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On a Generalization of the Tomita-Takesaki Theorem for a Quasifree State on a Self-Dual CCR Algebra

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Abstract

In this paper, we prove the Tomita-Takesaki theorem for an unbounded operator algebra given by the GNS-representation of a CCR algebra with respect to a quasifree state in the self-dual formalism.

1. Introduction

In quantum physics, the annihilation and creation operators for Bose particles satisfy the canonical commutation relations (CCR) and it is important to study the CCR algebras generated by those objects satisfying CCR. The CCR algebras have been studied by numerous physicists and mathematicians for a long time.

In the theory of bounded operator algebras, it is well-known that the Tomita-Takesaki theorem plays an important role for a study of structures of von Neumann algebras and for a study of KMS-states (matematical objects for equiliblium states) on C^*-dynamical systems.

If we represent a CCR algebra as an operator algebra in a Hilbert space, it is always unbounded. To avoid the difficulty coming from the unboundedness we usually consider a bounded operator algebra whose generators satisfy the Weyl-Segal commutation relations, but we can not directly observe the annihilation and creation operators in it. Thus it seems meaningful to study the unbounded operator algebra as itself and to develope the Tomita-Takesaki theory in it. A generalization of Tomita-Takesaki theorem to general unbounded operator algebras has been studied by several authors (for example, see [5]). But, for unbounded CCR algebras, as far as the authors know, it has been scarcely done except [4] and [6], in which the Tomita-Takesaki theorem was proved under a special condition. The aim of this paper is to prove the theorem for unbounded CCR algebras under more general situation by using Araki's self-dual formalism.

In section 2, a self-dual CCR algebra and a quasifree state are defined. And several results for these, which are known in [1] and [2], are stated.

In section 3, making use of the results in section 2, we prove the Tomita-Takesaki theorem for an unbounded operator algebra given by the GNS-representation of a CCR algebra with respect to a quasifree state in the self-dual formalism.

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2. Preliminaries

In this section, we introduce several notations and well-known results from [1] and [2].

1^{o}) Let K be a complex linear space. Let $r(\cdot,\cdot)$ be a hermitian form on $K \times K$ and Γ be an anti-linear operator on K such that

(1) $\quad \Gamma^2 = 1$ and $r(\Gamma f, \Gamma g) = - r(g, f)$ for $f, g \in K$.

For this triplet (K,r,Γ), let $\mathfrak{B}$ be the free complex $*$-algebra generated by $\{B(f) : f \in K\}$ and an identity $\mathbb{1}$, where the $*$-mapping is defined by $B(f)^* = B(\Gamma f)$ for $f \in K$. Futhermore, let $\mathcal{J}$ be the two-sided ideal of $\mathfrak{B}$ generated by the identity $\mathbb{1}$ and $\{B(\alpha f+\beta g) - \alpha B(f) - \beta B(g),\ B(f)^* B(g) - B(g)B(f)^* - r(f, g)\mathbb{1} : \alpha, \beta \in \mathbb{C}, f, g \in K\}$. Then $\mathcal{J}$ is $*$-invariant. The quotient $*$-algebra $\mathfrak{B}/\mathcal{J}$ is called a self-dual CCR algebra over (K,r,Γ) and denoted by $\mathfrak{A}(K,r,\Gamma)$. The self-dual CCR algebra $\mathfrak{A}(K,r,\Gamma)$ can be considered as a $*$-algebra generated by $\{B(f) : f \in K\}$ such that

$$(2) \quad \begin{cases} B(f)^* = B(\Gamma f), \\ B(\alpha f + \beta g) = \alpha B(f) + \beta B(g), \\ B(f)^* B(g) - B(g)B(f)^* = r(f, g)\mathbb{1} \quad \text{for } f, g \in K. \end{cases}$$

We set $\mathrm{Re}K = \{f \in K : \Gamma f = f\}$. Then it is clear that $B(f)$ is hermitian if and only if $f \in \mathrm{Re}K$.

Let (K',r',Γ') be an another triplet satisfying the same condition with (1). And let U be a linear operator from K into K' such that $\Gamma' U = U\Gamma$ and $r'(Uf, Ug) = r(f, g)$ for $f, g \in K$. Then it follows from the self-dual CCR algebras that there exists a unique $*$-homomorphism τ_U from $\mathfrak{A}(K,r,\Gamma)$ into $\mathfrak{A}(K',r',\Gamma')$ such that

$$(3) \qquad \tau_U(B(f)) = B(Uf) \quad \text{for} \quad f \in K.$$

Then it is easily checked that, if U is injective (resp. surjective), then τ_U is also injective (resp. surjective). In particular, if U is a bijective linear mapping of K satisfying $\Gamma U = U\Gamma$ and $r(Uf, Ug) = r(f, g)$ for $f, g \in K$, then U is called a Bogoliubov transformation for (K,r,Γ) and τ_U is also called a Bogoliubov automorphism of $\mathfrak{A}(K,r,\Gamma)$.

A linear functional φ on a $*$-algebra $\mathfrak{A}$ with an identity $\mathbb{1}$ is called a state if φ satisfies

$$\varphi(\mathbb{1}) = 1 \quad \text{and} \quad \varphi(A^*A) \geq 0 \quad \text{for all} \quad A \in \mathfrak{A}.$$

For any state φ on $\mathfrak{A}$, there exist a Hilbert space $\mathfrak{H}_\varphi$, a representation π_φ of $\mathfrak{A}$, and a cyclic vector Ω_φ for $\pi_\varphi(\mathfrak{A})$ in $\mathfrak{H}_\varphi$ such that $\pi_\varphi(A)$ is a closable linear operator with the domain $\pi_\varphi(\mathfrak{A})\Omega_\varphi$, $\pi_\varphi(A)^* \supset \pi_\varphi(A^*)$, and $\varphi(B^*A) = (\pi_\varphi(B)\Omega_\varphi, \pi_\varphi(A)\Omega_\varphi)$ for $A, B \in \mathfrak{A}$. In this paper, the triplet $(\mathfrak{H}_\varphi,\pi_\varphi,\Omega_\varphi)$ is called a GNS representation of $\mathfrak{A}$ with respect to φ.

