A Banach algebra which is an ideal in the second dual space

By

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1. Introduction

The second dual space A^{**} of a Banach algebra A can also be considered as a Banach algebra by the use of Arens muliplication [Arens, 1]. When A is embedded in A^{**} by the canonical mapping, A is only a subalgabra of A^{**} but is not an ideal in A^{**} in general. When does A^{**} contain A as an ideal? It is well-known that when A is a C*-algebra, A is a dual C*-algebra if and only if A is a two-sided ideal in A^{**}. Recently many authors obtained other characterizations in this case. In this paper, we shall consider the above problem for general Banach algebras. In §2, we shall show that a Banach algebra which is an ideal in the second dual space is characterized by the weak compactness of left or right multiplications on A. In §3, we shall show that for a group algebra A of a locally compact topological group G, A is a two-sided ideal in A^{**} if and only if G is compact. Moreover we shall show analogous result for a certain subalgebra in A^{**}.

2. Preliminaries

Let A be a Banach algebra. Denote by A^* the dual space of A, and denote by A^{**} the second dual space of A. Throughout we denote by π the canonical embedding of A into A^{**}. Let $x, y \in A, f \in A^*$ and $F, G \in A^{**}$. Then we define the following functions:

$$(f, x) \longrightarrow f \circ x : A^* \times A \longrightarrow A^* \quad \text{where } (f \circ x)(y) = f(xy),$$

$$(F, f) \longrightarrow F \circ f : A^{**} \times A^* \longrightarrow A^* \quad \text{where } (F \circ f)(x) = F(f \circ x),$$

and

$$(F, G) \longrightarrow F \circ G : A^{**} \times A^{**} \longrightarrow A^{**} \quad \text{where } (F \circ G)(f) = F(G \circ f).$$

The mutiplication $F\circ G$ thus defined on A^{**}, called Arens multiplication, extends the multiplication on A, is weak *-continuous in F for fixed G, and makes A^{**} into a Banach algebra. $\pi(A)$ is a closed subalgebra of A^{**}. Similarly, we define the following functions:

$$(x, f)\longrightarrow x*f\colon A\times A^{*}\longrightarrow A^{*}$$
$$\text{where } x*f(y)=f(yx),$$
$$(f, F)\longrightarrow f*F\colon A^{*}\times A^{**}\longrightarrow A^{*}$$
$$\text{where } (f*F)(x)=F(x*f),$$

and

$$(F, G)\longrightarrow F*G\colon A^{**}\times A^{**}\longrightarrow A^{**}$$
$$\text{where } F*G(f)=G(f*F).$$

Again this multiplication $F*G$ makes A^{**} into a Banach algebra. The two Arens multiplications agree if one of the factors is in A.

3. Characterizations

Let A be a Banach algebra. Denote by L_a(resp. R_a) the left (resp. right) multiplication on A. A operator T on a Banach space X is called weakly compact on X if for every bounded net $\{a_\alpha\}\subset X$, there exists a subnet $\{a_\beta\}$ of $\{a_\alpha\}$ and an element $a\in X$ such that $T(a_\beta)\longrightarrow a$ weakly. We have the following characterization.

PROPOSITION 3. 1. *Let A be a Banach algebra and $a\in A$. Then the following two statements are equivalent.*

1) *L_a (resp. R_a) is a weakly compact operator on A.*

2) $\pi(a)\circ A^{**}\subset\pi(A)$ (*resp.* $A^{**}\circ\pi(a)\subset\pi(A)$).

