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Weyl Normal Connections of Weyl Submanifolds

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Abstract

We study the Weyl normal connections of Weyl submanifolds. We show that if the Weyl normal connection is flat, then the induced 1-form of a Weyl submanifold is closed and the normal connection is also flat. Next, we investigate a compact Weyl submanifold of an Einstein-Weyl manifold with flat Weyl normal connection.

1. Introduction

Let M^n be a manifold with a conformal structure $[g]$ and a torsion-free affine connection D. A triplet $(M^n,[g],D)$ is called a Weyl manifold if $Dg=\omega\otimes g$ for a 1-form ω. A Weyl manifold $(M^n,[g],D)(n\geqq 3)$ is said to be Einstein-Weyl if the symmetrized Ricci tensor of D is proportional to a representative metric g in $[g]$. A compact Weyl manifold has a unique, up to homothety, metric g in the conformal class such that the 1-form ω is co-closed. We call this metric the Gauduchon metric. If furthermore the manifold is Einstein-Weyl, then the corresponding vector field $\omega^\sharp$ is Killing [11]. Compact Einstein-Weyl manifolds $(M^n,[g],D)(n\geqq 3)$ with closed 1-form ω are classified by Gauduchon in [5]. In [9], Pedersen, Poon and Swann studied Weyl submanifolds of Weyl manifolds. In the previous paper [8], we investigated Weyl space forms and their Weyl submanifolds.

In this paper, we shall study the Weyl normal connections of Weyl submanifolds. Let $(M^n,[g],D)$ be a Weyl submanifold of a Weyl manifold $(\bar{M}^m,[\bar{g}],\bar{D})$ and $\bar{\omega}$ and ω be the corresponding 1-forms of $\bar{g}$ and g respectively. In Section 3, we give the relation between the curvatures of the Weyl normal connection and the normal connection. We show that if the Weyl normal connection is flat, then the induced 1-form ω is closed and the normal connection is also flat. For a hypersurface (M^n,g) isometrically immersed in a Riemannian manifold $(\bar{M}^{n+1},\bar{g})$ the normal connection is flat, but for a Weyl hypersurface $(M^n,[g],D)$ of a Weyl manifold $(\bar{M}^{n+1},[\bar{g}],\bar{D})$ the Weyl normal connection is not necessarily flat. In co-dimension one, we show that the Weyl normal connection is flat if and only if the induced 1-form ω is closed. Next, we show that for a Weyl submanifold of a Weyl manifold with vanishing Weyl conformal curvature, the Weyl normal connection is flat if and only if the induced 1-form ω is closed and all of the Weyl second fundamental forms are simultaneously diagonalizable. Let $s^{\bar{D}}$ be the scalar curvature of $\bar{D}$ with respect to $\bar{g}$. For a compact Weyl submanifold of an Einstein-Weyl manifold with flat Weyl normal connection, we get the following result.

Theorem. *Let $(\bar{M}^m, [\bar{g}], \bar{D})$ be an Einstein-Weyl manifold with vanishing Weyl conformal curvature, and let $(M^n, [g], D)$ be a compact Weyl submanifold of $(\bar{M}^m, [\bar{g}], \bar{D})$ with flat Weyl normal connection ($m > n \geqq 2$). Suppose that the induced 1-form ω of g is not exact. Then we have*
(i) *If $s^{\bar{D}} > 0$ on M^n, then $(M^n, [g], D)$ is not Weyl totally umbilical.*
(ii) *If $s^{\bar{D}} \equiv 0$ on M^n and $(M^n, [g], D)$ is Weyl totally umbilical, then $(M^n, [g], D)$ is Weyl totally geodesic and Weyl flat.*

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2. Weyl manifolds

Let $(M^n, [g], D)$ be a Weyl manifold with $Dg = \omega \otimes g$. Let ∇ be the Levi-Civita connection of g. We define a vector field $B = \omega^\sharp$ by $g(X, B) = \omega(X)$. Then

$$D_X Y = \nabla_X Y - \frac{1}{2}\omega(X)Y - \frac{1}{2}\omega(Y)X + \frac{1}{2}g(X,Y)B, \tag{1}$$

where ∇ denotes the Levi-Civita connection of g.