Let φ be a state on a self-dual CCR algebra $\mathfrak{A}(K,r,\Gamma)$. If $\pi_\varphi(B(f))$ is essentially selfadjoint for every f in $\mathrm{Re}K$ and $\{W_\varphi(f) = \exp i\overline{\pi_\varphi(B(f))} : f \in \mathrm{Re}K\}$ satisfies the Weyl-Segal relations, that is,

$$W_\varphi(f)W_\varphi(g) = W_\varphi(f + g)\exp\frac{1}{2}r(g, f) \quad \text{for every} \quad f, g \in \mathrm{Re}K,$$

then φ is said to be regular. Further, let $\mathfrak{R}_\varphi$ be the von Neumann algebra generated by $\{W_\varphi(f) : f \in \mathrm{Re}K\}$.

2^o) Let φ be a state on a self-dual CCR algebra $\mathfrak{A}(K,r,\Gamma)$. We define a hermitian form $s(\cdot,\cdot)$ and a positive semi-definite form $(\cdot,\cdot)_s$ on $K \times K$, respectively, by

$$(4) \qquad s(f, g) = \varphi(B(f)^*B(g)),$$

$$(f, g)_s = s(f, g) + s(\Gamma g, \Gamma f) \quad \text{for} \quad f, g \in K.$$

Put $N_s = \{f \in K : (f, f)_s = 0\}$ and let K_s be the completion of K/N_s with respect to the inner product $(\cdot,\cdot)_s$ on K/N_s. Put $\overline{f} = f + N_s \in K_s$ for $f \in K$ and $\overline{K} = \{\overline{f} : f \in K\}$. We can respectively construct r_s and Γ_s from r and Γ such that the triplet (K_s,r_s,Γ_s) satisfies the same conditions as (1) for (K,r,Γ). Futhermore, the form s on $K \times K$ is also canonically extended to a positive semi-definite form on $K_s \times K_s$. We denote it by the same notation s. Since the two hermitian forms r_s and s on $K_s \times K_s$ are continuous with respect to the inner product $(\cdot,\cdot)_s$ on $K_s \times K_s$, there exist bounded selfadjoint operators R_s and S on K_s such that

$$
(5)\quad \begin{cases} r_s(\overline{f}, \overline{g}) = (\overline{f}, R_s\overline{g})_s \quad (f, g \in K), \\ s(\overline{f}, \overline{g}) = (\overline{f}, S\overline{g})_s \quad (f, g \in K), \\ 0 \le S \le 1,\ \Gamma_s S \Gamma_s = 1 - S, \text{ and } S - \Gamma_s S \Gamma_s = R_s. \end{cases}
$$

Since $\pi_\varphi(B(f)) = 0$ if and only if $(f, f)_s = 0$ (i.e. $f \in N_s$) by Lemma 3.4 of [1], we can define the $*$-representation $\overline{\pi}_\varphi$ of $\mathfrak{A}(\overline{K}, r_s, \Gamma_s)$ by

$$
\overline{\pi}_\varphi(B(\overline{f})) = \pi_\varphi(B(f)) \quad \text{for all} \quad f \in K.
$$

If we put a state $\overline{\varphi}$ on $\mathfrak{A}(\overline{K}, r_s, \Gamma_s)$ by the equation

$$
\overline{\varphi}(A) = (\Omega_\varphi, \overline{\pi}_\varphi(A)\Omega_\varphi) \quad \text{for every} \quad A \in \mathfrak{A}(\overline{K}, r_s, \Gamma_s),
$$

then we can show that $(\overline{\pi}_\varphi, \mathfrak{H}_\varphi, \Omega_\varphi)$ is the GNS-representation of $\mathfrak{A}(\overline{K}, r_s, \Gamma_s)$ for $\overline{\varphi}$.

A state φ on $\mathfrak{A}(K, r, \Gamma)$ is called a quasifree state if φ satisfies the following conditions: for every $n = 1, 2, \cdots$,

$$
\varphi(B(f_1)\cdots B(f_{2n-1})) = 0
$$

$$
\varphi(B(f_1)\cdots B(f_{2n})) = \sum \prod_{j=1}^{n} \varphi(B(f_{s(j)})B(f_{s(j+n)})),
$$

where the sum is over all permutations s satisfying $s(1) < s(2) < \cdots < s(n)$, $s(j) < s(j+n)$, $j = 1, 2, \cdots, n$.

If φ is a quasifree state on $\mathfrak{A}(K,r,\Gamma)$, then the mapping

$$(f_1,\cdots,f_n) \longrightarrow \varphi(B(f_1)\cdots B(f_n)): \quad K\times\cdots\times K \longrightarrow \mathbb{C}$$

is continuous for $(\cdot,\cdot)_s$ on K; that is, if $(f_j^{(m)}- f_j^{(m')},f_j^{(m)}- f_j^{(m')})_s$ converges to 0 as m, $m' \longrightarrow \infty$ $(j = 1,\cdots,n)$, then $\{\varphi(B(f_1^{(m)}) \cdots B(f_n^{(m)}))\}_{m=1}^{\infty}$ is a Cauchy sequence in $\mathbb{C}$. Thus, for any h_j in K_s $(j = 1,\cdots,n)$, if each sequence $\overline{\{f_j^{(m)}\}}_{m=1}^{\infty}$ $(f_j^{(m)} \in K)$ converges to h_j for $(\cdot,\cdot)_s$, then we can define a quasifree state $\tilde{\varphi}$ on $\mathfrak{A}(K_s,r_s,\Gamma_s)$ by the equation