PROOF. For each $F\in A^{**}$, there exists a bounded net $\{a_\alpha\}\subset A$ such that F is a weak *-limit of $\pi(a_\alpha)$. Then $\pi(a)\circ F=\text{w}^*\text{-limit } \pi(a)\circ\pi(a_\alpha)$.
If there exists a subnet $\{a_\beta\}$ of $\{a_\alpha\}$ and an element $b\in A$ such that b is a weak limit of $a\,a_\beta$. Then

$$\pi(b)=\text{w}^*\text{-}\lim_{\beta}\pi(aa_\beta)=\pi(a)\circ F$$

Thus the implication 1) $\Rightarrow$ 2) is proved.
Next we shall show the converse implication 2) $\Rightarrow$ 1). Let $\{a_\alpha\}$ be a net such that $\|a_\alpha\|\leqq 1$. From the Alaoglu's theorem there exists a subnet $\{a_\beta\}\subset\{a_\alpha\}$ and $F\in A^{**}$ such that F is a weak *-limit of $\pi(a_\beta)$. Since $\pi(a)\circ A^{**}\subset\pi(A)$, there exists an element $b\in A$ such that $\pi(b)=\pi(a)\circ F$. Then $b=\text{weak}\lim_{\beta} L_a(a_\beta)$. Hence L_a is a weakly compact operator on A. We can prove for R_a in a similar way. This completes the proof.

COROLLARLY 3. 2. *Let A be a Banach algebra. Then $\pi(A)$ is a two-sided ideal in A^{**} if and only if L_a and R_a are weakly compact operator for each $a\in A$.*

Now we define bounded linear operators on A^* by the following manner:

$$A^* \longrightarrow A^*$$

$$T_x: f \longrightarrow f \circ x \qquad (x \in A),$$

$$S_x: f \longrightarrow x * f \qquad (x \in A).$$

Moreover we use the following notations:

$$I_r(A) = \{F \in A^{**};\ \pi(A) \circ F \subset \pi(A)\},$$

$$I_l(A) = \{F \in A^{**};\ F \circ \pi(A) \subset \pi(A)\}.$$

Then we have the following proposition.

PROPOSITION 3. 3. *Let A be a Banach algebra and B be a subset of A^{**} such that $B \supset \pi(A)$. Then $B \subset I_r(A)$ (resp. $B \subset I_l(A)$) if and only if T_x (resp. S_x) is a $\sigma(A^*, B)$-compact operator for each $x \in A$.*

PROOF. Suppose that T_x is a $\sigma(A^*, B)$-compact operator for each $x \in A$. Suppose $B \not\subset I_r(A)$. Then there exists $x \in A$ and $F \in B$ such that $\pi(x) \circ F \notin \pi(A)$.

From the Hahn-Banach theorem there exists $G \in A^{***}$ (the third dual space of A) such that $\|G\| = 1$, $G(\pi(A)) = (0)$ and $G(\pi(x) \circ F) = 1$. Then by the Goldstein's theorem there exists a net $\{f_\alpha\} \subset A^*$ such that $G = \text{w}^*\text{-}\lim_\alpha \rho(f_\alpha)$ and $\|f_\alpha\| \leqq 1$ where ρ denotes the canonical mapping of A^* into A^{***}. From the assumption we can choose a subnet $\{f_\beta\}$ of $\{f_\alpha\}$ and $f \in A^*$ such that $T_x(f_\beta)$ converges to f in $\sigma(A^*, B)$-topology.

Then
$$\begin{aligned} F(f) &= \lim_\beta F(f_\beta \circ x) = \lim_\beta F \circ f_\beta(x) \\ &= \lim_\beta \pi(x) \circ F(f_\beta) = \lim_\beta \rho(f_\beta)(\pi(x) \circ F) \\ &= G(\pi(x) \circ F) = 1. \end{aligned}$$

But for each $y \in A$,
$$\begin{aligned} f(y) &= \pi(y)(x) = \lim_\beta f_\beta(xy) \\ &= \lim_\beta \rho(f_\beta)(\pi(xy)) = G(\pi(xy)) = 0. \end{aligned}$$

This is a contradiction. Thus $B \subset I_r(A)$.

We can prove for the case of $I_l(A)$ similarly.

