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A REMARK OF THE NUMERICAL RANGES OF OPERATORS ON HILBERT SPACES

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ABSTRACT. The usual numerical ranges of an operator a acting on a Hilbert space $\mathcal{H}$ is defined as $W(a) = \{(a\xi|\xi); \xi \in \mathcal{H}, \ \|\xi\| = 1\}$. This numerical range depends on the Hilbert space. In this paper, we consider a von Neumann algebra containing a and introduce another notion of numerical range for a. Under this consideration, we shall show that the numerical range of every operator does not depend on the Hilbert space in a sense.

We deal with bounded linear operators on a separable Hilbert space with inner product $(\,|\,)$. Let $\mathcal{B}(\mathcal{H})$ be the von Neumann algebra consisting of all bounded operators on $\mathcal{H}$.

The numerical range of an operator a is the subset of the complex numbers $\mathbb{C}$, given by

$$W(a) = \{(a\xi|\xi); \xi \in \mathcal{H}, \ \|\xi\| = 1\}.$$

Then $W(a)$ is a convex set (for example see, [6, p.314]).

Many authors gave the properties of numerical ranges of the operators concerned with the spectrum $\sigma(a)$ and the norm $\|a\|$. We know many results for these properties.

Furthermore, Berberian and Orland [1] introduce the following notion by which they gave a result on the closure of the numerical range of an operator: Let a be an operator on $\mathcal{H}$ and $\mathcal{A}$ a C^*-algebra containing a and the identity operator 1 on $\mathcal{H}$. Let write $S(\mathcal{A})$ for the set of all states of $\mathcal{A}$, that is the set of all linear functional on $\mathcal{A}$ such that $\phi(1) = 1$ and $\phi(x^*x) \geq 0$ for all $x \in \mathcal{A}$; then $S(\mathcal{A})$ is a convex subset of the dual space of $\mathcal{A}$, and is compact in the weak*-topology. Let $U(a) = \{\phi(a); \phi \in S(\mathcal{A})\}$, then $U(a)$ is a compact and convex subset of $\mathbb{C}$.

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And Berberian and Orland gave a characterization of the closure $\overline{W(a)}$ of $W(a)$ by showing $U(a) = \overline{W(a)}$.

By considering the notion of numerical range of every operator a introduced by Berberian and Orland and von Neumann algebra containing a, we shall introduce the following another notion of numerical range for an operator a.

Let a be an operator on $\mathcal{H}$ and $\mathcal{M}$ a von Neumann algebra containing a and 1. Let $NS(\mathcal{M})$ denote the set of all normal states of $\mathcal{M}$ and define $V(a) = \{\phi(a); \phi \in NS(\mathcal{M}).\}$ Then, since $NS(\mathcal{M})$ is a convex set in the dual space $\mathcal{M}^*$ of $\mathcal{M}$, $V(a)$ is a convex subset of $\mathbb{C}$ and has the properties as the usual numerical range;

$$V(\alpha 1 + \beta a) = \alpha + \beta V(a) \text{ for } \alpha, \beta \in \mathbb{C},$$
$$V(a^*) = \{\overline{\lambda}; \lambda \in V(a)\},$$
$$V(u^* a u) = V(a) \text{ for every unitary } u \in \mathcal{B}(\mathcal{H}).$$

Furthermore, the linear functional (so called vector states) ω_ξ $(\xi \in \mathcal{H}, \|\xi\| = 1)$ defined by

$$\omega_\xi(x) = (x\xi|\xi) \text{ for all } x \in \mathcal{M}$$

is a normal state of $\mathcal{M}$, thus $W(a) \subset V(a)$. And also, we have the relation

$$W(a) \subset V(a) \subset U(a).$$

Let $\mathcal{M}$ be a von Neumann algebra acting on $\mathcal{H}$ and $\mathcal{N}$ a von Neumann subalgebra of $\mathcal{M}$, then $NS(\mathcal{N}) = \{\phi|_{\mathcal{N}}; \phi \in NS(\mathcal{M})\}$. If b is an element of $\mathcal{N}$, then $V_{\mathcal{M}}(b) = V_{\mathcal{N}}(b)$. Furthermore, we have the following fundamental properties.

Let $\mathcal{M}$ be a von Neumann algebra containing a and 1. If $\mathcal{M}$ has a cyclic and separating vector, then any normal state is a vector state (for example, see [4, III, Chapter 1, Theorem 4]) and so $W(a) = V(a)$. Furthermore, if the Hilbert space $\mathcal{H}$ on which a is acting is finite dimensional, then $W(a)$ is a compact subset (for example see [5; Theorem 5.1-1]) and so $W(a) = V(a) = U(a)$.

By considering the definition for $V(a)$, we have the following property.