$$\tilde{\varphi}(B(h_1) \cdots B(h_n)) = \lim_{m\to\infty} \overline{\varphi}(B(\overline{f_1^{(m)}}) \cdots B(\overline{f_n^{(m)}}))$$

$$= \lim_{m\to\infty} \varphi(B(f_1^{(m)}) \cdots B(f_n^{(m)})).$$

Let $(\pi_{\tilde{\varphi}},\mathfrak{H}_{\tilde{\varphi}},\Omega_{\tilde{\varphi}})$ be the GNS-representation of $\mathfrak{A}(K_s,r_s,\Gamma_s)$ with respect to $\tilde{\varphi}$. We may identify $\mathfrak{H}_{\tilde{\varphi}} = \mathfrak{H}_{\varphi}$, and $\Omega_{\tilde{\varphi}} = \Omega_{\varphi}$. Since $\pi_{\varphi}(\mathfrak{A}(K,r,\Gamma))\Omega_{\varphi} = \pi_{\overline{\varphi}}(\mathfrak{A}(\overline{K},r_s,\Gamma_s))\Omega_{\overline{\varphi}} \subsetneqq \pi_{\tilde{\varphi}}(\mathfrak{A}(K_s,r_s,\Gamma_s))\Omega_{\tilde{\varphi}}$, the domain of $\pi_{\tilde{\varphi}}$ is different from that of π_{φ} and, for every $A \in \mathfrak{A}(K,r,\Gamma)$, the restriction $\pi_{\tilde{\varphi}}(\overline{A})|_{\pi_{\varphi}(\mathfrak{A}(K,r,\Gamma))\Omega}$ of $\pi_{\tilde{\varphi}}(\overline{A})$ to $\pi_{\varphi}(\mathfrak{A}(K,r,\Gamma))\Omega$ equals to $\pi_{\varphi}(A)$.

Conversely, let $\tilde{\varphi}$ be a state on $\mathfrak{A}(K_s,r_s,\Gamma_s)$ such that

$$(h_1, h_2)_s = \tilde{\varphi}(B(h_1)^*B(h_2)) + \tilde{\varphi}(B(h_2)B(h_1)^*) \quad (h_1, h_2 \in K_s),$$

and let α be a $*$-homomorphism such that

$$\alpha(B(f)) = B(f + N_s) \quad (f \in K).$$

Then $\varphi = \tilde{\varphi} \circ \alpha$ is a state on $\mathfrak{A}(K,r,\Gamma)$ such that

$$(f_1, f_2)_s = \varphi(B(f_1)^* B(f_2)) + \varphi(B(f_2)B(f_1)^*) \quad (f_1, f_2 \in K).$$

We may identify $\mathfrak{H}_{\tilde{\varphi}} = \mathfrak{H}_\varphi$, and $\Omega_{\tilde{\varphi}} = \Omega_\varphi$. But the domain of $\pi_{\tilde{\varphi}}$ is different from that of π_φ. If $\tilde{\varphi}$ is a quasifree state on $\mathfrak{A}(K_s,r_s,\Gamma_s)$, then φ is also a quasifree state on $\mathfrak{A}(K,r,\Gamma)$.

Therefore, this implies that every representation of $\mathfrak{A}(K,r,\Gamma)$ with respect to a quasifree state φ is almost equivalent to that of $\mathfrak{A}(K_s,r_s,\Gamma_s)$ with respect to a quasifree state $\tilde{\varphi}$.

3^o) Next we will summarize some main results in [1] and [2].

Theorem. Keep the notations in 1^o) and 2^o).

(i) There exists a one-to-one correspondense between quasifree states φ on $\mathfrak{A}(K,r,\Gamma)$ and hermitian forms s on $K \times K$ by the equality (4).

(ii) Any quasifree state on $\mathfrak{A}(K,r,\Gamma)$ is regular.

(iii) Let φ be a quasifree state on $\mathfrak{A}(K,r,\Gamma)$. If the associated operator S on K_s does not have an eigenvalue 0, then Ω_φ is a cyclic and separating vector for $\mathfrak{R}_\varphi$.

It follows from (iii) that there exist a modular operator Δ and a modular conjugation J on $\mathfrak{H}_\varphi$ such that

$$(6) \qquad J\,\mathfrak{R}_\varphi J = \mathfrak{R}_\varphi''' \quad \text{and} \quad \Delta^{it}\mathfrak{R}_\varphi\Delta^{-it} = \mathfrak{R}_\varphi \quad \text{for all} \quad t \in \mathbb{R}.$$

The aim of this paper is to show that Δ and J coincide with Δ_u and J_u which are a modular operator and a modular conjugation constructed by the unbounded CCR algebra $\pi_\varphi(\mathfrak{A}(K,r,\Gamma))$, respectively. Further we show that the same equalities for $\pi_\varphi(\mathfrak{A}(K,r,\Gamma))$ as in (6) hold. To prove those we need a few more notations which are used in [1] and [2] for the proof of the above theorem.