Conversely suppose that $B \subset I_r(A)$. Let $\{f_\alpha\}$ be a net in A^* such that $\|f_\alpha\| \leqq 1$. By the Alaoglu's theorem there exists a subnet $\{f_\beta\}$ of $\{f_\alpha\}$ and a linear functional $f \in A^*$ such that $\{f_\beta\}$ converges to f in w*-toplolgy. Let $x \in A$ and $F \in A^{**}$. There exists $y \in A$ such that $\pi(x) \circ F = \pi(y)$. Then

$$F(f_\beta \circ x) = \pi(x) \circ F(f_\beta) = f_\beta(y)$$

$$\longrightarrow f(y) = \pi(x) \circ F(f) = F(f \circ x).$$

Hence T_x is a $\sigma(A^*, B)$-compact operator for each $x \in A$. This completes the proof.

COROLLARY 3. 4. *Let A be a Banach algebra. Then $\pi(A)$ is a two-sided ideal in A^{**} if* *and only if T_x and S_x are weakly compact operators for each $x \in A$.*

When A is a B*-algebra, a left (or right) ideal $\pi(A)$ is also a right (or left) ideal in A^{**}. Therefore we have the following corollary.

COROLLARY 3. 5. *If A is a B^*-algebra, the following five statements are equivalent.*

1) *$\pi(A)$ is a two-sided ideal in A^{**}.*
2) *L_x(resp. 2)' R_x) is a weakly compact operator on A for each $x \in A$.*
3) *T_x(resp. 3)' S_x) is a weakly compact operator on A^* for each $x \in A$.*

4. A group algebra which is an ideal in the second dual space

Throughout this section, G will denote a locally compact topological group and μ a left Haar measure on G. In this section, we shall consider the second dual space $L^1(G)^{**}$ of the group algebra $L^1(G)$ of G. Let $L^\infty(G)$ be the space of all essentially bounded functions on G. Let $f \in L^\infty(G)$ and x, $y \in L^1(G)$. We denote by $\otimes$ the convolution of x and y. Then

$$\begin{aligned} S_x(f)(y) = x*f(y) = f(y \otimes x) &= \int_G f(s)\, y \otimes x(s)\, d\mu(s) \\ &= \int_G f(s) \int_G y(t) x(t^{-1} s) d\mu(t) d\mu(s) \\ &= \int_G \{ \int_G f(s) x(t^{-1} s) d\mu(s) y(t) d\mu(t) \} \\ &= \int_G x \tilde{*} f(t) y(t) d\mu(t) \\ &= x * f(y) \end{aligned}$$

where $x \tilde{*} f$ is defined by $x \tilde{*} f(t) = \int_G f(s) x(t^{-1} s) d\mu(s)$.

We have $\quad |x \tilde{*} f(t)| \leqq \|f\|_\infty \|x\|_1$.

Moreover

$$\begin{aligned} |x \tilde{*} f(t) - x \tilde{*} f(s)| &\leqq \int_G |f(r)|\ |x_t(r) - x_s(r)|\, d\mu(r) \\ &\leqq \|f\|_\infty \|x_t - x_s\|_1 \end{aligned}$$

where x_g is defined by $x_g(h) = x(g^{-1}h)$ $(g, h \in G)$.

Therefore $x \tilde{*} f$ is a bounded continuous function and hence $x*f \in L^\infty(G)$. Consequently we may identifiy $x \tilde{*} f$ to the realized function of a bounded linear functional $x*f$. In [Civin, 2] it is shown that for a locally compact abelian group, the following proposition is hold.

PROPOSITION 4. 1. *Let G be a locally compact topological group and μ be a left Haar measure on G. Then π $(L^1(G))$ is a two-sided ideal in $L^1(G)^{**}$ if and only if G is compact.*

PROOF. Since the sufficiency was shown in [Wong, 9], we have only to prove the necessity. Suppose that G is not compact. Then from the regularity of μ, there exists a compact set $C_1 \subset G$ such that $\mu(C_1) > 1$. From the regularity of μ, we have again a compact set $C_2 \subset G - C_1$ such that $\mu(C_2) > 1$.

