Nihonkai Math. J.
Vol.12(2001), 123-133

Integration Operators On Weighted Bloch Spaces

Rikio Yoneda

Abstract

Let g be an analytic function on the open unit disk D in the complex plane C. We shall study the following operator

$$J_g(f)(z) := \int_0^z f(\zeta)g'(\zeta)d\zeta$$

on the Bloch space B. We show that the operator J_g is bounded on B if and only if

$$\sup_{z\in D}(1-|z|^2)\left(\log\frac{1}{1-|z|^2}\right)|g'(z)| < +\infty,$$

and the operator J_g is compact on B if and only if

$$\lim_{|z|\to 1^-}(1-|z|^2)\left(\log\frac{1}{1-|z|^2}\right)|g'(z)| = 0.$$

And we shall also characterize the boundedness and compactness of J_g on weighted Bloch spaces.

Key Words and Phrases : integration operator, Bloch space, compactness, boundedness.

§1. Introduction

Let $D = \{z \in C; |z| < 1\}$ denote the open unit disk in the complex plane C and let $\partial D = \{z \in C; |z| = 1\}$ denote the unit circle. Let $H(D)$ denote the space of analytic functions on D. Let $dA(z)$ be the normalized area measure on D. Let $1 \le p < +\infty$. The Hardy space H^p is defined to be the Banach space of analytic functions f on D with the norm

$$\| f \|_p := \left(\sup_{0<r<1}\frac{1}{2\pi}\int_0^{2\pi}|f(re^{i\theta})|^p d\theta\right)^{\frac{1}{p}} < +\infty.$$

For $z, w \in D$, let $\beta(z,w) := \frac{1}{2}\log\frac{1+|\varphi_z(w)|}{1-|\varphi_z(w)|}$, where $\varphi_z(w) = \frac{z-w}{1-\bar{z}w}$. We will frequently use the following properties of φ_z :

$$1-|\varphi_z(w)|^2 = \frac{(1-|z|^2)(1-|w|^2)}{|1-\bar{z}w|^2},$$

2000 Mathematics Subject Classification : Primary 30 D 45 .

$$\varphi_z(z)=0, \qquad \varphi_z(0)=z, \qquad \varphi_z\circ\varphi_z(w)=w.$$

For $0<r<+\infty$, let $D(z)=D(z,r)=\{w\in D;\ \beta(z,w)<r\}$ denote the Bergman disk. Then $D(z,r)$ is a Euclidean disk with Euclidean center C and radius R

$$C=\frac{1-t^2}{1-t^2|z|^2}z\,,\quad R=\frac{1-|z|^2}{1-t^2|z|^2}t$$

respectively, where $t=\dfrac{e^r-e^{-r}}{e^r+e^{-r}}\in(0,1)$. We denote by $|D(z,r)|$ the normalized area of $D(z,r)$. Then $|D(z,r)|$ is comparable to $(1-|z|^2)^2$.

The Bloch space B is the space of functions $f\in H(D)$ such that

$$\|f\|_B:=\sup\{(1-|z|^2)\,|f'(z)|:z\in D\}<+\infty\,.$$

This is a semi-norm on B and it is Möbius invariant in the sense of $\|f\circ\varphi\|_B=\|f\|_B$ for all $f\in B$ and $\varphi\in Aut(D)$, where $Aut(D)$ is the Möbius group of bi-analytic mappings of D. The Bloch space B is a Banach space with the norm $\|f\|=|f(0)|+\|f\|_B$. The little Bloch space of D, denoted B_0, is the closed subspace of B consisting of functions f with $(1-|z|^2)f'(z)\to 0$ $(|z|\to 1^-)$. The space of analytic functions on D of bounded mean oscillation, denoted by $BMOA$, is the set of functions f in H^2 such that

$$\|f\|_{BMOA}:=\sup\{\|f\circ\varphi_z-f(z)\|_2\colon z\in D\}<+\infty\,.$$

It is clear that $|g'(0)|\le\|g\|_2$ for every function $g\in H(D)$. Applying $g=f\circ\varphi_z-f(z)$, it follows that $(1-|z|^2)|f'(z)|\le\|f\circ\varphi_z-f(z)\|_2$ for $f\in H(D)$ and $z\in D$. Thus it follows that $BMOA\subset B$.

Let $\alpha\ge 1$. The α-Bloch space B^α is defined to be the space of functions $f\in H(D)$ such that

$$\|f\|_{B^\alpha}:=\sup\{(1-|z|^2)^\alpha\,|f'(z)|:z\in D\}<+\infty.$$

And the little α-Bloch space B_0^α is the closed subspace of B^α consisting of functions f with $(1-|z|^2)^\alpha f'(z)\to 0$ $(|z|\to 1^-)$. Note that B^1 and B_0^1 are the Bloch space and the little Bloch space, respectively.