Keep the notations in 1^0) and 2^0) and put $K_s' = K_s \oplus K_s$, $r_s'(f_1\oplus g_1,\ f_2\oplus g_2) = r_s(f_1,\ f_2) - r_s(g_1,\ g_2)$, $\Gamma_s' = \Gamma_s \oplus \Gamma_s$, and

$$\begin{aligned}(f_1\oplus g_1,\ f_2\oplus g_2)_s' = (f_1,\ f_2)_s &+ (g_1,\ g_2)_s \\ &+ 2(f_1,\ S^{1/2}(1 - S)^{1/2}g_2)_s \\ &+ 2(g_1,\ S^{1/2}(1 - S)^{1/2}f_2)_s,\end{aligned}$$

respectively. Then the triplet (K_s',r_s',Γ_s') and the positive semidefinite form $(\cdot,\cdot)_s'$ satisfy the same conditions as (K,r,Γ) and $(\cdot,\cdot)_s$, respectively. In 2^0), the triplet (K_s,r_s,Γ_s) and a inner product $(\cdot,\cdot)_s$ on K_s are induced from (K,r,Γ) and a positive semidefinite form $(\cdot,\cdot)_s$ on K, respectively. By the same way, we can construct a triplet $(\hat{K}_s,\hat{r}_s,\hat{\Gamma}_s)$ and a inner product $(\cdot,\cdot)_s^\wedge$ on $\hat{K}_s$ from (K_s',r_s',Γ_s') and $(\cdot,\cdot)_s'$ on K_s', respectively. Further we can define operators $\hat{R}_s$ and Π_s and a positive semidefinite form $\Pi_s(\cdot,\cdot)$ which satisfy the following equalities which correspond to (5):

$$\hat{r}_s(h_1, h_2) = (h_1, \hat{R}_s h_2)^\wedge_s,$$

$$\pi_s = \frac{1}{2}(1 + \hat{R}_s),$$

$$\pi_s(h_1, h_2) = (h_1, \pi_s h_2)^\wedge_s,$$

$$0 \le \pi_s \le 1, \quad \hat{\Gamma}_s \pi_s \hat{\Gamma}_s = 1 - \pi_s.$$

For the proof of the theorem in 3^0) a construction of one more triplet $(\tilde{K}, \tilde{r}, \tilde{\Gamma})$ is needed. But it is not needed for our purpose, so we will not mention further results in [1] and [2] except the following proposition.

Proposition. Let φ be a quasifree state on a self-dual CCR algebra $\mathfrak{A}(K,r,\Gamma)$ and keep the notations in 2^0) and 3^0). We denote by α the canonical $*$-homomorphism from $\mathfrak{A}(K,r,\Gamma)$ into $\mathfrak{A}(\hat{K}_s,\hat{r}_s,\hat{\Gamma}_s)$ defined by (3). Then there exists a quasifree (precisely, Fock type (c.f. [1],[2])) state $\hat{\varphi}$ on $\mathfrak{A}(\hat{K}_s,\hat{r}_s,\hat{\Gamma}_s)$ such that $\hat{\varphi} \circ \alpha = \varphi$. Furthermore, if the operator S does not have an eigenvalue 0, then the cyclic vector $\Omega_{\hat{\varphi}}$ is also cyclic for $\pi_{\hat{\varphi}}(\alpha(\mathfrak{A}(K,r,\Gamma)))$, where $(\mathfrak{H}_{\hat{\varphi}},\pi_{\hat{\varphi}},\Omega_{\hat{\varphi}})$ denotes the GNS-representation of $\mathfrak{A}(\hat{K}_s,\hat{r}_s,\hat{\Gamma}_s)$ with respect to $\hat{\varphi}$.

The above proposition implies that we may identify the GNS-representation spaces $(\mathfrak{H}_{\varphi},\Omega_{\varphi})$ of $\mathfrak{A}(K,r,\Gamma)$ with respect to φ and $(\mathfrak{H}_{\hat{\varphi}},\Omega_{\hat{\varphi}})$ of $\mathfrak{A}(\hat{K}_s,\hat{r}_s,\hat{\Gamma}_s)$ with respect to $\hat{\varphi}$. But we must remark that the domain of $\pi_{\varphi}(\mathfrak{A}(K,r,\Gamma))$ is strictly included that of $\pi_{\hat{\varphi}}(\mathfrak{A}(\hat{K}_s,\hat{r}_s,\hat{\Gamma}_s))$ in general.

3. A Generalization of the Tomita-Takesaki Theorem

In this section, we prove the Tomita-Takesaki theorem for an unbounded operator algebra given by the GNS-representation of a CCR algebra with respect to a quasifree state in the self-dual formalism.

Let φ be a quasifree state on $\mathfrak{A}(K,r,\Gamma)$. Suppose that the associated operator S on K_s does not have an eigenvalue 0. If we put $H_s = \log(S(1-S)^{-1})$, then we can uniquely construct a one-parameter group $\{\tau(\exp itH_s)\}_{t\in\mathbb{R}}$ of Bogoliubov $*$-automorphisms (c.f. (3)) and a one-parameter unitary group $\{T_s(\exp itH_s)\}_{t\in\mathbb{R}}$ such that

$$\tau(\exp itH_s)(B(f)) = B(e^{itH_s}f) \quad (f \in K_s),$$

$$T_s(\exp itH_s)\pi_{\tilde{\varphi}}(A)\Omega_{\tilde{\varphi}} = \pi_{\tilde{\varphi}}(\tau(\exp itH_s)A)\Omega_{\tilde{\varphi}} \quad (A \in \mathfrak{A}(K_s,r_s,\Gamma_s)),$$

respectively. Let Θ be an infinitesimal generator of $T_s(\exp itH_s)$. Then we have

$$e^{it\Theta} = T_s(\exp itH_s).$$

If we define a mapping ω_0 on K_s' by the equation

$$\omega_0(f\oplus g) = \Gamma_s g\oplus\Gamma_s f \quad \text{for every} \quad f,\ g \in K_s,$$

then we can canonically construct a mapping ω on $\hat{K}_s$ from ω_0 on K_s'. Let $\tau(\omega)$ be a mapping on $\mathfrak{A}(\hat{K}_s,\hat{r}_s,\hat{\Gamma}_s)$ defined by

$$\tau(\omega)\Big(\sum_i c_i B(h_1^i)\cdots B(h_{n_i}^i)\Big) = \sum_i \bar{c}_i B(\omega h_1^i)\cdots B(\omega h_{n_i}^i) \quad (h_j^i \in \hat{K}_s).$$

Then there uniquely exists an antiunitary involution $T_{\pi_s}(\omega)$ on $\mathfrak{H}_{\hat{\varphi}}$ determined by

$$T_{\pi_s}(\omega)\pi_{\hat{\varphi}}(A)\Omega_{\hat{\varphi}} = \pi_{\hat{\varphi}}(\tau(\omega)A)\Omega_{\hat{\varphi}} \quad (A \in \mathfrak{A}(\hat{K}_s,\hat{r}_s,\hat{\Gamma}_s)).$$

For a modular operator and a modular conjugation constructed by the unbounded CCR algebra $\pi_\varphi(\mathfrak{A}(K,r,\Gamma))$, we have the following result.