The curvature tensor R of ∇ is defined by $R(X,Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z$. Let R^D be the curvature tensor of D. Then we have

$$\begin{aligned} R^D(X,Y)Z = {} & R(X,Y)Z - \frac{1}{2}\{[(\nabla_X\omega)Z + \frac{1}{2}\omega(X)\omega(Z)]Y \\ & - [(\nabla_Y\omega)Z + \frac{1}{2}\omega(Y)\omega(Z)]X + ((\nabla_X\omega)Y)Z - ((\nabla_Y\omega)X)Z \\ & - g(Y,Z)(\nabla_X B + \frac{1}{2}\omega(X)B) + g(X,Z)(\nabla_Y B + \frac{1}{2}\omega(Y)B)\} \\ & - \frac{1}{4}|\omega|^2(g(Y,Z)X - g(X,Z)Y), \end{aligned} \tag{2}$$

where X, Y and Z are any vector fields on M^n (cf. [6]).

We set $R^D(V,Z,X,Y) = g(R^D(X,Y)Z,V)$. Let Ric and s_g be the Ricci tensor and the scalar curvature of ∇ respectively, and $S(\nabla\omega)$ be the symmetric part of $\nabla\omega$.

For two 2−tensors p and l, we set

$$(p \owedge l)(V,Z,X,Y) = p(V,X)l(Z,Y) + p(Z,Y)l(V,X) - p(V,Y)l(Z,X) - p(Z,X)l(V,Y).$$

Then, from (2) we have the following decomposition into irreducible components

$$R^D = W + [\frac{1}{n-2}Ric_0 + \frac{1}{2}S_0(\nabla\omega) + \frac{1}{4}\omega \otimes_0 \omega] \owedge g + [\frac{1}{2n(n-1)}s_g - \frac{n-2}{8n}|\omega|^2 - \frac{1}{2n}d^*\omega]g \owedge g - [\frac{1}{2}(d\omega \owedge g) + g \otimes d\omega], \tag{3}$$

where W is the Weyl conformal curvature and $Ric_0, S_0, \otimes_0$ are trace-free parts and $2d\omega(X,Y) = X\omega(Y) - Y\omega(X) - \omega([X,Y])$ (cf. [9], [8]). There is no Weyl component W, when $n = 3$.

A Weyl manifold $(M^n, [g], D)$ is called Weyl flat if $R^D = 0$.

Let Ric^D be the Ricci tensor of D. Using (2), we obtain

$$Ric^D(X,Y) = Ric(X,Y) + \frac{1}{2}(n-1)(\nabla_X\omega)Y - \frac{1}{2}(\nabla_Y\omega)X + \frac{1}{4}(n-2)\omega(X)\omega(Y) + [\frac{1}{2}\mathrm{div}B - \frac{1}{4}(n-2)|\omega|^2]g(X,Y). \tag{4}$$

The conformal scalar curvature $s^D = tr_g Ric^D$ is the scalar curvature of D with respect to g, whose sign is conformal invariant, and satisfies

$$s^D = s_g - \frac{(n-1)(n-2)}{4}|\omega|^2 - (n-1)d^*\omega.$$

A Weyl manifold $(M^n, [g], D)(n \geq 3)$ is said to have an Einstein-Weyl structure if there exists a function $\tilde{\Lambda}$ on M^n such that

$$Ric^D(X,Y) + Ric^D(Y,X) = \tilde{\Lambda}g(X,Y). \tag{5}$$

From (3), we have the following

Lemma 1. *Let $(M^n, [g], D)$ be an Einstein-Weyl manifold with vanishing Weyl conformal curvature. Then we have*

$$R^D(V,Z,X,Y) = h[g(Y,Z)g(X,V) - g(X,Z)g(Y,V)] - [\frac{1}{2}(d\omega \owedge g) + g \otimes d\omega](V,Z,X,Y),$$

where $h = \frac{1}{n(n-1)}s^D$.

