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SOME SPECTRAL PROPERTIES OF ANALYTIC ELEMENTARY OPERATORS

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Abstract. In this paper, we shall deal with analytic elementary operators on $\mathcal{L}(\mathfrak{X})$. On the approximate point and defect spectra of them, we shall give a parallel result to the Lumer-Rosenblum's spectral theorem. Moreover, we shall give a sufficient condition for that the spectrum and the right (or left) spectrum of an analytic elementary operator on $\mathcal{L}(\mathfrak{H})$ coinside.

1. Introduction

Let $\mathcal{L}(\mathfrak{X})$ be the Banach algebra of all bounded linear operators on a complex Banach space $\mathfrak{X}$. It has been a problem of essential importance to study the structures of elementary operators acting on $\mathcal{L}(\mathfrak{X})$. An elementary operator $\Phi_{\mathbf{A},\mathbf{B}}$ is defined by $\Phi_{\mathbf{A},\mathbf{B}}(X) = A_1XB_1 + \cdots + A_nXB_n$ for all $X \in \mathcal{L}(\mathfrak{X})$, where n is a natural number, $\mathbf{A} = (A_1, \ldots, A_n)$ and $\mathbf{B} = (B_1, \ldots, B_n)$ are both n-tuples of mutually commuting operators in $\mathcal{L}(\mathfrak{X})$. $\Phi_{\mathbf{A},\mathbf{B}}$ is a bounded linear operator on $\mathcal{L}(\mathfrak{X})$. Elementary operators have already been studied from various viewpoints. In particular, the spectral properties of them have been researched deeply by many authors and been analyzed in great detail. (Especially, in the case where $\mathfrak{X}$ is a Hilbert space.) In order to state some of those results, we first recall the notations and the terminologies.

For an operator T acting on a Banach space, $\sigma(T)$ denotes the spectrum of T, the set of all complex numbers λ such that $\lambda - T$ is not invertible. $\sigma_l(T)$ is the *left spectrum* of T, the set of all complex numbers λ such that $\lambda - T$ is not left-invertible. The *right spectrum* $\sigma_r(T)$ is also defined analogously. Clearly, $\sigma(T) = \sigma_l(T) \cup \sigma_r(T)$. Following [2], $\sigma_\pi(T)$ denotes the *approximate point spectrum* of T, the set of all complex numbers λ such that $\lambda - T$ is not bounded below, and $\sigma_\delta(T)$ is the *approximate defect spectrum* of T, the set of all complex numbers λ such that $\lambda - T$ is not surjective. It also follows that $\sigma(T) = \sigma_\pi(T) \cup \sigma_\delta(T)$. Note that $\sigma_\pi(T) = \sigma_l(T)$ and $\sigma_\delta(T) = \sigma_r(T)$ hold for Hilbert space operators, but these relations do not hold in general. We have also to prepare the joint spectrum of $\mathbf{T} = (T_1, \ldots, T_n) \in \mathcal{L}(\mathfrak{X})^{(n)}$. The *left joint spectrum* of $\mathbf{T}$ is defined by

$$\sigma_l(\mathbf{T}) = \left\{ \lambda \in \mathbb{C}^n \;\middle|\; \text{There is no solution } \mathbf{X} \in \mathcal{L}(\mathfrak{X})^{(n)} \text{ to } \sum_{j=1}^{n} X_j(\lambda_j - T_j) = I \right\}.$$

(I stands for the identity operator.) The *right joint spectrum* $\sigma_r(\mathbf{T})$ is defined analogously.

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In the following statements, $\mathfrak{H}$ denotes a Hilbert space. For $\sigma, \tau \subset \mathbb{C}^n$, $\sigma \circ \tau$ is the subset $\{\alpha \circ \beta = \sum_{j=1}^n \alpha_j \beta_j \mid \alpha = (\alpha_1, \ldots, \alpha_n) \in \sigma,\ \beta = (\beta_1, \ldots, \beta_n) \in \tau\}$ of $\mathbb{C}$. The spectrum of an elementary operator $\Phi_{\mathbf{A},\mathbf{B}}$ on $\mathcal{L}(\mathfrak{H})$ was completely determined by R. Harte [7].