Let ω be analytic on $\{\zeta\ ;\ |1-\zeta|<1\}$. Assume that $|\omega(1-|z|^2)|\to 0$ as $z\in D$ and $|z|\to 1^-$. Then the weighted Bloch space B_ω is the space of functions $f\in H(D)$ such that

$$\|f\|_{B_\omega}:=\sup\{|\omega(1-|z|^2)|\,|f'(z)|:z\in D\}<+\infty.$$

We define the following

$$B_{\log}:=\{f\in H(D):\|g\|_{B_{\log}}:=\sup_{z\in D}(1-|z|^2)\left(\log\frac{1}{1-|z|^2}\right)|g'(z)|<+\infty\},$$

$$B_{\log,0}:=\{f\in H(D):\lim_{|z|\to 1^-}(1-|z|^2)\left(\log\frac{1}{1-|z|^2}\right)|g'(z)|=0\}.$$

For a Banach space X, let $S : X \to X$ be a linear operator. Then the operator S is said to be compact if for every bounded sequence $\{x_n\}$ in X, $\{S(x_n)\}$ has a convergent subsequence. On the other hand, the operator S is said to be weakly compact if for every bounded sequence $\{x_n\}$ in X, $\{S(x_n)\}$ has a weakly convergent subsequence. Then it is known that S is weakly compact if and only if $S^{**}(X^{**}) \subset X$, where S^{**} is the second adjoint of S and X is identified with its image under the natural embedding into its second dual X^{**}.

For $g \in H(D)$, the operator J_g is defined on the weighted Bloch space by the following:

$$J_g(f)(z) := \int_0^z f(\zeta)g'(\zeta)d\zeta \quad (f \in H(D)).$$

If $g(z) = z$, then J_g is the integration operator. If $g(z) = \log \frac{1}{1-z}$, then J_g is the Cesáro operator.

In [4], Ch. Pommerenke showed that J_g is a bounded operator on the Hardy space H^2 if and only if g belongs to $BMOA$, and this result was extended to the other Hardy spaces H^p, $1 \leq p < +\infty$, in [1]. In [2], A. Aleman and A. G. Siskakis studied the operator J_g defined on the weighted (radial weight) Bergman space. Recently, in [5], A. G. Siskakis and R. Zhao showed the following theorem:

Theorem A. *The operator J_g is bounded on $BMOA$ if and only if*

$$\sup_{I \subset \partial D} \left(\frac{(\log \frac{2}{|I|})^2}{|I|} \int_{S(I)} |g'(z)|^2 (1-|z|^2) dA(z) \right) < +\infty,$$

and J_g is compact on $BMOA$ if and only if

$$\lim_{|I| \to 0} \left(\frac{(\log \frac{2}{|I|})^2}{|I|} \int_{S(I)} |g'(z)|^2 (1-|z|^2) dA(z) \right) = 0,$$

where $S(I) = \{z : 1 - |I| \leq |z| < 1, \frac{z}{|z|} \in I\}$ for an arc I in ∂D.

In this paper, we shall study the boundedness and compactness of the operator J_g defined on the Bloch space , the α-Bloch space and the weighted Bloch space. Some of the techniques used to prove our theorems come from [2] and [5].

Throughout this paper, positive constants C and K are not necessary the same as the one in at each occurrence.

§2. The boundedness and compactness of J_g on the Bloch space

In this section, we study the boundedness and compactness of the operator J_g defined on the Bloch space.

Theorem 2.1. *The operator J_g is bounded on B if and only if $g \in B_{\log}$.*

Proof. Suppose that $g \in B_{\log}$. Then $\| g \|_{\log} = \sup_{z \in D}(1-|z|^2)\left(\log \frac{1}{1-|z|^2}\right)|g'(z)| < +\infty$. Let $f \in B$. Then

$$(1-|z|^2)|(J_g f)'(z)| = (1-|z|^2)|f(z)||g'(z)| = (1-|z|^2)\left(\log \frac{1}{1-|z|^2}\right)|g'(z)|\frac{|f(z)|}{\log \frac{1}{1-|z|^2}}.$$

Since $|f(z)| \leq C \| f \|_B \log \frac{1}{1-|z|^2}$ (see [7, Theorem 5.1.6]), we have

$$\begin{aligned} \| J_g f \|_B &\leq C \sup_{z \in D}(1-|z|^2)\left(\log \frac{1}{1-|z|^2}\right)|g'(z)| \, \| f \|_B \\ &= C \| g \|_{B_{\log}} \| f \|_B . \end{aligned}$$

To prove the converse, suppose that J_g is bounded on B. For $a \in D$, put $f_a(z) = \log \frac{1}{1-\bar{a}z}$. Then $f_a \in B$. For $z \in D(a,r)$, we have $\log \frac{1}{1-|a|^2} \leq C \left|\log \frac{1}{1-\bar{a}z}\right|$. Since the subharmonicity of $|g'(z)|$, we see that $(1-|a|^2)^2|g'(a)|^2 \leq \int_{D(a,r)} |g'(z)|^2 dA(z)$ (see [7, Proposition 4.3.8]) So by using the fact that there is a constant $C_1 > 0$ (depending only on r) such that $\int_{D(a,r)} \frac{1}{(1-|z|^2)^2} dA(z) \leq C_1 < \infty$, we have