3. Weyl submanifolds

Let $(\bar{M}^m, [\bar{g}], \bar{D})$be a Weyl manifold with $\bar{D}\bar{g} = \bar{\omega}\otimes\bar{g}$ and $i : M^n \to \bar{M}^m$ an immersed submanifold. A torsion-free connection D on M is given by $D_X Y = \pi(\bar{D}_X Y)$, where π is the orthogonal projection from $i^*T\bar{M}^m$ to TM^n and X, Y are vector fields on M^n. Since $\bar{D}\bar{g} = \bar{\omega}\otimes\bar{g}$, we obtain $Dg = \omega\otimes g$, where $g = i^*\bar{g}$ and $\omega = i^*\bar{\omega}$. The second fundamental form β of the Weyl structure is defined by

$$\bar{D}_X Y = D_X Y + \beta(X, Y). \tag{6}$$

Let $\bar{B}$ be the vector field dual to $\bar{\omega}$. The vector field B dual to ω satisfies the decomposition $\bar{B} = B + B^\perp$, where $B^\perp$ is normal component with respect to M^n. Let α be the second fundamental form of the isometric immersion $i : (M^n, g) \to (\bar{M}^m, \bar{g})$, i.e.,

$$\bar{\nabla}_X Y = \nabla_X Y + \alpha(X, Y), \tag{7}$$

where $\bar{\nabla}$ and ∇ are the Levi-Civita connections of $\bar{g}$ and g respectively. From (1), (6) and (7), we obtain $\beta = \alpha + \frac{1}{2} g \otimes B^\perp$. Let ξ be a normal vector field on M^n and X be a tangent vector field on M^n. We have the Weingarten equations

$$\bar{D}_X \xi = -A^\beta_\xi X + D^N_X \xi, \tag{8}$$

$$\bar{\nabla}_X \xi = -A^\alpha_\xi X + \nabla^N_X \xi, \tag{9}$$

where $-A^\beta_\xi X$ and $-A^\alpha_\xi X$ are the tangential components and $D^N_X\xi$ and $\nabla^N_X\xi$ are the normal components of $\bar{D}_X\xi$ and $\bar{\nabla}_X\xi$ respectively. D^N is called the Weyl normal connection and A^β_ξ the Weyl second fundamental form. From $\bar{D}\bar{g} = \bar{\omega}\otimes\bar{g}$ and (8), we get $\bar{g}(\beta(X,Y), \xi) = g(A^\beta_\xi X, Y)$ and $D^N g^\perp = \omega\otimes g^\perp$, where $g^\perp$ is the induced metric of the normal bundle $T(M^n)^\perp$.

The mean curvature vector H^α of M^n is defined to be $H^\alpha = \frac{1}{n} tr_g \alpha$, $n = \dim M^n$. Since $\beta = \alpha + \frac{1}{2} g \otimes B^\perp$, the corresponding mean curvature vectors are related by $H^\beta = H^\alpha + \frac{1}{2}B^\perp$.

A Weyl submanifold $(M^n, [g], D)$ is said to be Weyl totally geodesic if $\beta = 0$. For a normal section ξ on $(M^n, [g], D)$, if $A^\beta_\xi = \lambda I$ for some function λ, then ξ is called a Weyl umbilical section on $(M^n, [g], D)$. If the Weyl submanifold $(M^n, [g], D)$ is Weyl umbilical with respect to every local normal section of $(M^n, [g], D)$, then $(M^n, [g], D)$ is said to be Weyl totally umbilical. A Weyl submanifold $(M^n, [g], D)$ is said to be Weyl minimal if $H^\beta = 0$. These notions are conformally invariant(cf. [9]).