Proposition 1. *Let a be an operator on $\mathcal{H}$. Let $\mathcal{H}^\infty$ be the Hilbert space of direct summand $\mathcal{H} \oplus \mathcal{H} \oplus \mathcal{H} \oplus \cdots$ and (a_n) the operator on $\mathcal{H}^\infty$ with $a_n = a$ for every n. Then $V(a) = W((a_n))$.*

Proof. Let $\mathcal{M}$ be a von Neumann algebra generated by a and 1. Let λ be an element of $V(a)$, then there exists an element ϕ of $NS(\mathcal{M})$ with $\phi(a) = \lambda$. Put $\phi = \sum_{n=1}^{\infty} \omega_{\xi_n}$ with $\phi(1) = \sum_{n=1}^{\infty} \|\xi_n\|^2 = 1$ that ξ_n is not zero vector. Furthermore, put $\alpha_n = \|\xi_n\|$ and $\eta_n = \frac{1}{\alpha_n}\xi_n$ for $n = 1, 2, \cdots$, then $\|\eta_n\| = 1$ and

$$\phi(a) = \sum_{n=1}^{\infty} (a\xi_n|\xi_n) = \sum_{n=1}^{\infty} (a(\alpha_n\eta_n)|\alpha_n\eta_n) = \lambda.$$

Let $\eta = (\alpha_n\eta_n)$ in $\mathcal{H}^\infty$, then $\|\eta\| = 1$ and

$$\omega_\eta((a_n)) = ((a_n)\eta|\eta) = \sum_{n=1}^{\infty} (a(\alpha_n\eta_n)|\alpha_n\eta_n) = \phi(a) = \lambda.$$

Thus λ is an element of $W((a_n))$.

On the other hand, let λ be an element of $W((a_n))$, then there exists an element $\zeta = (\zeta_n)$ of $\mathcal{H}^\infty$ with $\omega_\zeta((a_n)) = \lambda$ and $\|\zeta\| = \sqrt{\sum_{n=1}^{\infty} \|\zeta_n\|^2} = 1$. Define a normal state φ of $\mathcal{M}$ with $\varphi = \sum_{n=1}^{\infty} \omega_{\zeta_n}$, then

$$\varphi(a) = \sum_{n=1}^{\infty} (a\zeta_n|\zeta_n) = \omega_\zeta((a_n)) = \lambda.$$

Thus λ is an element of $V(a)$. Therefore, $V(a) = W((a_n))$. □

Furthermore, we can show the relation $W(a) = V(a)$. For the proof of this fact, we have the following considerations.

Lemma 2. *Let $\mathcal{M}$ be a von Neumann algebra containing an operator a. If $\{\lambda_n\}_{n=1}^{\infty}$ be a sequence of element in $V(a)$ and $\{\alpha_n\}_{n=1}^{\infty}$ a sequence of non-negative numbers with $\sum_{n=1}^{\infty}\alpha_n = 1$, then $\sum_{n=1}^{\infty}\alpha_n\lambda_n$ is an element of $V(a)$. In particular, $V(a) = \left\{\sum_{j=1}^{\infty}\alpha_j\lambda_j; \lambda_j \in W(a), \alpha_j \geq 0, \sum_{j=1}^{\infty}\alpha_j = 1\right\}$.*

Proof. Since $\{\lambda_n\}_{n=1}^{\infty}$ is a sequence of elements in $V(a)$, there exists a normal state ϕ_n with $\phi_n(a) = \lambda_n$ for every n. Then, we have the relation;

$$\sum_{n=1}^{\infty}\|\alpha_n\phi_n\| = \sum_{n=1}^{\infty}\alpha_n\|\phi_n\| = \sum_{n=1}^{\infty}\alpha_n = 1.$$

Since $NS(\mathcal{M})$ is a Banach space, $\sum_{n=1}^{\infty}\alpha_n\phi_n$ converges to an element of $NS(\mathcal{M})$. Put $\phi = \sum_{n=1}^{\infty}\alpha_n\phi_n$, then $\phi \in NS(\mathcal{M})$ and

$$\phi(a) = \sum_{n=1}^{\infty}\alpha_n\phi_n(a) = \sum_{n=1}^{\infty}\alpha_n\lambda_n$$

is an element of $V(a)$. $\square$

By considering the properties appeared in Lemma 2, we shall give a property for the bounded convex sets in the complex plane $\mathbb{C}$.