Lemma 3.1. Let φ be a quasifree state on $\mathfrak{A}(K,r,\Gamma)$. If the associated operator S on K_s does not have an eigenvalue 0, then the mapping

$$S_u : A\Omega_\varphi \longrightarrow A^*\Omega_\varphi \qquad (A \in \pi_\varphi(\mathfrak{A}(K,r,\Gamma)))$$

is closable and the polar decomposition of $\bar{S}_u$ is given by the following

$$\overline{S}_u = J_u \Delta_u^{1/2},$$

where $J_u = T_{\Pi_s}(\omega)$, and $\Delta_u = e^{-\theta}$.

Proof. For any $f \in \mathcal{D}(S^{-1/2})$, we have

$$\|\pi_{\hat{\varphi}}[B((\{-(1-S)^{1/2}S^{-1/2}\Gamma_s f\}\oplus\Gamma_s f)^{\wedge})]\Omega_{\hat{\varphi}}\|^2$$

$$= \varphi_{\hat{\varphi}}(B(h)^*B(h)) \quad (\text{where } h = ((\{-(1-S)^{1/2}S^{-1/2}\Gamma_s f\}\oplus\Gamma_s f)^{\wedge}))$$

$$= \Pi_s(h, h)$$

$$= \frac{1}{2}\{(h, h)_s^{\wedge} + \hat{\Gamma}_s(h, h)\}$$

$$= \frac{1}{2}\{\|-(1-S)^{1/2}\Gamma_s f+(1-S)^{1/2}\Gamma_s f\|_s^2 + \|-(1-S)S^{-1/2}\Gamma_s f+S^{1/2}\Gamma_s f\|_s^2$$

$$+ \|-(1-S)^{1/2}\Gamma_s f+(1-S)^{1/2}\Gamma_s f\|_s^2 - \|-(1-S)S^{-1/2}\Gamma_s f+S^{1/2}\Gamma_s f\|_s^2\}$$

$$= 0.$$

Hence we have

$$\pi_{\hat{\varphi}}[B((\{-(1-S)^{1/2}S^{-1/2}\Gamma_s f\}\oplus\Gamma_s f)^{\wedge})]\Omega_{\hat{\varphi}} = 0.$$

From this equality, we have

$$T_{\Pi_s}(\omega)\pi_{\hat{\varphi}}[B((f\oplus 0)^\wedge)]\Omega_{\hat{\varphi}} = \pi_{\hat{\varphi}}[B((0\oplus\Gamma_s f)^\wedge)]\Omega_{\hat{\varphi}}$$

$$= \pi_{\hat{\varphi}}[B(((1-S)^{1/2}S^{-1/2}\Gamma_s f\oplus 0)^\wedge)]\Omega_{\hat{\varphi}}$$

$$= e^{-\theta/2}\pi_{\hat{\varphi}}[B((f\oplus 0)^\wedge)]^*\Omega_{\hat{\varphi}}.$$

Note that, for $f, g \in \mathcal{D}(S^{-1/2})$,

$$B((0\oplus f)^\wedge)B((g\oplus 0)^\wedge) = B((g\oplus 0)^\wedge)B((0\oplus f)^\wedge) + \hat{\Gamma}_s(\hat{\Gamma}_s((0\oplus f)^\wedge), (g\oplus 0)^\wedge)$$

$$= B((g\oplus 0)^\wedge)B((0\oplus f)^\wedge).$$

Hence we have, for $f, g \in \mathcal{D}(S^{-1/2})$,

$$T_{\Pi_s}(\omega)\pi_{\hat{\varphi}}[B((f\oplus 0)^\wedge)]\pi_{\hat{\varphi}}[B((g\oplus 0)^\wedge)]\Omega_{\hat{\varphi}}$$

$$= T_{\Pi_s}(\omega)\pi_{\hat{\varphi}}[B((f\oplus 0)^\wedge)]T_{\Pi_s}(\omega)T_{\Pi_s}(\omega)\pi_{\hat{\varphi}}[B((g\oplus 0)^\wedge)]\Omega_{\hat{\varphi}}$$

$$= T_{\Pi_s}(\omega)\pi_{\hat{\varphi}}[B((f\oplus 0)^\wedge)]T_{\Pi_s}(\omega)\pi_{\hat{\varphi}}[B(((1-S)^{1/2}S^{-1/2}\Gamma_s g\oplus 0)^\wedge)]\Omega_{\hat{\varphi}}$$

$$= \pi_{\hat{\varphi}}[B((0\oplus\Gamma_s f)^\wedge)]\pi_{\hat{\varphi}}[B(((1-S)^{1/2}S^{-1/2}\Gamma_s g\oplus 0)^\wedge)]\Omega_{\hat{\varphi}}$$

$$= \pi_{\hat{\varphi}}[B(((1-S)^{1/2}S^{-1/2}\Gamma_s g\oplus 0)^\wedge)]\pi_{\hat{\varphi}}[B((0\oplus\Gamma_s f)^\wedge)]\Omega_{\hat{\varphi}}$$