Let $(M^n, [g], D)$ be a Weyl submanifold of a Weyl manifold $(\bar{M}^m, [\bar{g}], \bar{D})$. Let $R^{\bar{D}}$ and R^D be curvature tensors of $\bar{D}$ and D respectively. Let $\bar{\omega}$ and ω be the corresponding 1-forms of $\bar{g}$ and g respectively. Then we have the equation of Gauss

$$
\begin{aligned}
(10) \qquad R^{\bar{D}}(V,Z,X,Y) = R^D(V,Z,X,Y) &+ \bar{g}(\beta(X,Z),\beta(Y,V)) \\
&- \bar{g}(\beta(Y,Z),\beta(X,V)).
\end{aligned}
$$

From (1), (8) and (9), we have

$$-A^{\beta}_{\xi}X + D^N_X\xi = -A^{\alpha}_{\xi}X + \nabla^N_X\xi - \frac{1}{2}\bar{\omega}(X)\xi - \frac{1}{2}\bar{\omega}(\xi)X.$$

Thus we have

$$(11) \qquad A^{\beta}_{\xi}X = A^{\alpha}_{\xi}X + \frac{1}{2}\bar{\omega}(\xi)X, \qquad D^N_X\xi = \nabla^N_X\xi - \frac{1}{2}\bar{\omega}(X)\xi.$$

Using (11), we give a relation between the curvatures of D^N and ∇^N. We define the curvature $R^{D\perp}$ of the Weyl normal connection D^N by

$$R^{D\perp}(X,Y)\xi = D^N_X D^N_Y\xi - D^N_Y D^N_X\xi - D^N_{[X,Y]}\xi,$$

where X, Y are any tangent vector fields and ξ is any normal vector field on M^n. Then we have

$$
\begin{aligned}
R^{D\perp}(X,Y)\xi &= \nabla^N_X\nabla^N_Y\xi - \nabla^N_Y\nabla^N_X\xi - \nabla^N_{[X,Y]}\xi \\
&\quad - \frac{1}{2}(X\bar{\omega}(Y))\xi + \frac{1}{2}(Y\bar{\omega}(X))\xi + \frac{1}{2}\bar{\omega}([X,Y])\xi \\
&= R^{\perp}(X,Y)\xi - d\omega(X,Y)\xi.
\end{aligned}
$$

Therefore we have

Lemma 2. *Let $(M^n, [g], D)$ be a Weyl submanifold of a Weyl manifold $(\bar{M}^m, [\bar{g}], \bar{D})$ and $R^{D\perp}$ and $R^{\perp}$ be the curvatures of D^N and ∇^N respectively. Then we have*

$$R^{D\perp}(X,Y)\xi = R^{\perp}(X,Y)\xi - d\omega(X,Y)\xi.$$

Theorem 1. *Let $(M^n, [g], D)$ be a Weyl submanifold of a Weyl manifold $(\bar{M}^m, [\bar{g}], \bar{D})$. If the Weyl normal connection D^N is flat, then the induced 1-form ω is closed and the normal connection ∇^N is also flat. In particular, when $(M^n, [g], D)$ is a Weyl hypersurface of a Weyl manifold $(\bar{M}^{n+1}, [\bar{g}], \bar{D})$, the Weyl normal connection D^N is flat if and only if the induced 1-form ω is closed.*

Proof. Since ∇^N is a metric connection in the normal bundle $T(M^n)^\perp$ with respect to the induced metric $g^\perp$ on $T(M^n)^\perp$, we have

$$R^\perp(\eta,\xi,X,Y)+R^\perp(\xi,\eta,X,Y)=0, \tag{12}$$

where X,Y are tangent vector fields and ξ,η normal vector fields on M^n. Thus, from Lemma 2, we obtain

$$R^{D\perp}(\eta,\xi,X,Y)+R^{D\perp}(\xi,\eta,X,Y)=-2d\omega(X,Y)\bar{g}(\xi,\eta). \tag{13}$$

Thus if the Weyl normal connection D^N is flat, then the induced 1-form ω is closed. Using Lemma 2 again, we obtain that the normal connection ∇^N is flat.