Theorem A-1 (Harte [7, Theorem 3.5]**).**

The left and right spectra of $\Phi_{\mathbf{A},\mathbf{B}}$ *on* $\mathcal{L}(\mathfrak{H})$ *are given by*

$$\sigma_l(\Phi_{\mathbf{A},\mathbf{B}}) = \sigma_l(\mathbf{A}) \circ \sigma_r(\mathbf{B}) \tag{1}$$

and

$$\sigma_r(\Phi_{\mathbf{A},\mathbf{B}}) = \sigma_r(\mathbf{A}) \circ \sigma_l(\mathbf{B}). \tag{2}$$

Thus

$$\sigma(\Phi_{\mathbf{A},\mathbf{B}}) = \sigma_l(\mathbf{A}) \circ \sigma_r(\mathbf{B}) \cup \sigma_r(\mathbf{A}) \circ \sigma_l(\mathbf{B}). \tag{3}$$

Subsequently, L. A. Fialkow [5] characterized the approximate point and defect spectra of $\Phi_{\mathbf{A},\mathbf{B}}$ on $\mathcal{L}(\mathfrak{H})$.

Theorem A-2 (Fialkow [5, §2, Theorems 2.3,2.8]**).**

The approximate point (resp. defect) spectrum and the left (resp. right) spectrum of $\Phi_{\mathbf{A},\mathbf{B}}$ on $\mathcal{L}(\mathfrak{H})$ *coinside.*

$$\sigma_\pi(\Phi_{\mathbf{A},\mathbf{B}}) = \sigma_l(\Phi_{\mathbf{A},\mathbf{B}}) \tag{4}$$

and

$$\sigma_\delta(\Phi_{\mathbf{A},\mathbf{B}}) = \sigma_r(\Phi_{\mathbf{A},\mathbf{B}}). \tag{5}$$

In the present note, we shall deal with analytic elementary operators on $\mathcal{L}(\mathfrak{X})$. ($\mathfrak{X}$ is an arbitrary Banach space.) Let A and B be in $\mathcal{L}(\mathfrak{X})$ and let $f_1, \ldots, f_n$ (resp. $g_1, \ldots, g_n$) be in $\mathcal{A}(\sigma(A))$ (resp. $\mathcal{A}(\sigma(B))$). Here $\mathcal{A}(\sigma(T))$ denotes the set of all complex-valued functions analytic in a neighborhood of the spectrum $\sigma(T)$ of $T \in \mathcal{L}(\mathfrak{X})$. An analytic elementary operator Ψ on $\mathcal{L}(\mathfrak{X})$ is defined by

$$\Psi(X) = \sum_{j=1}^{n} f_j(A) X g_j(B) \quad (X \in \mathcal{L}(\mathfrak{X})). \tag{6}$$

Needless to say, Ψ is an elementary operator on $\mathcal{L}(\mathfrak{X})$. (The terminology "*analytic* elementary operator" was used by Fialkow [6].) For the spectrum of Ψ on $\mathcal{L}(\mathfrak{X})$, there exists a different characterization from Theorem A-1. Indeed, G. Lumer and M. Rosenblum had obtained the following spectral theorem for Ψ.

Theorem A-3 (Lumer-Rosenblum [8, §V, Theorem 10]**).**

The spectrum of Ψ *on* $\mathcal{L}(\mathfrak{X})$ *is given by*

$$\sigma(\Psi) = \left\{ \sum_{j=1}^{n} f_j(\alpha) g_j(\beta) \;\middle|\; \alpha \in \sigma(A), \beta \in \sigma(B) \right\}. \tag{7}$$

The formula (7) claims that the equation $f_1(A)Xg_1(B) + \cdots + f_n(A)Xg_n(B) = Y$ has a unique solution X for each Y if and only if the complex-valued function H of two variables of the form $H(z,w) = f_1(z)g_1(w) + \cdots + f_n(z)g_n(w)$ has no zero on the Cartesian product $\sigma(A) \times \sigma(B)$. In this paper, for the approximate point and defect spectra of Ψ, we shall give the following parallel result to Theorem A-3.