$$= \pi_{\hat{\varphi}}[B(((1-S)^{1/2}S^{-1/2}\Gamma_s g\oplus 0)^\wedge)]\pi_{\hat{\varphi}}[B(((1-S)^{1/2}S^{-1/2}\Gamma_s f\oplus 0)^\wedge)]\Omega_{\hat{\varphi}}$$

$$= e^{-\theta/2}\pi_{\hat{\varphi}}[B((g\oplus 0)^\wedge)]^*\pi_{\hat{\varphi}}[B((f\oplus 0)^\wedge)]^*\Omega_{\hat{\varphi}}$$

$$= e^{-\theta/2}\left(\pi_{\hat{\varphi}}[B((f\oplus 0)^\wedge)]\pi_{\hat{\varphi}}[B((g\oplus 0)^\wedge)]\right)^*\Omega_{\hat{\varphi}}.$$

Put

$$A_0 = \sum_i c_i \pi_{\hat{\varphi}}[B((f_1^i\oplus 0)^\wedge)]\cdots\pi_{\hat{\varphi}}[B((f_{n_i}^i\oplus 0)^\wedge)] \quad (f_j^i \in \mathcal{D}(S^{-1/2})).$$

By the same way, we have

$$T_{\pi_s}(\omega)A_0\Omega_{\hat{\varphi}} = e^{-\theta/2}A_0^*\Omega_{\hat{\varphi}}.$$

Thus we have

$$(e^{-\theta/2}\psi, A_0^*\Omega_{\hat{\varphi}}) = (\psi, T_{\pi_s}(\omega)A_0\Omega_{\hat{\varphi}}) \quad \text{for all} \quad \psi \in \mathcal{D}(e^{-\theta/2}).$$

Put

$$A_1 = \sum_i c_i \pi_{\hat{\varphi}}[B((g_1^i\oplus 0)^\wedge)]\cdots\pi_{\hat{\varphi}}[B((g_{n_i}^i\oplus 0)^\wedge)]\Omega_{\hat{\varphi}} \quad (g_j^i \in K_s).$$

Since $\hat{\varphi}$ is a quasifree state and $((f\oplus 0)^\wedge, (f\oplus 0)^\wedge)_s^\wedge = (f, f)_s$ for all $f \in K_s$, there exist sequences $\{f_{j,m}^i\}_{m=1}^\infty$ $(f_{j,m}^i \in \mathcal{D}(S^{-1/2}))$ such that

$$\sum_i c_i \pi_{\hat\varphi}[B((f^i_{1,m}\oplus 0)^\wedge)]\cdots\pi_{\hat\varphi}[B((f^i_{n_i,m}\oplus 0)^\wedge)]\Omega_{\hat\varphi}$$

converges to A_1 in $\mathfrak{H}_{\hat\varphi}$. Thus we have

$$(e^{-\theta/2}\psi,\ A_1^*\Omega_{\hat\varphi}) = (\psi,\ T_{\pi_s}(\omega)A_1\Omega_{\hat\varphi}) \quad \text{for all} \quad \psi \in \mathcal{D}(e^{-\theta/2}).$$

This implies that

$$A_1^*\Omega_{\hat\varphi} \in \mathcal{D}(e^{-\theta/2})$$

and so

$$T_{\pi_s}(\omega)A_1\Omega_{\hat\varphi} = e^{-\theta/2}A_1^*\Omega_{\hat\varphi}.$$

By replacing A_1 with A_1^*, we have

$$T_{\pi_s}(\omega)A_1^*\Omega_{\hat\varphi} = e^{-\theta/2}A_1\Omega_{\hat\varphi}.$$

Hence we have, for all $A_2 \in \pi_{\tilde\varphi}(\mathfrak{A}(K_s, r_s, \Gamma_s))$,

$$T_{\pi_s}(\omega)A_2^*\Omega_{\tilde\varphi} = e^{-\theta/2}\ A_2\Omega_{\tilde\varphi}.$$

From this, it follows that

$$T_{\pi_s}(\omega)e^{-\theta/2}A_2\Omega_{\tilde{\varphi}} = A_2^*\Omega_{\tilde{\varphi}}.$$

Moreover we have, for all $A \in \pi_\varphi(\mathfrak{A}(K,r,\Gamma))$,

$$T_{\pi_s}(\omega)e^{-\theta/2}A\Omega_\varphi = A^*\Omega_\varphi.$$

Let $S_{u,2}$ be the mapping on $\mathfrak{H}_{\tilde{\varphi}}$ defined by

$$S_{u,2}(A_2\Omega_{\tilde{\varphi}}) = A_2^*\Omega_{\tilde{\varphi}} \quad \text{for every} \quad A_2 \in \pi_{\tilde{\varphi}}(\mathfrak{A}(K_s,r_s,\Gamma_s)).$$

Since $S_{u,2} \subset T_{\pi_s}(\omega)e^{-\theta/2}$, $S_{u,2}$ is closable and

$$S_{u,2} = T_{\pi_s}(\omega)(e^{-\theta/2}|_{\mathfrak{D}(S_{u,2})}).$$

Hence we have

$$\overline{S}_{u,2} = T_{\pi_s}(\omega)\overline{(e^{-\theta/2}|_{\mathfrak{D}(S_{u,2})})}.$$