Repeating this arguments, we have a infinite sequence $\{C_n\}$ of compact sets such that $C_n \cap C_m = \phi$ $(n \neq m)$ and $\mu(C_n) > 1$ for all n. Clearly no compact set contains all C_n. We define $C'_n = \bigcup_{p=1}^{n} C_p$ and $C_\infty = \bigcup_{p=1}^{\infty} C_p$. Let f_n, x and f be a characteristic functions of $C'_n C_1$, C_1 and $C_\infty C_1$ respectively.

Then $f_n, f \in L^\infty(G)$ and $x \in L^1(G)$. By the definition of C'_n and C_∞, $f_n(t)$ converges to $f(t)$ pointwise. Therefore by the Lebesgue's bounded convergence theorem f_n converges to f in the weak $*$-topology in $(L^1(G))^* = L^\infty(G)$. From the assumption of our theorem there exists an element $h \in L^1(G)$ such that $\pi(h) = F * \pi(x)$. Then

$$F(x * f_n) = f_n * F(x) = F * \pi(x)(f_n) = \pi(h)(f_n) = f_n(h)$$

$$\longrightarrow f(h) = \pi(h)(f) = F(x * f).$$

Therefore $x * f$ is a weak limit of $x * f_n$.
Now for any $t \in C_\infty$

$$x \tilde{*} f(t) = \int_G f(ts) x(s) d\mu(s) = \int_{C_1} f(ts) d\mu(s) = \mu(C_1) > 1.$$

Hence $x \tilde{*} f$ does not vanish at infinity.

On the other hand, for any $t \notin C'_n C_1 C_1^{-1}$ (compact set)

$$x \tilde{*} f_n(t) = \int_{C_1} f_n(ts) d\mu(s) = 0.$$

This means that the support of $x \tilde{*} f_n$ is compact.
Hence $x * f_n$ vanishes at infinity. But the class of $x * f$ belongs to the space $C_0(G)$, as an element of $L^\infty(G)$, since $x * f$ is a weak limit of $x * f_n$ and $C_0(G)$ is weakly closed (in fact norm closed). Since $x * f$ is continuous, it vanishes at infinity. This contradiction leads to the fact that G is compact. Thus our proposition is completely proved.

Let $f \in L^\infty(G)$ and $x, y \in L^1(G)$. We shall consider the realization of bounded linear functional $f \circ x$.

$$T_x(f)(y) = \int_G f(s) x \otimes y(s) d\mu(s) = \int_G f(s) \int_G x(st) y(t^{-1}) d\mu(t) d\mu(s)$$

$$= \int_G \{ \int_G f(s) x(st) d\mu(s) \} y(t^{-1}) d\mu(t) = \int_G \Delta(t^{-1}) f \tilde{\circ} x(t^{-1}) y(t^{-1}) d\mu(t)$$

$$= \int_G f \tilde{\circ} x(t) y(t) d\mu(t) = f \circ x(y)$$

where $\Delta(t)$ is a modular function on G and $\widetilde{f\circ x}$ is defined by $\widetilde{f\circ x}(t)=\int_G f(s)x(st^{-1})d\mu(s)$.

We have $|f\circ x(t)|\leqq\|f\|_\infty\Delta(t^{-1})^2\|x\|_1$.

Since $\widetilde{f\circ x}$ is continuous on G, if it is bounded $\widetilde{f\circ x}\in L^\infty(G)$. Then we may identify $\widetilde{f\circ x}$ to the realized function of a bounded linear functional $f\circ x$.

Next we consider the certain subalgebra of the second dual algebra. [Flanders 4]. Let B be a Banach algebra and A be a closed subalgebra of B. A is called a block subalgebra of B if $ABA\subset A$.

In [Flanders 4] it is shown that for a B*-algebra A the following conditions are equivalent:

1) $\pi(A)$ is a block subalgebra in A^{**}.

2) $\pi(A)$ is a two-sided ideal in A^{**}.

