad+\partial_1^{p_1}\phi_1(x_1)\partial_2^{p_2}\Psi_{2,Y}(x_2)\cdots\partial_n^{p_n}\Psi_{n,Y}(x_n)\\
&\quad-\partial_1^{p_1}\phi_1(x_1)\partial_2^{p_2}\phi_2(x_2)\cdots\partial_n^{p_n}\Psi_{n,Y}(x_n)\\
&\quad+\partial_1^{p_1}\phi_1(x_1)\partial_2^{p_2}\phi_2(x_2)\cdots\partial_n^{p_n}\Psi_{n,Y}(x_n)\\
&\qquad\cdots\\
&\quad+\partial_1^{p_1}\phi_1(x_1)\partial_2^{p_2}\phi_2(x_2)\cdots\partial_{n-1}^{p_{n-1}}\phi_{n-1}(x_{n-1})\partial_n^{p_n}\Psi_{n,Y}(x_n)\\
&\quad-\partial_1^{p_1}\phi_1(x_1)\partial_2^{p_2}\phi_2(x_2)\cdots\partial_{n-1}^{p_{n-1}}\phi_{n-1}(x_{n-1})\partial_n^{p_n}\phi_n(x_n)]|\\
&\le|exp(\frac{1}{n}e^{\frac{k}{\sqrt{n}}|x_1|})\partial_1^{p_1}(\Psi_{1,Y}(x_1)-\phi_1(x_1))exp(\frac{1}{n}e^{\frac{k}{\sqrt{n}}|x_2|})\partial_2^{p_2}\Psi_{2,Y}(x_2)\\
&\quad\cdots exp(\frac{1}{n}e^{\frac{k}{\sqrt{n}}|x_n|})\partial_n^{p_n}\Psi_{n,Y}(x_n)|\\
&\quad+exp(\frac{1}{n}e^{\frac{k}{\sqrt{n}}|x_1|})\partial_1^{p_1}\phi_1(x_1)exp(\frac{1}{n}e^{\frac{k}{\sqrt{n}}|x_2|})\partial_2^{p_2}(\Psi_{2,Y}(x_2)-\phi_2(x_2))\\
&\quad\cdots exp(\frac{1}{n}e^{\frac{k}{\sqrt{n}}|x_n|})\partial_n^{p_n}\Psi_{n,Y}(x_n)|\\
&\quad+\cdots+|exp(\frac{1}{n}e^{\frac{k}{\sqrt{n}}|x_1|})\partial_1^{p_1}\phi_1(x_1)exp(\frac{1}{n}e^{\frac{k}{\sqrt{n}}|x_2|})\partial_2^{p_2}\phi_2(x_2)|\\
&\quad\cdots|exp(\frac{1}{n}e^{\frac{k}{\sqrt{n}}|x_{n-1}|})\partial_{n-1}^{p_{n-1}}\phi_{n-1}(x_{n-1})exp(\frac{1}{n}e^{\frac{k}{\sqrt{n}}|x_n|})\\
&\quad\times\partial_n^{p_n}(\Psi_{n,Y}(x_n)-\phi_n(x_n))|
\end{aligned}$$

By (10) and taking $\alpha=\frac{k}{\sqrt{n}}<0$, it follows that

$$(14)\qquad \max_{x_j\in R}exp(\frac{1}{n}e^{\frac{k}{\sqrt{n}}|x_j|})|\partial_j^{p_j}(\Psi_{j,Y}(x_j)-\phi_j(x_j))|\to 0,$$

as $Y \to +\infty$, for $1 \le j \le n$. Also, for $1 \le j \le n$,

$$\max_{x_j \in R} exp(\frac{1}{n} e^{\frac{k}{\sqrt{n}}|x_j|}) |\partial_j^{p_j}(\Psi_{j,Y}(x_j))|$$
$$\le \max_{x_j \in R} exp(\frac{1}{n} e^{\frac{k}{\sqrt{n}}|x_j|}) \partial_j^{p_j} |(\Psi_{j,Y}(x_j) - \phi_j(x_j))|$$
$$+ \max_{x_j \in R} exp(\frac{1}{n} e^{\frac{k}{\sqrt{n}}|x_j|}) |\partial_j^{p_j}(\phi_j(x_j))|$$

Since $\phi_j \in \mathcal{D}(R^n)$, there exists a $Q_j > 0$ such that

$$\max_{x_j \in R} exp(\frac{1}{n} e^{\frac{k}{\sqrt{n}}|x_j|}) |\partial_j^{p_j}(\phi_j(x_j))| \le Q_j.$$

Taking into account (14), there exists a $P_j > 0, 1 \le j \le n$, such that

$$\max_{x_j \in R} exp(\frac{1}{n} e^{\frac{k}{\sqrt{n}}|x_j|}) |\partial_j^{p_j}(\Psi_{j,Y}(x_j))| \le P_j,$$

for any $Y \ge 0$, and so,

$$q_{k,p}(\Psi_{1,Y}(x_1)\Psi_{2,Y}(x_2) \cdots \Psi_{n,Y}(x_n) - \phi_1(x_1)\phi_2(x_2) \cdots \phi_n(x_n)$$
$$\le \max_{x_1 \in R} exp(\frac{1}{n} e^{\frac{k}{\sqrt{n}}|x_1|}) |\partial_1^{p_j}(\Psi_{1,Y}(x_1) - \phi_1(x_1))| \cdot P_1 \cdots P_n$$
$$+ Q_1 \cdot \max_{x_2 \in R} exp(\frac{1}{n} e^{\frac{k}{\sqrt{n}}|x_2|}) |\partial_2^{p_2}(\Psi_{2,Y}(x_2) - \phi_2(x_2))| \cdot P_3 \cdots P_n$$
$$+ \cdots Q_1 \cdots Q_{n-1} \cdot \max_{x_n \in R} exp(\frac{1}{n} e^{\frac{k}{\sqrt{n}}|x_n|}) |\partial_n^{p_n}(\Psi_{n,Y}(x_n) - \phi_n(x_n))|$$