$$\begin{aligned} (1-|a|^2)^2 \left(\log \frac{1}{1-|a|^2}\right)^2 |g'(a)|^2 &\leq \left(\log \frac{1}{1-|a|^2}\right)^2 \int_{D(a,r)} |g'(z)|^2 dA(z) \\ &\leq C \int_{D(a,r)} \left|\log \frac{1}{1-\bar{a}z}\right|^2 |g'(z)|^2 dA(z) \\ &= C \int_{D(a,r)} \frac{1}{(1-|z|^2)^2}(1-|z|^2)^2 \left|\log \frac{1}{1-\bar{a}z}\right|^2 |g'(z)|^2 dA(z) \\ &\leq C \sup_{z \in D(a,r)} (1-|z|^2)^2 \left|\log \frac{1}{1-\bar{a}z}\right|^2 |g'(z)|^2 \int_{D(a,r)} \frac{1}{(1-|z|^2)^2} dA(z) \\ &\leq CC_1 \sup_{z \in D}(1-|z|^2)^2 \left|\log \frac{1}{1-\bar{a}z}\right|^2 |g'(z)|^2 \\ &\leq CC_1 \sup_{a \in D} \| J_g f_a \|_B^2 \\ &\leq CC_1 \| J_g \|^2 \sup_{a \in D} \| f_a \|_B^2 . \end{aligned}$$

Since $\| f_a \|_B = \sup_{z \in D}(1-|z|^2)\frac{1}{|1-\bar{a}z|}|a| \leq 2 < +\infty$ for any $a \in D$, we see $\sup_{a \in D} \| f_a \|_B < \infty$. Hence we have $\sup_{a \in D}(1-|a|^2)\left(\log \frac{1}{1-|a|^2}\right)|g'(a)| < \infty$. Thus $g \in B_{\log}$. □

Lemma 2.2. *For $f \in H(D)$ and $0 < r < 1$, put $f_r(z) = f(rz)$, $z \in D$. Let $f \in B_{\log}$. Then $\lim_{r \to 1^-} \| f_r - f \|_{B_{\log}} = 0$ if and only if $f \in B_{\log,0}$.*

Proof. Suppose that $f \in B_{\log}$ and $\lim_{r \to 1^-} \| f_r - f \|_{B_{\log}} = 0$. Then for any $\epsilon > 0$, there is a $\delta_0 \in (0,1)$ such that $\| f_r - f \|_{B_{\log}} < \epsilon$ for $\delta_0 < r < 1$. By using the fact $|a+b|^2 \leq 2|a|^2 + 2|b|^2$

and the definition of $\| * \|_{B_{\log}}$, we have

$$\begin{aligned}&(1-|a|^2)^2\left(\log\frac{1}{1-|a|^2}\right)^2|f'(a)|^2\\&\le 2(1-|a|^2)^2\left(\log\frac{1}{1-|a|^2}\right)^2|f'(a)-f_r'(a)|^2+2(1-|a|^2)^2\left(\log\frac{1}{1-|a|^2}\right)^2|f_r'(a)|^2\\&\le 2\epsilon+2(1-|a|^2)^2\left(\log\frac{1}{1-|a|^2}\right)^2|f_r'(a)|^2.\end{aligned}$$

Since $f_r \in B_{\log,0}$, we have $(1-|a|^2)^2\left(\log\frac{1}{1-|a|^2}\right)^2|f_r'(a)|^2 \to 0$ ($|a| \to 1^-$). Hence we see $f \in B_{\log,0}$.

To prove the converse, suppose $f \in B_{\log,0}$. Then for arbitrary enough small $\epsilon > 0$, there is a $\delta \in (0,1)$ such that $(1-|z|^2)\log\frac{1}{1-|z|^2}|f'(z)| < \epsilon$ for all $\delta^2 < |z| < 1$ and $(1-|z|^2)\log\frac{1}{1-|z|^2}$ is a decreasing function on $\delta^2 < |z| < 1$. For $0 < r < 1$, we have

$$\begin{aligned}\| f_r - f \|_{B_{\log}} &= \sup\{(1-|z|^2)\log\frac{1}{1-|z|^2}|rf'(rz)-f'(z)| : z \in D\}\\&\le \sup\{(1-|z|^2)\log\frac{1}{1-|z|^2}|rf'(rz)-f'(z)| : \delta < |z| < 1\}\\&\quad+\sup\{(1-|z|^2)\log\frac{1}{1-|z|^2}|rf'(rz)-f'(z)| : |z| \le \delta\}.\end{aligned}$$

Since $rf'(rz) \to f'(z)$ uniformly for $|z| \le \delta$, the second term in the above approaches to zero as $r \to 1^-$. If $\delta < r < 1$ and $\delta < |z| < 1$, then we have $\delta^2 < r|z| < 1$. Since $(1-|z|^2)\log\frac{1}{1-|z|^2}$ is a decreasing function on $|z| \in (\delta^2, 1)$, for $\delta^2 < r|z| < |z| < 1$,

$$(1-|z|^2)\log\frac{1}{1-|z|^2}|rf'(rz)| \le (1-r^2|z|^2)\log\frac{1}{1-r^2|z|^2}|f'(rz)| < \epsilon.$$

Hence

$$\sup\{(1-|z|^2)\log\frac{1}{1-|z|^2}|rf'(rz)-f'(z)| : \delta < |z| < 1\} \le 2\epsilon$$

for all $\delta < r < 1$. So we see $\limsup_{r\to 1^-} \| f_r - f \|_{B_{\log}} \le 2\epsilon$. Thus we have $\lim_{r \to 1^-} \| f_r - f \|_{B_{\log}} = 0$. □