Here we shall show the above conditions 1) and 2) are equivalent for $L^1(G)$. We have

Proposition 4. 2. *Let G be a locally compact unimodular group. Then the following conditions are equivaleut.*

1) *$\pi(L^1(G))$ is a two-sided ideal in $L^1(G)^{**}$.*

2) *$\pi(L^1(G))$ is a block subalgeba of $L^1(G)^{**}$.*

3) *G is compact.*

Proof. It is sufficient to prove the implication 2)$\Rightarrow$3). Since G is unimodular, $\Delta(t)=1$. We shall show the outine of the proof. Suppose that G is not compact. We choose a sequence $\{C_n\}$ of compact sets such that $C_n\cap C_m=\phi$ $(m\neq n)$ and $\mu(C_n)>1$ for all n. Let f_n, x and f be characteristic functions of $C_1C'_nC_1$, C_1 and $C_1C_\infty C_1$ respectively. Then $f_n\longrightarrow f$ in weak *-topology in $L^\infty(G)$. From the assumption, for each $x\in L^1(G)$ and $F\in L^1(G)^{**}$, there exists $g\in L^1(G)$ such that

$$\pi(g)=\pi(x)\circ F*\pi(x).$$

$$\begin{aligned}F((x*f_n)\circ x)&=F\circ(x*f_n)(x)=\pi(x)\circ F(x*f_n)\\&=f_n*(\pi(x)\circ F)(x)=\pi(x)\circ F*\pi(x)(f_n)=f_n(g)\\&\longrightarrow f(g)=F((x*f)\circ x)\end{aligned}$$

Hence $(x*f)\circ x=$weak lim $(x*f_n)\circ x$

Now for any $t\in C_\infty$,

$$\begin{aligned}x*(f\circ x)(t)&=\int_G f\circ x(ts)x(s)d\mu(s)\\&=\int_{C_1}\int_G f(rts)x(r)d\mu(r)d\mu(s)\\&=\int_{C_1}\int_{C_1}f(rts)d\mu(r)d\mu(s)\end{aligned}$$

$$=\int_{C_1}\mu(C_1)d\mu(s)$$

$$=\mu(C_1)^2>1.$$

Hence $x*(f\circ x)\notin C_0(G)$. However for any $t\notin C_1^{-1}C_1C'_nC_1C_1^{-1}$ (compact set),

$$x*(f_n\circ x)(t)=\int_G f_n\circ x(ts)x(s)d\mu(s)$$

$$=\int_{C_1}\int_G f_n(rts)x(r)d\mu(r)d\mu(s)$$

$$=\int_{C_1}\int_{C_1} f_n(rts)d\mu(r)d\mu(s)$$

$$=0.$$

This is a contradiction by the same reason as the proof of Proposition 4.1, and so G is compact.

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References

[1] R. ARENS: *The adjoint of a bilinear operation.* Proc. Amer. Math. Soc. 2 (1951), 839–848.

[2] P. CIVIN: *Ideals in the second conjugate algebra of a group algebra.* Math. Scand. 11 (1972), 161–174.

[3] P. CIVIN and B. YOOD: *The second conjugate space of a Banach algebra as an algebra.* Pacific J. Math., 11 (1961), 847–870.

[4] M. C. FLANDERS: *Ideal C*-algebras.* Doctoral disertation, Tulane Univ. 1968.

[5] S. A. MCKILIGAN: *Duality in B*-algebras.* Proc. Amer. Math. Soc., 37 (1973), 86–88.

[6] T. OGASAWARA and K. YOSHINAGA: *Weakly completely continuous Banach *-algebras.* J. Sci. Hiroshima Univ., Ser. A. 18 (1954), 15–36.

[7] ————; *A characterization of dual B*-algebras.* J. Sci. Hiroshima Univ., Ser. A. 18 (1954), 179–182.

[8] P. K. WONG: *The Arens Product and duality in B*-algebras II.* Proc. Amer. Math. Soc., 27 (1971), 535–538.

[9] P. K. WONG: *On the Arens product and annihilator algebras.* Proc. Amer Math. Soc., 30 (1971), 79–83.

[10] B. J. TOMIUK-P. K. WONG: *The Arens product and duality in B*-algebras.* Proc. Amer. Math. Soc., 25 (1970), 529–534.