Next, we assume that $(M^n,[g],D)$ is a Weyl hypersurface of a Weyl manifold $(\bar{M}^{n+1},[\bar{g}],\bar{D})$. From (12), we obtain $(R^\perp(X,Y)\xi)^\perp=0$. Lemma 2 implies that $R^{D\perp}(X,Y)\xi=-d\omega(X,Y)\xi$. Therefore we get that the Weyl normal connection D^N is flat if and only if the induced 1-form ω is closed. ∎

We recall the equation of Ricci

$$R^{\bar{D}}(\eta,\xi,X,Y)=R^{D\perp}(\eta,\xi,X,Y)+g([A^\beta_\eta,A^\beta_\xi]X,Y), \tag{14}$$

where X,Y are tangent vector fields and ξ,η normal vector fields on M^n(cf. [3]). Let $(\bar{M}^m,[\bar{g}],\bar{D})$ be a Weyl manifold with vanishing Weyl conformal curvature. From (3), we obtain

$$R^{\bar{D}}(\eta,\xi,X,Y)=-d\omega(X,Y)\bar{g}(\xi,\eta).$$

So we have

$$R^{D\perp}(\eta,\xi,X,Y)=-d\omega(X,Y)\bar{g}(\xi,\eta)+g([A^\beta_\xi,A^\beta_\eta]X,Y). \tag{15}$$

Therefore, from Theorem 1 and (15), we have the following result.

Theorem 2. *Let $(M^n,[g],D)$ be a Weyl submanifold of a Weyl manifold $(\bar{M}^m,[\bar{g}],\bar{D})$ with vanishing Weyl conformal curvature. Then the Weyl normal connection D^N is flat if and only if the induced 1-form ω is closed and all of the Weyl second fundamental forms A^β_ξ are simultaneously diagonalizable.*

Corollary 1. *Let $(M^n,[g],D)$ be a Weyl totally umbilical submanifold of a Weyl manifold $(\bar{M}^m,[\bar{g}],\bar{D})$ with vanishing Weyl conformal curvature. Then the Weyl normal connection D^N is flat if and only if the induced 1-form ω is closed.*

Remark 1. Let (M^n, g) be a submanifold isometrically immersed in a Riemannian manifold $(\bar{M}^m, \bar{g})$. For a Riemannian manifold $(\bar{M}^m, \bar{g})$, we have

$$\bar{R} = \bar{W} + \frac{1}{m-2}[\bar{R}ic - \frac{1}{m}s_{\bar{g}}\bar{g}] \owedge \bar{g} + \frac{1}{2m(m-1)}s_{\bar{g}}\bar{g} \owedge \bar{g}. \tag{16}$$

We assume that the Weyl conformal curvature $\bar{W}$ is zero. From the equation of Ricci and (16), we have

$$R^{\perp}(\eta, \xi, X, Y) = g([A^{\alpha}_{\xi}, A^{\alpha}_{\eta}]X, Y). \tag{17}$$

Thus the normal connection ∇^N is flat if and only if all of the second fundamental forms A^{α}_{ξ} are simultaneously diagonalizable.

We study a compact Weyl submanifold of an Einstein-Weyl manifold with flat Weyl normal connection.

Theorem 3. *Let $(\bar{M}^m, [\bar{g}], \bar{D})$ be an Einstein-Weyl manifold with vanishing Weyl conformal curvature, and let $(M^n, [g], D)$ be a compact Weyl submanifold of $(\bar{M}^m, [\bar{g}], \bar{D})$ with flat Weyl normal connection $(m > n \geq 2)$. Suppose that the induced 1-form ω of g is not exact. Then we have*
(i) *If $s^{\bar{D}} > 0$ on M^n, then $(M^n, [g], D)$ is not Weyl totally umbilical.*
(ii) *If $s^{\bar{D}} \equiv 0$ on M^n and $(M^n, [g], D)$ is Weyl totally umbilical, then $(M^n, [g], D)$ is Weyl totally geodesic and Weyl flat.*

Proof. We assume that $(M^n, [g], D)$ is a Weyl totally umbilical submanifold. Then we have $\beta(X, Y) = g(X, Y)H^{\beta}$. From Lemma 1 and (10), we have

$$Ric^D(X, Y) = (n-1)(h + |H^{\beta}|^2)g(X, Y) + \frac{n}{2}d\omega(X, Y), \tag{18}$$

where $h = \frac{1}{m(m-1)}s_{\bar{g}} - \frac{m-2}{4m}|\bar{\omega}|^2 - \frac{1}{m}d^*\bar{\omega}$.