Moreover, in the case where $\mathfrak{X}$ is a Hilbert space, we shall give a sufficient condition for that $\sigma(\Psi)$ and $\sigma_r(\Psi)$ (or $\sigma(\Psi)$ and $\sigma_l(\Psi)$) coinside.

Theorem 1. *The approximate point and defect spectra of Ψ on $\mathcal{L}(\mathfrak{X})$ satisfy*

$$\sigma_\pi(\Psi) \supseteq \left\{ \sum_{j=1}^{n} f_j(\alpha)g_j(\beta) \;\middle|\; \alpha \in \sigma_\pi(A), \beta \in \sigma_\delta(B) \right\} \tag{8}$$

and

$$\sigma_\delta(\Psi) \supseteq \left\{ \sum_{j=1}^{n} f_j(\alpha)g_j(\beta) \;\middle|\; \alpha \in \sigma_\delta(A), \beta \in \sigma_\pi(B) \right\}. \tag{9}$$

2. Main Result

In terms of applications, the most important elementary operator is $\delta_{A,B}$ on $\mathcal{L}(\mathfrak{X})$ defined by $\delta_{A,B}(X) = AX - XB$, where A and B are two fixed operators in $\mathcal{L}(\mathfrak{X})$. $\delta_{A,B}$ is often called a *generalized derivation*, and it is an analytic elementary operator. The approximate point and defect spectra of $\delta_{A,B}$ on $\mathcal{L}(\mathfrak{X})$ have very simple structures. The next result is due to C. Davis and P. Rosenthal.

Theorem B-1 (Davis-Rosenthal [2, §4, Theorems 3-5]**).**

The approximate point and defect spectra of $\delta_{A,B}$ on $\mathcal{L}(\mathfrak{X})$ satisfy

$$\sigma_\pi(\delta_{A,B}) = \sigma_\pi(A) - \sigma_\delta(B) \; (= \{\alpha - \beta \mid \alpha \in \sigma_\pi(A), \beta \in \sigma_\delta(B)\}) \tag{10}$$

and

$$\sigma_\delta(\delta_{A,B}) \supseteq \sigma_\delta(A) - \sigma_\pi(B) \; (= \{\alpha - \beta \mid \alpha \in \sigma_\delta(A), \beta \in \sigma_\pi(B)\}). \tag{11}$$

Moreover, if $\mathfrak{X}$ is a Hilbert space, then the inclusion (11) *can be replaced by* "=".

Remark 1. In general, the inclusion (11) can be proper. That was already pointed out in [2].

Our Theorem 1 asserts that Theorem B-1 can be partly extended to analytic elementary operators on $\mathcal{L}(\mathfrak{X})$. (See the previous section for the definition.) Before showing our result, we prepare some auxiliary lemmas.

The following well-known duality was already used in the proof of Theorem B-1 in [2].

Lemma 1. *For all $T \in \mathcal{L}(\mathfrak{X})$,*

$$\sigma_\pi(T^\dagger) = \sigma_\delta(T) \tag{12}$$

and

$$\sigma_\delta(T^\dagger) = \sigma_\pi(T), \tag{13}$$

where $T^\dagger$ denotes the Banach space adjoint of T.

The next claim may also be well-known, but we dare to give here a brief proof of it.

Lemma 2. *If $T \in \mathcal{L}(\mathfrak{X})$ and $\|(\lambda - T)x_k\| \to 0$ for a sequence of unit vectors $\{x_k\}$, then $\|(f(\lambda) - f(T))x_k\| \to 0$ for all $f \in \mathcal{A}(\sigma(T))$.*

proof. From the definition of the functional calculus, $f(T) = (1/2\pi i)\int_\Gamma f(z)(z-T)^{-1}dz$ for a suitable system of curves Γ surrounding $\sigma(T)$. Since $\lambda \in \sigma_\pi(T) \subseteq \sigma(T)$, $f(\lambda) = (1/2\pi i)\int_\Gamma f(z)(z-\lambda)^{-1}dz$ by the Cauchy formula. Therefore

$$\begin{aligned} f(\lambda) - f(T) &= (1/2\pi i)\Big(\int_\Gamma f(z)(z-\lambda)^{-1}dz - \int_\Gamma f(z)(z-T)^{-1}dz\Big) \\ &= (1/2\pi i)\int_\Gamma f(z)(z-\lambda)^{-1}(\lambda - T)(z-T)^{-1}dz \\ &= \Big\{(1/2\pi i)\int_\Gamma f(z)(z-\lambda)^{-1}(z-T)^{-1}dz\Big\}(\lambda - T). \end{aligned}$$