We shall show that $\overline{(e^{-\theta/2}|_{\mathfrak{D}(S_{u,2})})}$ is selfadjoint. At first, note that $(e^{-\theta/2}|_{\mathfrak{D}(S_{u,2})})$ is symmetric. Moreover the range of $(e^{-\theta/2}|_{\mathfrak{D}(S_{u,2})}) \pm i1$ is dense in $\mathfrak{H}_{\tilde{\varphi}}$, because

$$\text{Range}\Big((e^{-\theta/2}|_{\mathfrak{D}(S_{u,2})}) \pm i1\Big)$$

$$\supset \mathrm{Range}\left((e^{-\theta/2} \pm i1)|_{\pi_{\tilde{\varphi}}(\mathfrak{A}(\mathcal{D}(S^{-1/2}),r_s,\Gamma_s))\Omega_{\tilde{\varphi}}}\right)$$

$$= \pi_{\tilde{\varphi}}(\mathfrak{A}(K_s,r_s,\Gamma_s))\Omega_{\tilde{\varphi}}.$$

This implies that $(e^{-\theta/2}|_{\mathcal{D}(S_{u,2})})$ is essentially selfadjoint and so $\overline{(e^{-\theta/2}|_{\mathcal{D}(S_{u,2})})}$ is selfadjoint. Thus we have

$$\overline{(e^{-\theta/2}|_{\mathcal{D}(S_{u,2})})} = e^{-\theta/2}.$$

Further, since $\overline{(e^{-\theta/2}|_{\mathcal{D}(S_{u,2})})} = \overline{(e^{-\theta/2}|_{\mathcal{D}(S_u)})}$, we have

$$\overline{(e^{-\theta/2}|_{\mathcal{D}(S_u)})} = e^{-\theta/2}.$$

By the uniqueness of the polar decomposition, this completes the proof.

In the following proposition, we show that Δ_u and J_u coincide with a modular operator and a modular conjugation constructed by R_φ, respectively.

Proposition 3.2. Let φ be a quasifree state on $\mathfrak{A}(K,r,\Gamma)$. If the associated operator S on K_s does not have an eigenvalue 0, then the mapping

$$S_b : B\Omega_\varphi \longrightarrow B^*\Omega_\varphi \qquad (B \in R_\varphi)$$

is closable, and $\bar{S}_b = \bar{S}_u$.

Proof. In order to prove Proposition 3.2, it is sufficient to show that the graph $G(\bar{S}_b)$ of $\bar{S}_b$ coincides with that of $\bar{S}_u$.

First, we shall show that $G(S_b) \subset G(\bar{S}_u)$. At first, note that $G(S_b) \subset \overline{G(S_b|_{\mathfrak{R}})}$ (where $\mathfrak{R}$ is the set of all polynomials of elements of $\{W(f) : f \in \mathrm{Re}K\}$). Since

$$W_\varphi(f)\Omega_\varphi = \sum_{n=0}^{\infty}(n!)^{-1}i^n\pi_\varphi(B(f))^n\Omega_\varphi \quad (f \in \mathrm{Re}K),$$

by the Weyl-Segal relations, it follows that $G(S_b|_{\mathfrak{R}}) \subset G(\bar{S}_u)$. Thus we have $G(S_b) \subset G(\bar{S}_u)$. This implies that S_b is closable and $G(\bar{S}_b) \subset G(\bar{S}_u)$.

Next, we prove that $G(S_u) \subset G(\bar{S}_b)$. To do this, we note that, for each $\xi \in \pi_\varphi(\mathfrak{A}(K,r,\Gamma))\Omega_\varphi$,

$$\xi = \sum_i c_i\pi_\varphi(B(f_1^i))\cdots\pi_\varphi(B(f_{n_i}^i))\Omega_\varphi \quad (f_j^i \in \mathrm{Re}K,\ c_i \in \mathbb{C}).$$

Moreover

$$\frac{(W_\varphi(t_1f_1)-W_\varphi(0))\cdots(W_\varphi(t_mf_m)-W_\varphi(0))\Omega_\varphi}{(it_1)^{-1}\cdots(it_m)^{-1}}$$

tends to $\pi_\varphi(B(f_1))\cdots\pi_\varphi(B(f_m))\Omega_\varphi$ in $\mathfrak{H}_\varphi$ as t_j tends to 0 in $\mathbb{R}$

for each $j = 1,\cdots,m$. Thus we have $G(S_u) \subset G(\overline{S}_b)$ and so $G(\overline{S}_u) \subset G(\overline{S}_b)$. This completes the proof.

We now introduce the commutant of $\pi_\varphi(\mathfrak{A}(K,r,\Gamma))$.

Definition 3.3. Suppose that φ is a state on $\mathfrak{A}(K,r,\Gamma)$. The commutant of $\pi_\varphi(\mathfrak{A}(K,r,\Gamma))$, denoted by $\pi_\varphi(\mathfrak{A}(K,r,\Gamma))'''$, consists of all bounded operators C on $\mathfrak{H}_\varphi$ such that

$$(\psi_1, C\pi_\varphi(A)\psi_2) = (\pi_\varphi(A^*)\psi_1, C\psi_2)$$

for all $\psi_1, \psi_2 \in \pi_\varphi(\mathfrak{A}(K,r,\Gamma))\Omega_\varphi$ and $A \in \mathfrak{A}(K,r,\Gamma)$.

Lemma 3.4. Let φ be a quasifree state on $\mathfrak{A}(K,r,\Gamma)$. If the associated operator S on K_s does not have an eigenvalue 0, then $\pi_\varphi(\mathfrak{A}(K,r,\Gamma))''' = \mathfrak{R}_\varphi'''$.

Proof. If $C \in \pi_\varphi(\mathfrak{A}(K,r,\Gamma))'''$, then we have

$$(\pi_\varphi(B(g))^n\xi_1, C\xi_2) = (\xi_1, C\pi_\varphi(B(g))^n\xi_2)$$

for all $\xi_1, \xi_2 \in \pi_\varphi(\mathfrak{A}(K,r,\Gamma))\Omega_\varphi$ and $g \in \mathrm{Re}K$. Thus we have

$$(i^n\pi_\varphi(B(-g))^n\xi_1, C\xi_2) = (\xi_1, Ci^n\pi_\varphi(B(g))^n\xi_2),$$

and so

$$(W_\varphi(-g)\xi_1, C\xi_2) = (\xi_1, CW_\varphi(g)\xi_2).$$

Hence we have $W_\varphi(g)C = CW_\varphi(g)$. This implies that $C \in \mathfrak{R}_\varphi'''$.