Theorem 2.3. *The operator J_g is compact on B if and only if*

$$\lim_{|z|\to 1^-}(1-|z|^2)\left(\log\frac{1}{1-|z|^2}\right)|g'(z)| = 0.$$

Proof. Since $|f(z)| \le C \| f \|_B \log\frac{1}{1-|z|^2}$ for $f \in B$, the unit ball of B is a normal family of analytic functions. By the normal family argument, J_g is a compact operator on B if and only if every sequence $\{f_n\}$ in B with $\| f_n \|_B \le 1$ and $f_n \to 0$ $(n \to +\infty)$ uniformly on compact subsets of D has a subsequence $\{f_{n_k}\}$ such that $\| J_g f_{n_k} \|_B \to 0$ $(n \to +\infty)$.

Suppose that $\lim_{|z|\to 1^-}(1-|z|^2)\log\frac{1}{1-|z|^2}|g'(z)| = 0$. By the proof of Theorem 2.1, we have $\| J_g f \|_B \leq C \| g \|_{B_{\log}} \| f \|_B$ for $f \in B$. Then by Lemma 2.2, there exist polynomials P_n such that $\| g - P_n \|_{B_{\log}} \to 0$. Since $\| (J_g - J_{P_n})(f) \|_B \leq C \| g - P_n \|_{B_{\log}} \| f \|_B$, thus we have $\| J_g - J_{P_n} \| \leq C \| g - P_n \|_{B_{\log}} \to 0$ $(n \to \infty)$. For any polynomials P, J_P is a compact operator on B (see [2, p.342]). Hence we see that J_g is a compact operator on B.

To prove the converse, suppose that J_g is compact on B. Let $a_n \to a \in \partial D$ and put $f_n(z) := \log \frac{1}{1-\overline{a_n}z}$, $f(z) := \log \frac{1}{1-\overline{a}z}$. Then $f_n(z) \to f(z)$ uniformly on compact subsets of D. By the proof of Theorem 2.1 and the fact $|a+b|^2 \leq 2|a|^2 + 2|b|^2$, we have

$$
\begin{aligned}
&(1-|a_n|^2)^2 \left(\log \frac{1}{1-|a_n|^2}\right)^2 |g'(a_n)|^2 \\
&\leq C \sup_{z \in D(a_n,r)} (1-|z|^2)^2 \left|\log \frac{1}{1-\overline{a_n}z}\right|^2 |g'(z)|^2 \\
&\leq 2C \sup_{z \in D(a_n,r)} \left|\log \frac{1-\overline{a}z}{1-\overline{a_n}z}\right|^2 |g'(z)|^2 (1-|z|^2)^2 \\
&\qquad\qquad +2C \sup_{z \in D(a_n,r)} \left|\log \frac{1}{1-\overline{a}z}\right|^2 |g'(z)|^2 (1-|z|^2)^2 \\
&\leq 2C \| J_g(f_n - f) \|_B^2 + 2C \sup_{z \in D(a_n,r)} \left|\log \frac{1}{1-\overline{a}z}\right|^2 |g'(z)|^2 (1-|z|^2)^2 \\
&=: M_1 + M_2.
\end{aligned}
$$

By the compactness of J_g, we have $M_1 \to 0$ $(n \to \infty)$. Since B_0 is a subspace of B and they share the same norm, the compactness of J_g on B implies its compactness on B_0 (see [10, Lemma 8] or [5, Theorem 3.6]). Hence we see that J_g is weakly compact on B_0. Since $(B_0)^{**} = B$ and $J_g^{**} = J_g$, we have $J_g(B) \subset B_0$. Thus we have $J_g(f) \in B_0$. Thus we have

$$
M_2 = \sup_{z \in D(a_n,r)} \left|\log \frac{1}{1-\overline{a}z}\right|^2 |g'(z)|^2 (1-|z|^2)^2 = \sup_{z \in D(a_n,r)} \left((1-|z|^2)\left|(J_g(f))'(z)\right|\right)^2 \to 0 \ (n \to \infty).
$$

Hence we have $\lim_{|z| \to 1^-} (1-|z|^2) \log \frac{1}{1-|z|^2} |g'(z)| = 0$. □

§3. The boundedness and compactness of J_g on the α-Bloch space

In this section, we study the boundedness and compactness of the operator J_g defined on the α-Bloch space for $\alpha > 1$.