Setting $S = (1/2\pi i)\int_\Gamma f(z)(z-\lambda)^{-1}(z-T)^{-1}dz$ $(\in \mathcal{L}(\mathfrak{X}))$, $\|(f(\lambda) - f(T))x_k\| = \|S(\lambda - T)x_k\| \leq \|S\| \cdot \|(\lambda - T)x_k\| \to 0$. □

Particularly, from Lemmas 1 and 2, we obtain

$$f(\sigma_\pi(T)) \subseteq \sigma_\pi(f(T)) \ \ \text{and} \ \ f(\sigma_\delta(T)) \subseteq \sigma_\delta(f(T)) \tag{14}$$

for all $T \in \mathcal{L}(\mathfrak{X})$ and for all $f \in \mathcal{A}(\sigma(T))$.

Now, we can show the following main theorem. This is a partial extension of the Davis-Rosenthal's Theorem B-1, and also a parallel result to the Lumer-Rosenblum's Theorem A-3 in the previous section. The main idea of the following proof is based on the proofs of [2, Theorems 4,5] and [5, Lemma 2.6].

Theorem 1. *The approximate point and defect spectra of Ψ on $\mathcal{L}(\mathfrak{X})$ satisfy*

$$\sigma_\pi(\Psi) \supseteq \left\{\sum_{j=1}^n f_j(\alpha)g_j(\beta) \;\middle|\; \alpha \in \sigma_\pi(A), \beta \in \sigma_\delta(B)\right\} \tag{15}$$

and

$$\sigma_\delta(\Psi) \supseteq \left\{\sum_{j=1}^n f_j(\alpha)g_j(\beta) \;\middle|\; \alpha \in \sigma_\delta(A), \beta \in \sigma_\pi(B)\right\}. \tag{16}$$

proof. First, we shall prove (15). Let $\alpha \in \sigma_\pi(A)$ and let $\beta \in \sigma_\delta(B)$. Then there exists a sequence of unit vectors $\{x_k\}$ such that $\|(\alpha - A)x_k\| \to 0$ and also there exists a sequence of norm 1 functionals $\{\rho_k\}$ such that $\|(\beta - B^\dagger)\rho_k\| \to 0$ (from Lemma 1). Consider the sequence of norm 1 operators $\{x_k \otimes \rho_k\}$. (Here, for a vector x and a continuous functional ρ, $x \otimes \rho$ denotes the rank 1 operator defined by $(x \otimes \rho)y = \rho(y)x$ for all $y \in \mathfrak{X}$. $x \otimes \rho$ has the norm $\|x \otimes \rho\| = \|x\| \cdot \|\rho\|$.) Set $\mu = \sum_{j=1}^n f_j(\alpha)g_j(\beta)$. We can show $\|(\mu - \Psi)(x_k \otimes \rho_k)\| \to 0$. Indeed, by an easy calculation, it follows that

$$\begin{aligned} \|(\mu - \Psi)(x_k \otimes \rho_k)\| &= \left\| \sum_{j=1}^n (f_j(\alpha)x_k) \otimes (g_j(\beta)\rho_k) - \sum_{j=1}^n (f_j(A)x_k) \otimes (g_j(B^\dagger)\rho_k) \right\| \\ &\leq \sum_{j=1}^n |g_j(\beta)| \cdot \|(f_j(\alpha) - f_j(A))x_k\| \\ &\quad + \sum_{j=1}^n \|f_j(A)\| \cdot \|(g_j(\beta) - g_j(B^\dagger))\rho_k\|. \end{aligned}$$

Since each term of the right-hand side of the above inequality converges to 0 by Lemma 2, we conclude that $\mu \in \sigma_\pi(\Psi)$. Thus the inclusion (15) was proved.