Since the Weyl normal connection is flat, $d\omega = 0$. Thus, from the Weitzenböck formula we have

$$Ric(B, B) = \mathrm{div}(\nabla_B B) + (\mathrm{divB})^2 - |\nabla B|^2. \tag{19}$$

Since $(M^n, [g], D)$ is a compact Weyl manifold, if $n \geq 3$ we have the Gauduchon metric g such that $\mathrm{div}B \equiv 0$. When $n = 2$, a compact Weyl manifold has a unique, up to homothety, a metric g in the conformal class such that $\mathrm{div}B \equiv 0$ (cf. [2]).

We use a metric such that $\mathrm{div}B \equiv 0$. From (19), we get

$$Ric(B, B) = \mathrm{div}(\nabla_B B) - |\nabla B|^2. \tag{20}$$

From (4) and (18), we have

$$\text{(21)}\ Ric(X,Y) = \{(n-1)(h+|H^\beta|^2) + \frac{n-2}{4}|\omega|^2\}g(X,Y) - \frac{n-2}{4}\omega(X)\omega(Y)$$
$$-\frac{1}{2}(n-1)(\nabla_X\omega)Y + \frac{1}{2}(\nabla_Y\omega)X.$$

In the case where $n \geqq 3$, it follows from (18) that $(M^n, [g], D)$ is an Einstein-Weyl manifold. For a compact Einstein-Weyl manifold $(M^n, [g], D)$, since $d\omega = 0$, we get $\nabla B \equiv 0$ with respect to the Gauduchon metric g. Thus, from (21) we have

$$\text{(22)}\ Ric(X,Y) = \{(n-1)(h+|H^\beta|^2) + \frac{n-2}{4}|\omega|^2\}g(X,Y) - \frac{n-2}{4}\omega(X)\omega(Y).$$

In the case where $n = 2$, since $d\omega = 0$ from (21) we obtain

$$\text{(23)}\qquad Ric(X,Y) = (h+|H^\beta|^2)g(X,Y) - \frac{1}{2}(\nabla_X\omega)Y + \frac{1}{2}(\nabla_Y\omega)X$$
$$= (h+|H^\beta|^2)g(X,Y).$$

Thus, from (20), (22) and (23), we obtain

$$\text{(24)}\qquad -\int_{M^n}|\nabla\omega|^2 dV_g = (n-1)\int_{M^n}|\omega|^2(h+|H^\beta|^2)dV_g,$$

where dV_g denotes the volume element with respect to g.

Now, we suppose that $s^{\bar{D}} > 0$. Then $h > 0$. This is a contradiction because the induced 1-form ω is not exact.

Next, we assume that $s^{\bar{D}} \equiv 0$ on M^n and $(M^n, [g], D)$ is a Weyl totally umbilical submanifold. Since $s^{\bar{D}} = 0$, from (24) we obtain $H^\beta \equiv 0$, i.e., $(M^n, [g], D)$ is Weyl totally geodesic. Since $h = 0$ and $d\omega = 0$, from Lemma 1 and (10), $(M^n, [g], D)$ is Weyl flat. ∎

Remark 2. (I). It is known that for an Einstein-Weyl manifold $(\bar{M}^m, [\bar{g}], \bar{D})$ with vanishing Weyl conformal curvature, if $m \geqq 4$, then $\bar{\omega}$ is closed(cf. [4]). In three dimensions, the Weyl conformal curvature is zero, but the 1-form $\bar{\omega}$ of an Einstein-Weyl manifold $(\bar{M}^3, [\bar{g}], \bar{D})$ is not necessarily closed (cf. [8]).

In the above theorem, we don't need the assumption that the Weyl normal connection is flat when $n = \dim M \geqq 3$.