Lemma 3. *Let A be a bounded convex subset of the complex number plane $\mathbb{C}$, then we have the following relation*

$$A = \left\{\sum_{j=1}^{\infty}a_j\lambda_j; \lambda_j \in A, a_j \geq 0, \sum_{j=1}^{\infty}a_j = 1\right\}.$$

Proof. Let $B = \left\{\sum_{j=1}^{\infty}a_j\lambda_j; \lambda_j \in A, a_j \geq 0, \sum_{j=1}^{\infty}a_j = 1\right\}$, then B is contained the closure $\overline{A}$ of A. We suppose that $B \setminus A \neq \emptyset$. If λ is an element of $B \setminus A$, then λ is an element of $\overline{A} \setminus A$. Since A is a convex set and λ is not an element of A, there exists a line ℓ of support A at λ. We can in general suppose that $\lambda = 1$, $\ell = \{z \in \mathbb{C}; \mathrm{Re}\, z = 1\}$ and $A \subset \{z \in \mathbb{C}; \mathrm{Re}\, z \leq 1\}$. Put $1 = \lambda = \sum_{j=1}^{\infty} a_j \lambda_j$. Since $\mathrm{Re}\lambda_j \leq 1$, $a_j \geq 0\, (j = 1, 2, \cdots)$ and $\sum_{j=1}^{\infty} a_j = 1$, we can show the relation $\mathrm{Re}\lambda_j = 1\, (j = 1, 2, \cdots)$. Thus, each λ_j is on the line $\ell (= \{z \in \mathbb{C}; \mathrm{Re}\, z = 1\})$. Since all $\lambda_j\, (j = 1, 2, \cdots)$ are elements of A, A is a convex set and λ is not an element of A, each λ_j is on the half part of the line ℓ with respect to λ. Thus, we have the relation $\mathrm{Im}\lambda_j > 0$ for every j or $\mathrm{Im}\lambda_j < 0$ for every j. Therefore, we have the following relation;

$$0 = \mathrm{Im}\lambda = \mathrm{Im}(\sum_{j=1}^{\infty} a_j \lambda_j) = \sum_{j=1}^{\infty} a_j (\mathrm{Im}\lambda_j) \neq 0.$$

This is a contradiction. Therefore λ is an element of A. $\square$

We get the following properties from the above mentioned Lemma 2 and Lemma 3 that it is the main part in this paper.

Theorem 4. *The original numerical range $W(a)$ and our numerical range $V(a)$ for every operator a coincide.*

Corollary 5. *If ϕ is a normal state on a von Neumann algebra $\mathcal{M}$ acting on a Hilbert space $\mathcal{H}$ and a in $\mathcal{M}$, then there exists a vector ξ of $\mathcal{H}$ satisfying $\phi(a) = \omega_\xi(a)$.*

Remark. In general, $W(a)$ does not equal to $U(a)$ by [1]. The referee gave the following example.

Example. We consider the operator $a : l^2 \to l^2$ defined by $a(x_1, x_2, x_3, \cdots) = (x_1, \frac{1}{2}x_2, \frac{1}{3}x_3, \cdots)$. Then $W(a) = (0, 1]$. But $\phi(a) = 0$ for any singular state ϕ on $\mathcal{B}(l^2)$.

Let a be an operator acting on a Hilbert space $\mathcal{H}$ and $\mathcal{M}$ a von Neumann algebra containing a and 1. If π is a faithful normal $*$-representation of $\mathcal{M}$ onto a von

Neumann algebra $\mathcal{N}$ on a Hilbert space $\mathcal{K}$, then ${}^t\pi(NS(\mathcal{N})) = NS(\mathcal{M})$ and so $V(a) = V(\pi(a))$. Thus, by considering Theorem 4, we have the following theorem.

Theorem 6. *Let a be an operator acting on a Hilbert space $\mathcal{H}$ and $\mathcal{M}$ a von Neumann algebra containing a. Let π be a faithful normal representation of $\mathcal{M}$, then $W(a) = W(\pi(a))$.*

Berberian and Orland introduced the numerical ranges of elements of C^*-algebras. Furthermore, Bonsal and Duncan introduced the numerical ranges for elements of normed algebras and Banach algebras in [3] and [4]. As an application of Theorem, we can introduce the numerical ranges in a sense of notions in [1], [3] and [4]:

Let $\mathcal{M}$ be a W^*-algebra with the pre-dual space $\mathcal{M}_*$ and $(\mathcal{M}_*)_1^+ = \{\phi \in \mathcal{M}_*; \phi(1) = 1$ and $\phi(x^*x) \geq 0$ for every $x \in \mathcal{M}\}$. For every element a in $\mathcal{M}$, define $V_{\mathcal{M}}(a) = \{\phi(a); \phi \in (\mathcal{M}_*)_1^+\}$. Then we have the following corollary.

Corollary 7. *Let $\mathcal{M}$ be a W^*-algebra with pre-dual space $\mathcal{M}_*$ and a an element of $\mathcal{M}$. If π is a faithful normal $*$-representation of $\mathcal{M}$ to a Hilbert space $\mathcal{H}_\pi$, then $V_{\mathcal{M}}(a) = V(\pi(a)) = W(\pi(a))$.*

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