Next, to prove (16), let $\alpha \in \sigma_\delta(A)$ and let $\beta \in \sigma_\pi(B)$, and set $\mu = \sum_{j=1}^n f_j(\alpha)g_j(\beta)$. If we suppose that $\mu - \Psi$ is surjective, then we reach a contradiction. Indeed, if $\mu - \Psi$ is surjective, then there exists an $M > 0$ such that

$$\|(\mu - \Psi)(X)\| \geq M \cdot \inf\{\|Y - X\| \mid Y \in \ker(\mu - \Psi)\} \text{ for all } X \in \mathcal{L}(\mathcal{E}). \tag{17}$$

On the other hand, from the definition of the functional calculus,

$$f_j(A) = (1/2\pi i) \int_{\Gamma_j} f_j(z)(z - A)^{-1} dz \tag{18}$$

where Γ_j is a system of curves surrounding $\sigma(A)$ for $j = 1, \ldots, n$. Let

$$M_j = \max_{z \in \Gamma_j} |f_j(z)(z - \alpha)^{-1}| \cdot \|(z - A)^{-1}\| \tag{19}$$

for $j = 1, \ldots, n$ and set

$$\epsilon = \frac{2\pi M}{8n \cdot \max_{1 \leq j \leq n}(M_j + 1) \cdot \max_{1 \leq j \leq n}(\|g_j(B)\| + 1) \cdot \max_{1 \leq j \leq n}(l(\Gamma_j) + 1)}.$$

(Here, $l(\Gamma_j)$ denotes the length of Γ_j.) By the assumption (and by Lemma 2), there exists a norm 1 functional ρ such that $\|(\alpha - A^\dagger)\rho\| < \epsilon$. For this ρ,

$$\|(f_j(\alpha) - f_j(A^\dagger))\rho\| < \frac{M}{8n \cdot (\|g_j(B)\| + 1)} \quad (j = 1, \ldots, n). \tag{20}$$

By the same way, we can find a unit vector x such that

$$\|(g_j(\beta) - g_j(B))x\| < \frac{M}{8n \cdot (|f_j(\alpha)| + 1)} \quad (j = 1, \ldots, n). \tag{21}$$

For these x and ρ, we can find a norm 1 operator $C \in \mathcal{L}(\mathfrak{X})$ such that $|\langle Cx, \rho\rangle| > 1/2$. Then, by the assumption that $\mu - \Psi$ is surjective and by (17), there exists an $X \in \mathcal{L}(\mathcal{E})$ such that

$$C = (\mu - \Psi)(X), \ \|X\| \leq 2/M. \tag{22}$$

Then

$$
\begin{aligned}
1/2 &< |\langle Cx, \rho\rangle| \\
&= \Big|\Big\langle \Big[\Big(\sum_{j=1}^{n} f_j(\alpha)g_j(\beta)\Big)X - \sum_{j=1}^{n} f_j(A)Xg_j(B)\Big]x, \rho\Big\rangle\Big| \\
&\leq \sum_{j=1}^{n} |\langle (f_j(\alpha)g_j(\beta)X - f_j(A)Xg_j(B))x, \rho\rangle| \\
&\leq \sum_{j=1}^{n} |\langle f_j(\alpha)X(g_j(\beta) - g_j(B))x, \rho\rangle| \\
&\quad + \sum_{j=1}^{n} |\langle Xg_j(B)x, (f_j(\alpha) - f_j(A^\dagger))\rho\rangle| \\
&\leq \sum_{j=1}^{n} |f_j(\alpha)| \cdot \|X\| \cdot \|(g_j(\beta) - g_j(B))x\| \\
&\quad + \sum_{j=1}^{n} \|(f_j(\alpha) - f_j(A^\dagger))\rho\| \cdot \|g_j(B)\| \cdot \|X\| \\
&< 1/2
\end{aligned}
$$

because of (20),(21),(22). But $1/2 < 1/2$ is an evident contradiction. Thus $\mu - \Psi$ fails to be surjective, that is, $\mu \in \sigma_\delta(\Psi)$ and hence the inclusion (16) holds. □

Remark 2. It is regrettable that we do not know whether the inclusion (15) can be replaced by "=" or not. (On (16), the equality does not hold in general even though $\Psi = \delta_{A,B}$ because of Remark 1.) Also, we have not proved the equality of (16) in the case where $\mathfrak{X}$ is a Hilbert space.