(II). Let $(\bar{M}^m, \bar{g}, \bar{D})(m \geqq 3)$ be a compact Einstein-Weyl manifold with the Gauduchon metric $\bar{g}$ which has vanishing Weyl conformal curvature. If the 1-form $\bar{\omega}$ is non-zero, then we obtain $s^{\bar{D}} \geqq 0$. In particular, $s^{\bar{D}}$ is constant sign for compact Einstein-Weyl three-manifolds (cf. [10], [8]).

Examples.
(I). Let $(\phi, V, \eta, \bar{g})$ be a Sasakian structure of a Sasakian manifold $\bar{M}^{2n+1}$ with constant ϕ-sectional curvature $k \geqq 1$. Then $\bar{Ric}(X,Y) = \beta\bar{g}(X,Y) + \gamma\eta(X)\eta(Y)$, where $\beta = \frac{n+1}{2}k + \frac{3n-1}{2}$ and $\gamma = -\frac{n+1}{2}(k-1)$. We define a 1-form $\bar{\omega}$ by $\bar{\omega} = f\eta$, where $f^2 = -\frac{4}{2n-1}\gamma$. We set

$$\bar{D}_X Y = \bar{\nabla}_X Y - \frac{1}{2}\bar{\omega}(X)Y - \frac{1}{2}\bar{\omega}(Y)X + \frac{1}{2}\bar{g}(X,Y)\bar{\omega}^\sharp,$$

where $\bar{\nabla}$ denotes the Levi-Civita connection of $\bar{g}$. Then $(\bar{M}^{2n+1}, \bar{g}, \bar{D})$ is an Einstein-Weyl manifold with $s^{\bar{D}} = 2n(2n+1) > 0$ (cf. [7]).

Let M^2 be an anti-invariant submanifold of a Sasakian manifold $\bar{M}^3$ with constant ϕ-sectional curvature $k \geqq 1$. Since $\bar{g}(X, \phi Y) = d\eta(X,Y)$, for tangent vector fields X, Y on M^2, $d\eta(X,Y) = 0$ (cf. [12]). Thus the induced 1-form ω of a Weyl hypersurface (M^2, g, D) of the above Einstein-Weyl manifold $(\bar{M}^3, \bar{g}, \bar{D})$ is closed. From Theorem 1, the Weyl normal connection is flat. Since M^2 is an anti-invariant submanifold of $\bar{M}^3$, the structure vector field V is tangent to M^2. Thus we have $\phi X = -\bar{\nabla}_X V = -\alpha(X,V) = -\beta(X,V) + \frac{1}{2}g(X,V)B^\perp$, where X is any tangent vector field on M^2. We suppose that a Weyl hypersurface (M^2, g, D) of $(\bar{M}^3, \bar{g}, \bar{D})$ is Weyl totally umbilical. Since $\beta(X,V) = g(X,V)H^\beta$, we have $0 = \phi V = -\beta(V,V) + \frac{1}{2}g(V,V)B^\perp$, i.e., $H^\beta = \frac{1}{2}B^\perp$. This contradicts to the fact that $\phi X = -\beta(X,V) + \frac{1}{2}g(X,V)B^\perp \neq 0$. Therefore a Weyl hypersurface (M^2, g, D) of $(\bar{M}^3, \bar{g}, \bar{D})$ is not Weyl totally umbilical.
(II). $S^1 \times S^{n-1}$ is a Weyl minimal submanifold of an Einstein-Weyl manifold S^{n+1} but not Weyl totally umbilical. The Weyl normal connection of $S^1 \times S^{n-1}$ is flat and $S^1 \times S^{n-1}$ is tangent to the vector field $\bar{\omega}^\sharp$(cf. [9]).
(III). The Weyl curvature $R^{\bar{D}}$ of $S^1 \times S^n$ is Weyl flat. $S^1 \times S^{n-1}$ is a Weyl totally geodesic submanifold of $S^1 \times S^n$ with flat Weyl normal connection and $S^1 \times S^{n-1}$ is tangent to the vector field $\bar{\omega}^\sharp$(cf. [9]).

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