Theorem 3.1. *Let $\alpha > 1$. Then the operator J_g is bounded on B^α if and only if*

$$
\sup_{z \in D} (1-|z|^2)|g'(z)| < +\infty, \qquad i.e \quad g \in B.
$$

Proof. Suppose that $\sup_{z \in D} (1-|z|^2)|g'(z)| < +\infty$. Let $f \in B^\alpha$. Then we see

$$
(1-|z|^2)^\alpha |(J_g f)'(z)| = (1-|z|^2)^\alpha |f(z)||g'(z)| = (1-|z|^2)|g'(z)| \frac{|f(z)|}{(1-|z|^2)^{1-\alpha}}.
$$

Since $|f(z)| \leq C \parallel f \parallel_{B^\alpha} (1-|z|^2)^{1-\alpha}$ (see [9, Proposition 7]), we have

$$\parallel J_g f \parallel_{B^\alpha} \leq C \sup_{z\in D}(1-|z|^2)|g'(z)| \parallel f \parallel_{B^\alpha}.$$

To prove the converse, suppose that J_g is a bounded operator on B^α. For $a \in D$, put $f_a(z) = (1-\bar{a}z)^{1-\alpha}$. Then it is clear that $f_a \in B^\alpha$. By using the subharmonicity of $|g'(z)|$, the fact that $\int_{D(a,r)} \frac{1}{(1-|z|^2)^2} dA(z) \leq C_1 < \infty$, and the fact that $|1-\bar{a}z|$ is comparable to $(1-|z|^2)$ on $D(a,r)$, we have

$$\begin{aligned}
(1-|a|^2)^2|g'(a)|^2 &\leq \int_{D(a,r)} |g'(z)|^2 dA(z) \\
&= \int_{D(a,r)} (1-|z|^2)^{2\alpha}(1-|z|^2)^{2(1-\alpha)}|g'(z)|^2 \frac{dA(z)}{(1-|z|^2)^2} \\
&\leq C\int_{D(a,r)} (1-|z|^2)^{2\alpha}\,|1-\bar{a}z|^{2(1-\alpha)}\,|g'(z)|^2 \frac{dA(z)}{(1-|z|^2)^2} \\
&\leq C\int_{D(a,r)} \frac{dA(z)}{(1-|z|^2)^2}\left(\sup_{z\in D(a,r)}(1-|z|^2)^\alpha\,|1-\bar{a}z|^{1-\alpha}\,|g'(z)|\right)^2 \\
&\leq CC_1\left(\sup_{z\in D}(1-|z|^2)^\alpha\,|1-\bar{a}z|^{1-\alpha}\,|g'(z)|\right)^2 \\
&\leq CC_1 \sup_{a\in D} \parallel J_g f_a \parallel^2_{B^\alpha} \\
&\leq CC_1 \parallel J_g \parallel^2 \sup_{a\in D} \parallel f_a \parallel^2_{B^\alpha}.
\end{aligned}$$

Since $\parallel f_a \parallel_{B^\alpha} = \sup_{z\in D}(1-|z|^2)^\alpha \frac{(\alpha-1)|a|}{|1-\bar{a}z|^\alpha} \leq (\alpha-1)2^\alpha < +\infty$ for any $a \in D$, we see $\sup_{a\in D} \parallel f_a \parallel_{B^\alpha} < +\infty$. Hence we have $\sup_{a\in D}(1-|a|^2)|g'(a)| < +\infty$. □

Theorem 3.2. *Let $\alpha > 1$. Then the operator J_g is compact on B^α if and only if*

$$\lim_{|z|\to 1^-}(1-|z|^2)|g'(z)| = 0, \quad i.e \quad g \in B_0.$$

Proof. Since $|f(z)| \leq C \parallel f \parallel_{B^\alpha} (1-|z|^2)^{1-\alpha}$ for $f \in B^\alpha$, the unit ball of B^α is a normal family of analytic functions. By the normal family argument, J_g is a compact operator on B^α if and only if every sequence $\{f_n\}$ in B^α with $\parallel f_n \parallel_{B^\alpha} \leq 1$ and $f_n \to 0$ $(n \to +\infty)$ uniformly on compact subsets of D has a subsequence $\{f_{n_k}\}$ such that $\parallel J_g f_{n_k} \parallel_{B^\alpha} \to 0$ $(n \to +\infty)$.

Suppose that $\lim_{|z|\to 1^-}(1-|z|^2)|g'(z)| = 0$. By the proof of Theorem 3.1, we have

$$\parallel J_g f \parallel_{B^\alpha} \leq C \parallel g \parallel_B \parallel f \parallel_{B^\alpha}$$

for $f \in B^\alpha$. Then by [7 , Theorem 5.2.2.], there exist polynomials P_n such that $\parallel g - P_n \parallel_B \to 0$. Since $\parallel (J_g - J_{P_n})(f) \parallel_{B^\alpha} \leq C \parallel g - P_n \parallel_B \parallel f \parallel_{B^\alpha}$, we have

$$\parallel J_g - J_{P_n} \parallel \leq C \parallel g - P_n \parallel_B \to 0 \ (n \to \infty).$$

For any polynomials P, J_P is a compact operator on B^α (see [2 , p.342]). Hence we see that J_g is a compact operator on B^α.