3. Applications for Ψ on $\mathcal{L}(\mathfrak{H})$

In this section, we shall give a sufficient condition for that the spectrum and the right (or left) spectrum of an analytic elementary operator Ψ coinside. We shall deal with only the case where $\mathfrak{X}$ is a Hilbert space $\mathfrak{H}$ throughout this section. For $T \in \mathcal{L}(\mathfrak{H})$, $T^* \in \mathcal{L}(\mathfrak{H})$ denotes the Hilbert space adjoint of T.

Definition 1. *$T \in \mathcal{L}(\mathfrak{H})$ is said to be satisfying the condition* **(R)** *if $\sigma(T) = \sigma_r(T) = \sigma_\delta(T)$.*

Since $\mathfrak{H}$ is a Hilbert space, the second equality above is valid for an arbitrary $T \in \mathcal{L}(\mathfrak{H})$. The essential equality of the condition **(R)** is the first one. Clearly, if T^* satisfies the condition **(R)**, then $\sigma(T) = \sigma_l(T) = \sigma_\pi(T)$.

The operator T is called ***hyponormal*** if $T^*T \geq TT^*$. It is well-known that every hyponormal operator satisfies the condition **(R)**. Let $C_2(\mathfrak{H})$ be the two sided ideal of all Hilbert-Schmidt class operators in $\mathcal{L}(\mathfrak{H})$. $C_2(\mathfrak{H})$ is a Hilbert space with respect to the inner product $\langle S, T\rangle = \mathrm{tr}(T^*S)$. According to S. Y. Shaw [10, §3], if $A, B \in \mathcal{L}(\mathfrak{H})$ and A, B^* are both hyponormal, then $\delta_{A,B}|_{C_2(\mathfrak{H})}$ is also a hyponormal operator. Thus $\delta_{A,B}|_{C_2(\mathfrak{H})}$ satisfies the condition **(R)**, that is, $\sigma(\delta_{A,B}|_{C_2(\mathfrak{H})}) = \sigma_r(\delta_{A,B}|_{C_2(\mathfrak{H})})$. Moreover, according to [3, Theorem 3.20] and [4, Theorem 3.2], $\sigma(\delta_{A,B}) = \sigma(\delta_{A,B}|_{C_2(\mathfrak{H})})$ and $\sigma_r(\delta_{A,B}) = \sigma_r(\delta_{A,B}|_{C_2(\mathfrak{H})})$. Thus, if A and B^* are both hyponormal, then the spectrum and the right spectrum of $\delta_{A,B}$ (on $\mathcal{L}(\mathfrak{H})$)

coinside. We shall show an extension of the above result as a corollary of Theorem 1.

Corollary 1. *If both A and B^* satisfy the condition* **(R)**, *then the spectrum and the right spectrum of Ψ coinside.*

$$\sigma(\Psi) = \sigma_\delta(\Psi) = \sigma_r(\Psi). \tag{23}$$

proof. We have to show the first equality. (The second one is a part of the Fialkow's Theorem A-2.) We can verify it in the following way.

$$\begin{aligned} \sigma_\delta(\Psi) &\supseteq \left\{ \sum_{j=1}^{n} f_j(\alpha) g_j(\beta) \;\middle|\; \alpha \in \sigma_\delta(A), \beta \in \sigma_\pi(B) \right\} & (24)\\ &= \left\{ \sum_{j=1}^{n} f_j(\alpha) g_j(\beta) \;\middle|\; \alpha \in \sigma_r(A), \beta \in \sigma_l(B) \right\} & (25)\\ &= \left\{ \sum_{j=1}^{n} f_j(\alpha) g_j(\beta) \;\middle|\; \alpha \in \sigma(A), \beta \in \sigma(B) \right\} & (26)\\ &= \sigma(\Psi). & (27) \end{aligned}$$

(24) follows from (16) in Theorem 1, (25) follows from the assumption that $\mathfrak{H}$ is a Hilbert space, (26) follows from the assumption that A and B^* satisfy the condition **(R)** and finally, (27) is the Lumer-Rosenblum's Theorem A-3. Thus $\sigma(\Psi) = \sigma_\delta(\Psi)$ follows because $\sigma_\delta(\Psi)$ is always the subset of $\sigma(\Psi)$. □

Analogously, we can also show the next symmetric result.