Let $\mathfrak{R}_{\hat{\varphi}}$ be the von Neumann algebra generated by $\{W_{\hat{\varphi}}((\overline{f}\oplus 0)^\wedge)$: $f \in \mathrm{Re}K\}$. Then we have

$$\mathfrak{R}_\varphi''' = J_u\mathfrak{R}_\varphi J_u = T_{\pi_s}(\omega)\mathfrak{R}_\varphi T_{\pi_s}(\omega) = T_{\pi_s}(\omega)\mathfrak{R}_{\hat{\varphi}}T_{\pi_s}(\omega),$$

by the Tomita-Takesaki theorem and Proposition 3.2. Now, for any $f \in \mathrm{Re}K$, $g \in K$, and ψ_1, $\psi_2 \in \pi_\varphi(\mathfrak{A}(K,r,\Gamma))\Omega_\varphi$, we have

$$\left(\pi_\varphi(B(g)^*)\psi_1,\ T_{\pi_s}(\omega)W_{\hat{\varphi}}((\overline{f}\oplus 0)^\wedge)T_{\pi_s}(\omega)\psi_2\right)$$

$$= \lim_{n\to\infty}\left(\pi_\varphi(B(g)^*)\psi_1,\ T_{\pi_s}(\omega)\left(\sum_{m=1}^{n}(m!)^{-1}i^m\pi_{\hat{\varphi}}(B((\overline{f}\oplus 0)^\wedge))^m\right)T_{\pi_s}(\omega)\psi_2\right)$$

$$= \lim_{n\to\infty}\left(\pi_{\hat{\varphi}}(B((\overline{\Gamma g}\oplus 0)^\wedge))\psi_1,\ \left(\sum_{m=1}^{n}(m!)^{-1}(-i)^m\pi_{\hat{\varphi}}(B((0\oplus\overline{f})^\wedge))^m\right)\psi_2\right)$$

$$= \lim_{n\to\infty}\left(\psi_1,\ \left(\sum_{m=1}^{n}(m!)^{-1}(-i)^m\pi_{\hat{\varphi}}(B((0\oplus\overline{f})^\wedge))^m\right)\pi_{\hat{\varphi}}(B((\overline{g}\oplus 0)^\wedge))\psi_2\right)$$

$$= \lim_{n\to\infty}\left(\psi_1,\ T_{\pi_s}(\omega)\left(\sum_{m=1}^{n}(m!)^{-1}i^m\pi_{\hat{\varphi}}(B((\overline{f}\oplus 0)^\wedge))^m\right)T_{\pi_s}(\omega)\pi_\varphi(B(g))\psi_2\right)$$

$$= \left(\psi_1,\ T_{\pi_s}(\omega)W((\overline{f}\oplus 0)^\wedge)T_{\pi_s}(\omega)\pi_\varphi(B(g))\psi_2\right).$$

By the same process, we have, for any $A \in \mathfrak{A}(K,r,\Gamma)$,

$$(\pi_\varphi(A^*)\psi_1,\ T_{\pi_s}(\omega)PT_{\pi_s}(\omega)\psi_2) = (\psi_1,\ T_{\pi_s}(\omega)PT_{\pi_s}(\omega)\pi_\varphi(A)\psi_2),$$

where P is any polynomial generated by $\{W_{\hat{\varphi}}((\overline{f}\oplus 0)^\wedge) : f \in \mathrm{Re}K\}$. Hence, we have, for any $W''' \in \mathfrak{R}'''_\varphi$ and $A \in \mathfrak{A}(K,r,\Gamma)$,

$$(\pi_\varphi(A^*)\psi_1,\ W'''\psi_2) = (\psi_1,\ W'''\pi_\varphi(A)\psi_2).$$

This implies that $\pi_\varphi(\mathfrak{A}(K,r,\Gamma))''' \supset \mathfrak{R}'''_\varphi$. This completes the proof.

Finally, we prove the Tomita-Takesaki theorem for an unbounded operator algebra given by the GNS-representation of a CCR algebra with respect to a quasifree state in the self-dual formalism.

Theorem 3.5. Let φ be a quasifree state on $\mathfrak{A}(K,r,\Gamma)$. If the associated operator S on K_s does not have an eigenvalue 0, then

$$\overline{S}_u = \overline{S}_b,\ \text{and}\ \ J_u(\pi_\varphi(\mathfrak{A}(K,r,\Gamma))''')'''J_u = \pi_\varphi(\mathfrak{A}(K,r,\Gamma))'''.$$

Moreover, if e^{itH_s} maps $\overline{K}$ onto $\overline{K}$, then

$$\Delta_u^{it}\pi_\varphi(\mathfrak{A}(K,r,\Gamma))\Delta_u^{-it} = \pi_\varphi(\mathfrak{A}(K,r,\Gamma)) \quad \text{for all} \quad t \in \mathbb{R}.$$

Proof. The last equality follows from the fact that the mapping $A \longrightarrow \Delta_u^{it}A$ is a Bogoliubov $*$-automorphism on $\pi_\varphi(\mathfrak{A}(K,r,\Gamma))$.

Remark. Let K be the set of all rapidly decreasing functions on $\mathbb{R}$ as in [5]. In this case, they proved the generalized Tomita-Takesaki theorem for H_s with a trace class operator $e^{-\theta}$. Thus our result is a generalization of [5].

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