To prove the converse, suppose that J_g is compact on B^α. Let $a_n \to a \in \partial D$ and put $f_n(z) := (1-\overline{a_n}z)^{1-\alpha}$, $f(z) := (1-\bar{a}z)^{1-\alpha}$. Then $f_n(z) \to f(z)$ uniformly on compact subsets of D. By the proof of Theorem 3.1 and the fact $|a+b|^2 \le 2|a|^2 + 2|b|^2$, we have

$$\begin{aligned}
(1-|a_n|^2)^2|g'(a_n)|^2 &\le C\int_{D(a_n,r)} |1-\overline{a_n}z|^{2(1-\alpha)}\,|g'(z)|^2(1-|z|^2)^{2\alpha}\frac{dA(z)}{(1-|z|^2)^2} \\
&\le 2C\int_{D(a_n,r)} \left|(1-\overline{a_n}z)^{(1-\alpha)} - (1-\bar{a}z)^{(1-\alpha)}\right|^2 |g'(z)|^2(1-|z|^2)^{2\alpha}\frac{dA(z)}{(1-|z|^2)^2} \\
&\qquad +2C\int_{D(a_n,r)} |1-\bar{a}z|^{2(1-\alpha)}\,|g'(z)|^2(1-|z|^2)^{2\alpha}\frac{dA(z)}{(1-|z|^2)^2} \\
&\le K \parallel J_g(f_n - f) \parallel_{B^\alpha}^2 + K \sup_{z\in D(a_n,r)} |1-\bar{a}z|^{2(1-\alpha)}\,|g'(z)|^2(1-|z|^2)^{2\alpha} \\
&=: I_1 + I_2.
\end{aligned}$$

By the compactness of J_g, we have $I_1 \to 0$ $(n \to \infty)$. Since B_0^α is a subspace of B^α and they share the same norm, the compactness of J_g on B^α implies its compactness on B_0^α. Hence we see that J_g is weakly compact on B_0^α. Since $(B_0^\alpha)^{**} = B^\alpha$ (see [9]) and $J_g^{**} = J_g$, we have $J_g(B^\alpha) \subset B_0^\alpha$. Thus we have $J_g(f) \in B_0^\alpha$. Thus we have

$$\begin{aligned}
I_2 &= \sup_{z\in D(a_n,r)} |1-\bar{a}z|^{2(1-\alpha)}\,|g'(z)|^2(1-|z|^2)^{2\alpha} = \sup_{z\in D(a_n,r)} \left((1-|z|^2)^\alpha \left|(J_g(f))'(z)\right|\right)^2 \\
&= \sup_{z\in D} \left(\chi_{D(a_n,r)}(z)(1-|z|^2)^\alpha \left|(J_g(f))'(z)\right|\right)^2 \to 0 \ (n\to\infty).
\end{aligned}$$

Hence we have $\lim_{|z|\to 1^-}(1-|z|^2)|g'(z)| = 0$. □

§4. The boundedness and compactness of J_g on the weighted Bloch space B_ω

In this section, we study the boundedness and compactness of J_g on the weighted Bloch space B_ω.

Theorem 4.1. *Let $0 < r < +\infty$. Let ω be analytic, non-vanishing on $\{\zeta : |1-\zeta| < 1\}$, and $|\omega(1-|z|^2)| \to 0$ as $|z| \to 1^-$. Suppose that*
(i) $\sup_{z\in D} |\omega(1-|z|^2)| \int_0^{|z|} \frac{ds}{|\omega(1-s^2)|} < +\infty$,
(ii) $\sup_{z,a\in D} \frac{|\omega(1-|z|^2)|}{|\omega(1-\bar{a}z)|} < +\infty$,
(iii) $\int_0^{|z|} \frac{ds}{|\omega(1-s^2)|} < +\infty$ *for any* $z \in D$,
(iv) $\int_0^{|z|} \frac{ds}{|\omega(1-s^2)|} \to \infty$ $(|z| \to 1^-)$,
(v) for any $a \in D$ there is a constant $C > 0$ (independent on a) such that $\left|\frac{\omega(1-\bar{a}z)}{\omega(1-|z|^2)}\right| \le C$ *for all $z \in D(a,r)$,*

(vi) there is a constant $K > 0$(independent on z) such that $\int_0^{|z|} \frac{ds}{|\omega(1-s^2)|} \le K \left| \int_0^z \frac{1}{\omega(1-\bar{z}\eta)} d\eta \right|$ *for all $z \in D$.*

Then the operator J_g is bounded on B_ω if and only if

$$\| g \|_W := \sup_{z \in D} |\omega(1-|z|^2)| \int_0^{|z|} \frac{ds}{|\omega(1-s^2)|} |g'(z)| < +\infty.$$

Proof. Suppose that $\| g \|_W < \infty$. Let $f \in B_\omega$. Then we see

$$\begin{aligned} |f(a) - f(0)| &= |a \int_0^1 f'(at)dt| \\ &\le |a| \int_0^1 |\omega(1-|at|^2)||f'(at)| \frac{1}{|\omega(1-|at|^2)|} dt \\ &\le \| f \|_{B_\omega} \int_0^1 \frac{|a|}{|\omega(1-|at|^2)|} dt = \| f \|_{B_\omega} \int_0^{|a|} \frac{1}{|\omega(1-s^2)|} ds. \end{aligned}$$

Thus $|f(a)| \le C \| f \|_{B_\omega} \int_0^{|a|} \frac{1}{|\omega(1-s^2)|} ds$. Then

$$\begin{aligned} |\omega(1-|z|^2)| |(J_g f)'(z)| &= |\omega(1-|z|^2)||f(z)||g'(z)| \\ &\le C|\omega(1-|z|^2)| \int_0^{|z|} \frac{ds}{|\omega(1-s^2)|} |g'(z)| \, \| f \|_{B_\omega} \\ &\le C \| g \|_W \| f \|_{B_\omega}. \end{aligned}$$

Hence we have $\| J_g f \|_{B_\omega} \le C \| f \|_{B_\omega}$.