By using (14), we obtains the result. □

Theorem 3.3. *Let $f \in \mathcal{K}'_{e,k}, k \in Z, k < 0$, and set by $F(y) = (\mathcal{F}f)(y), y \in R^n$. Then for any $\phi_1, \phi_2, ..., \phi_n \in \mathcal{D}(R), t = (t_1, t_2, ..., t_n) \in R^n$, and $\phi(t) = \phi_1(t_1)\phi_2(t_2) \cdots \phi_n(t_n)$, one has*

$$< f(t), g(t) > = \lim_{Y \to +\infty} < \frac{1}{(2\pi)^n} \int_{C(0;Y)} F(y) e^{-ity} dy, \phi(t) > .$$

Here we mean by $C(0;Y)$ the n-tube $[-Y,Y] \times [-Y,Y] \times \cdots \times [-Y,Y]$.

Proof. If we only replace $(1+|x|^2)^k$ by $exp(e^{k|x|})$ in the proof of Lemma 2.2 in [2], we obtained that if $\phi \in \mathcal{K}_e$ and $f \in \mathcal{K}'_{e,k}, k \in Z, k < 0$, for any $Y > 0$,

$$\int_{C(0;Y)} < f(x), e^{ixy}\phi(y)dy =< f(x), \int_{C(0;Y)} \phi(y) e^{ixy} dy > .$$

Then, by applying Fubini's theorem and Lemma 3.2,

$$
\begin{aligned}
&< \frac{1}{(2\pi)^n}\int_{C(0;Y)} F(y)e^{-ity}dy, \phi(t) > \\
&\quad =< \frac{1}{(2\pi)^n}\int_{C(0;Y)} < f(x), e^{ixy} > e^{-ity}dy, \phi(t) > \\
&\quad = \frac{1}{(2\pi)^n}\int_{C(0;Y)} < f(x), e^{ixy} > dy \int_{R^n} \phi(t)e^{-ity}dt \\
&\quad =< f(x), \frac{1}{(2\pi)^n}\int_{C(0;Y)} e^{ixy}dy \int_{R^n} \phi(t)e^{-ity}dt > \\
&\quad =< f(x), \frac{1}{(2\pi)^n}\int_{R^n} \phi_1(t_1)\cdots\phi_n(t_n)dt \int_{C(0;Y)} e^{i(x-t)y}dy > \\
&\quad =< f(x), \frac{1}{\pi^n}\int_{R^n} \prod_{j=1}^{n} \phi_j(x_j+t_j)\frac{\sin Yt_j}{t_j}dt > \\
&\quad \longrightarrow < f(x), \phi(x) >
\end{aligned}
$$

as $Y \to +\infty$. □

Let $f, g \in \mathcal{K}'_{e,k}, k \in Z, k < 0$ and $F(y) = G(y)$, for any $y \in R^n$, where $F(y) = (\mathcal{F}f)(y)$, and $G(y) = (\mathcal{F}g)(y)$. Then, using the Theorem 3.3, we have

$$< f(x), \phi_1(x_1)\phi_2(x_2)\cdots\phi_n(x_n) > = < g(x), \phi_1(x_1)\phi_2(x_2)\cdots\phi_n(x_n) >,$$

for all $\phi_1, \phi_2, \cdots \phi_n \in \mathcal{D}(R)$. Let $\phi \in \mathcal{D}(R^n)$, by [4, Proposition 1, p.369], there exists a sequence whose terms are products of the form $\phi_{i_1}\phi_{i_2}\cdots\phi_{i_n}$, being $\phi_{i_j} \in \mathcal{D}(R)$, for $j = 1, 2, ..., n$ and $i_j \in N$, which converges to $\phi \in \mathcal{D}(R^n)$. Since convergence in $\mathcal{D}$ implies convergence in $\mathcal{K}_{e,k}$, it follows that $< f, \phi >=< g, \phi >$ for any $\phi \in \mathcal{D}(R^n)$. Since $\mathcal{D}$ is dense in $\mathcal{K}_{e,k}$, it follows that $f = g$ in $\mathcal{K}'_{e,k}$. Also, for all $y \in R^n$,

$$(\mathcal{F}(f \circledast g))(y) = < (f \circledast g)(x), e^{ixy} > = < f(t), < g(x), e^{iy(x+t)} >>$$

$$= < f(t), e^{ity} >< g(x), e^{ixy} > = F(y) \cdot G(y).$$

Hence it follows that for $f, g, h \in \mathcal{K}'_{e,k}, k \in Z, k < 0$,

$$f \circledast g = g \circledast f$$

and

$$f \circledast (g \circledast h) = (f \circledast g) \circledast h$$

in $\mathcal{K}'_{e,k}$. Furthermore the Dirac delta belongs to $\mathcal{K}'_{e,k}$ and

$$f \circledast \delta = \delta \circledast f = f.$$

This shows that $\mathcal{K}'_{e,k}, k \in Z, k < 0$ is a commutative convolution algebra with unit element with respect to $\circledast$ defined in (1).

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Recieved 5 August, 2004