Corollary 2. *If both A^* and B satisfy the condition* **(R)**, *then the spectrum and the left spectrum of Ψ coinside.*

$$\sigma(\Psi) = \sigma_\pi(\Psi) = \sigma_l(\Psi). \tag{28}$$

In the rest of this paper, we shall show that many classes of Hilbert space operators satisfy the condition **(R)**.

Definition 2. *$T \in \mathcal{L}(\mathfrak{H})$ is said to be satisfying the condition* **(N1)** *if $\bar{\lambda} \in \sigma_\pi(T^*)$ whenever $\lambda \in \sigma_\pi(T)$.*

Definition 3. *For $T \in \mathcal{L}(\mathfrak{H})$, a complex number λ is said to be belonging to $\sigma_{na}(T)$ if there exists a sequence of unit vectors $\{x_k\}$ such that $\|(\lambda - T)x_k\| \to 0$* and $\|(\lambda - T)^* x_k\| \to 0$ *simultaneously. $\sigma_{na}(T)$ is called the normal approximate point spectrum of T. $T \in \mathcal{L}(\mathfrak{H})$ is said to be satisfying the condition* **(N2)** *if $\sigma_\pi(T) = \sigma_{na}(T)$.*

Remark 3. If T satisfies the condition **(N2)**, then T satisfies also the condition **(N1)**. The relation $\sigma_\pi(T) \supseteq \sigma_{na}(T)$ is valid for an arbitrary operator $T \in \mathcal{L}(\mathfrak{H})$. But there exists an operator S such that $\sigma_{na}(S) = \emptyset$. On the contrary, $\sigma_\pi(T)$ is non-void for every T because $\sigma(T)$ has the (non-empty) boundary $\partial\sigma(T)$ and $\partial\sigma(T) \subseteq \sigma_\pi(T)$.

Proposition 1. *If T satisfies the condition* **(N1)**, *then T satisfies the condition* **(R)**.

proof. We have only to verify that $\sigma_\pi(T) \subseteq \sigma_\delta(T)$ whenever T satisfies the condition **(N1)**. Suppose that $\lambda \in \sigma_\pi(T)$. Then $\bar{\lambda} \in \sigma_\pi(T^*) = \sigma_l(T^*)$. Thus $\lambda \in \sigma_r(T) = \sigma_\delta(T)$. □

Example 1. $T \in \mathcal{L}(\mathfrak{H})$ is called *dominant* if $(\lambda - T)\mathfrak{H} \subseteq (\lambda - T)^*\mathfrak{H}$ for all $\lambda \in \mathbb{C}$. The dominant operator T satisfies the condition **(N2)** because of the following well-known fact. T is dominant if and only if there exists a $M_\lambda > 0$ for all $\lambda \in \mathbb{C}$ such that $M_\lambda \|(\lambda - T)x\| \geq \|(\lambda - T)^*x\|$ for all $x \in \mathfrak{H}$. The class of dominant operators is very large. (It is clear that every hyponormal operator is dominant.) See [11] for more informations for dominant operators.

Example 2. p-hyponormal operators and log-hyponormal operators satisfy the condition **(N2)**. $T \in \mathcal{L}(\mathfrak{H})$ is called *p-hyponormal* if $(T^*T)^p \geq (TT^*)^p$ for $0 < p < \infty$. M. Chō and T. Huruya showed that every p-hyponormal operator satisfies the condition **(N2)** in [1]. $T \in \mathcal{L}(\mathfrak{H})$ is called log-*hyponormal* if T is invertible and $\log(T^*T) \geq \log(TT^*)$. In [12], K. Tanahashi has shown that every log-hyponormal operator also satisfies the condition **(N2)**.

Example 3. Recently, S. M. Patel introduced a new class of Hilbert space operators in [9]. An operator $T \in \mathcal{L}(\mathfrak{H})$ is called *quasi-isometry* if $T^{*2}T^2 = T^*T$. For example, isometries and idempotents on $\mathfrak{H}$ enjoy this relation. Patel showed that every quasi-isometry satisfies the condition **(N1)**.

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