To prove the converse, suppose that J_g is bounded on B_ω. Put $h_a(z) := \int_0^z \frac{1}{\omega(1-\bar{a}\eta)} d\eta$. By the assumptions that there is a constant $C > 0$ (independent on a) such that $\left| \frac{\omega(1-\bar{a}z)}{\omega(1-|z|^2)} \right| \le C$ for all $z \in D(a,r)$, and that there is a constant $K > 0$ (independent on z) such that $\int_0^{|z|} \frac{ds}{|\omega(1-s^2)|} \le K \left| \int_0^z \frac{1}{\omega(1-\bar{z}\eta)} d\eta \right|$ for all $z \in D$, by using the subharmonicity of $(|\omega(1-\bar{a}z)| \left| \int_0^z \frac{1}{\omega(1-\bar{a}\eta)} d\eta \right| |g'(z)|)^2$, we have

$$\begin{aligned} &\left(|\omega(1-|a|^2)| \int_0^{|a|} \frac{ds}{|\omega(1-s^2)|} |g'(a)| \right)^2 \\ &\le \left(K|\omega(1-|a|^2)| \left| \int_0^a \frac{1}{\omega(1-\bar{a}\eta)} d\eta \right| |g'(a)| \right)^2 \\ &\le K^2 \frac{C_1}{(1-|a|^2)^2} \int_{D(a,r)} \left(|\omega(1-\bar{a}z)| \left| \int_0^z \frac{1}{\omega(1-\bar{a}\eta)} d\eta \right| |g'(z)| \right)^2 dA(z) \\ &\le K^2 C^2 \frac{C_1}{(1-|a|^2)^2} \int_{D(a,r)} \left(|\omega(1-|z|^2)| |h_a(z)| |g'(z)| \right)^2 dA(z) \\ &\le K^2 C^2 C_1 C_2 \sup_{z \in D(a,r)} |\omega(1-|z|^2)|^2 |h_a(z)|^2 |g'(z)|^2 \\ &\le K^2 C^2 C_1 C_2 \| J_g h_a \|_{B_\omega}^2 \end{aligned}$$

Since $\sup_{z,a \in D} \frac{|\omega(1-|z|^2)|}{|\omega(1-\bar{a}z)|} < +\infty$, we see $h_a \in B_\omega$. By the boundedness of J_g on B_ω, we have

$$\sup_{z \in D} |\omega(1-|z|^2)| \int_0^{|z|} \frac{ds}{|\omega(1-s^2)|} |g'(z)| < +\infty. \quad \square$$

For example, $\omega(\zeta) := \zeta$ and $\omega(\zeta) := \zeta^\alpha$ and $\omega(\zeta) := \zeta\log\zeta$ satisfy the conditions $(i) \sim (vi)$ of Theorem 4.1.

We define the following

$$B_W := \{f \in H(D) : \| g \|_{B_W} := \sup_{z\in D} |\omega(1-|z|^2)| \int_0^{|z|} \frac{ds}{|\omega(1-s^2)|} |g'(z)| < +\infty\},$$

$$B_{W,0} := \{f \in H(D) : \lim_{|z|\to 1^-} |\omega(1-|z|^2)| \int_0^{|z|} \frac{ds}{|\omega(1-s^2)|} |g'(z)| = 0\}.$$

Lemma 4.2. *Let ω be as in Theorem 4.1. Moreover suppose that $|\omega(1-|z|^2)| \int_0^{|z|} \frac{ds}{|\omega(1-s^2)|} \downarrow 0$ $(|z| \to 1^-)$. Then for $f \in B_W$, $\lim_{r\to 1^-} \| f_r - f \|_W = 0$ if and only if $f \in B_{W,0}$.*

Proof. Suppose that $f \in B_W$ and $\lim_{r\to 1^-} \| f_r - f \|_W = 0$. Then for any $\epsilon > 0$, there is a $\delta_0 \in (0,1)$ such that $\| f_r - f \|_W < \epsilon$ for $\delta_0 < r < 1$. By using the fact $|a+b|^2 \le 2|a|^2 + 2|b|^2$ and the definition of $\| * \|_W$, we have

$$\left(|\omega(1-|z|^2)| \int_0^{|z|} \frac{ds}{|\omega(1-s^2)|}\right)^2 |f'(z)|^2 \le 2\epsilon + 2\left(|\omega(1-|z|^2)| \int_0^{|z|} \frac{ds}{|\omega(1-s^2)|}\right)^2 |f_r'(z)|^2.$$

Since $f_r \in B_{W,0}$, we have $f \in B_{W,0}$.

To prove the converse, suppose $f \in B_{W,0}$. Then for arbitrary small $\epsilon > 0$, there is a $\delta \in (0,1)$ such that $|\omega(1-|z|^2)| \int_0^{|z|} \frac{ds}{|\omega(1-s^2)|} |f'(z)| < \epsilon$ for all $\delta^2 < |z| < 1$ and $|\omega(1-|z|^2)| \int_0^{|z|} \frac{ds}{|\omega(1-s^2)|}$ is a decreasing function on $\delta^2 < |z| < 1$. We have

$$\| f_r - f \|_W \le \sup\{ |\omega(1-|z|^2)| \int_0^{|z|} \frac{ds}{|\omega(1-s^2)|} |rf'(rz) - f'(z)| : \delta < |z| < 1 \}$$
$$+ \sup\{ |\omega(1-|z|^2)| \int_0^{|z|} \frac{ds}{|\omega(1-s^2)|} |rf'(rz) - f'(z)| : |z| \le \delta \}.$$

Since $rf'(rz) \to f'(z)$ uniformly for $|z| \le \delta$, the second term in the above approaches to zero as $r \to 1^-$. If $\delta < r < 1$ and $\delta < |z| < 1$, then we have $\delta^2 < r|z| < 1$. For $\delta^2 < r|z| < |z| < 1$,

$$|\omega(1-|z|^2)| \int_0^{|z|} \frac{ds}{|\omega(1-s^2)|} |rf'(rz)| \le |\omega(1-|rz|^2)| \int_0^{|rz|} \frac{ds}{|\omega(1-s^2)|} |f'(rz)| < \epsilon.$$

Hence

$$\sup\{ |\omega(1-|z|^2)| \int_0^{|z|} \frac{ds}{|\omega(1-s^2)|} |rf'(rz) - f'(z)| : \delta < |z| < 1 \} \le 2\epsilon$$

for all $\delta < r < 1$. So we see $\limsup_{r\to 1^-} \| f_r - f \|_W \le 2\epsilon$. Thus we have $\lim_{r\to 1^-} \| f_r - f \|_W = 0$. □

We also see that examples $\omega(\zeta) := \zeta$ and $\omega(\zeta) := \zeta^\alpha$ and $\omega(\zeta) = \zeta\log\zeta$ satisfy the condition of Lemma 4.2.

Proposition 4.3. *Let ω be as in Theorem 4.1. Suppose that for any $a \in D$, $|\omega(1-|a|^2)|$ is comparable to $|\omega(1-|z|^2)|$ on $D(a,r)$, and that $|\omega(1-|z|^2)| \int_0^{|z|} \frac{ds}{|\omega(1-s^2)|} \downarrow$*

0 $(|z| \to 1^-)$. *If* $\lim_{|z|\to 1^-} |\omega(1-|z|^2)| \int_0^{|z|} \frac{ds}{|\omega(1-s^2)|} |f'(z)| = 0$, *then the operator* J_g *is compact on* B_ω.

Proof. By the proof of Theorem 4.1, we have $\| J_g f \|_{B_\omega} \leq C \| g \|_W \| f \|_{B_\omega}$. Suppose that $\lim_{|z|\to 1^-} |\omega(1-|z|^2)| \int_0^{|z|} \frac{ds}{|\omega(1-s^2)|} |g'(z)| = 0$. Then by Lemma 4.2, there exist polynomials P_n such that $\| g - P_n \|_W \to 0$. Since $\| (J_g - J_{P_n})(f) \|_{B_\omega} \leq C \| g - P_n \|_W \| f \|_{B_\omega}$, we have $\| J_g - J_{P_n} \| \leq C \| g - P_n \|_W \to 0$ $(n \to \infty)$. For any polynomials P, J_P is a compact operator on B_ω (see [2, p.342]). Hence we see that J_g is a compact operator on B_ω. □

Acknowledgment. The author wishes to express his sincere gratitude to Professor Takahiko Nakazi for his many helpful suggestions and advices.

References

[1] A. Aleman and A. G. Siskakis, An integral operator on H^p, Complex Variables, **28**(1995), 149-158.

[2] A. Aleman and A. G. Siskakis, Integration operators on Bergman spaces, Indiana Univ. Math. J. **46**(1997), 337-356.

[3] N. Dunford and J. T. Schwarz, Linear Operators I, Intersciense Publishers, John Wiley and Sons, New York 1958.

[4] Ch. Pommerenke, Schlichte Funktionen und analytische Functionen von beschrankter mittlerer Oszillation, Comment. Math. Helv. **52**(1977), 591-602.

[5] A. G. Siskakis and R. Zhao, A Volterra type operator on spaces of analytic functions, Contemporary Mathematics. **232**(1999), 299-311.

[6] K. Stroethoff, The Bloch space and Besov spaces of analytic functions, Bull. Austral. Math. Soc. **54**(1996), 211-219.

[7] K. Zhu, Operator Theory in Function Spaces, Marcel Dekker, New York 1990.

[8] K. Zhu, Analytic Besov Spaces, J. Math. Anal. Appl. **157**(1991), 318-336.

[9] K. Zhu, Bloch type spaces of analytic functions, Rocky Mout. J. Math. **23**(1993), 1143-1177.

[10] K. Zhu, Multipliers of BMO in the Bergman Metric with Applications to Toeplitz Operators, J. Funct. Anal. **87**(1989), 31-50.

Tokyo Metropolitan College of Technology
Tokyo 140-0011 , Japan
yoneda@tokyo-tmct.ac.jp

Received March 7, 2001 Revised July 13, 2001