Notes on Certain Hermitian Spaces

By

Hideo MIZUSAWA

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The purpose of this note is to generalize some theorems which have been obtained in a Kählerian space [11], [2] to a certain Hermitian space, that is, a Hermitian space with a condition $\nabla_r F_i^{\cdot r}=0$, where ∇_r denotes the covariant derivative with respect to the Riemannian connection. we shall call such a space a semi-Kählerian space or an Apte's space [1], [6]. In this space we shall consider an infinitesimal holomorphically projective transformation, the conformally flatness and a constant sectional curvatur. Next, we shall show that if this space be conformal to a Kählerian space, then it coincides with a Kählerian space.

As to the notations and conventions, we follow J. A, Schouten [4].

§1. Preliminaries

In a 2n-dimensional differentiable space, if an almost Hermitian structure is defined by assigning to the space a tensor field $F_j^{\cdot i}$ and a positive definite Riemannian meric tensor field g_{ji} such that

$$F_j^{\cdot r} F_r^{\cdot i}=-\delta_j^i, \tag{1.1}$$

$$g_{ji}=F_j^{\cdot b} F_i^{\cdot a} g_{ba}. \tag{1.2}$$

then the space is called an almost Hermitian space.

An almost Hermitian space is called a Hermitian space if the Nijenhuis tensor identically vanishes, that is

$$N_{ji}{}^h \equiv F_j^{\cdot r}(\nabla_r F_i^{\cdot h}-\nabla_i F_r^{\cdot h})-F_i^{\cdot r}(\nabla_r F_j^{\cdot h}-\nabla_j F_r^{\cdot h})=0. \tag{1.3}$$

Taking account of the relation

$$N_{jih}+2N_{h(ji)}=2(F_j^{\cdot r}\nabla_r F_{ih}+F_i^{\cdot r}\nabla_j F_{rh}),$$

we see that (1.3) is equivalent to the following [6]

$$\nabla_j F_{ih}-F_j^{\cdot b} F_i^{\cdot a}\nabla_b F_{ah}=0 \tag{1.4}$$

or

$$F_j^{\cdot r}\nabla_r F_{ih}+F_i^{\cdot r}\nabla_j F_{rh}=0$$

where

$$N_{jih}=N_{ji}{}^r g_{rh},\quad F_{jh}=g_{hr}F_j^{\cdot r}.$$

If a Hermitian space satisfies

$$\nabla_r F_j^{\cdot r}=0, \tag{1.5}$$

then the space is ealled a semi-Kählerian space or an Apte's space.

It is easily verified that the condition (1.5) is equivalent to the following, with respect to a complex coordinates $(Z^\alpha,\bar{Z}^\alpha)$

$$\left\{{\alpha \atop \alpha\bar{\lambda}}\right\}=0,\ \text{Conj.}\quad \alpha=1,\ 2,\ \ldots,\ n.;\ \bar{\lambda}=\bar{1},\ \bar{2},\ \ldots,\ \bar{n}.$$

Next, we shall define the following operations for any tensor T_{jih}, $T_{ji}{}^h$ in an almost Hermitian space.

$$\begin{cases} O_{ji}T_{jih}=\dfrac{1}{2}(T_{jih}-F_j^{\cdot b}F_i^{\cdot a}T_{bah}), & O_j^hT_{ji}{}^h=\dfrac{1}{2}(T_{ji}{}^h-F_j^{\cdot b}F_a^{\cdot h}T_{bi}{}^a), \\ {}^*O_{ji}T_{jih}=\dfrac{1}{2}(T_{jih}+F_j^{\cdot b}F_i^{\cdot a}T_{bah}), & {}^*O_j^hT_{ji}{}^h=\dfrac{1}{2}(T_{ji}{}^h+F_j^{\cdot b}F_a^{\cdot h}T_{bi}{}^a). \end{cases} \tag{1.6}$$

As to the two operations with the same indices, we have

$$OO=O,\ {}^*O{}^*O={}^*O,\ {}^*OO=O{}^*O=0. \tag{1.7}$$

A tensor is called pure (hybrid) in two indices if it vanishes by transvection of $^*O(O)$ on these indices.

By the definition, (1.4) is written

$$O_{ji}\nabla_j F_{ih}=0. \tag{1.8}$$

In an almost Hermitian space we denote the Riemannian curvature tensor by $K_{kji}{}^h$ and put

$$\begin{cases} K_{kjih}=K_{kji}{}^r g_{rh},\ K_{ji}=K_{kji}{}^k,\ \widetilde{K}_{ji}=F_j^{\cdot r}K_{ir}, \\ H_{ji}=\dfrac{1}{2}F^{ba}K_{abji},\ \widetilde{H}_{ji}=F_j^{\cdot r}H_{ir},\ K=g^{ji}K_{ji},\ H=F^{ji}H_{ji}. \end{cases} \tag{1.9}$$

By the definition (1.9) and the first Bianchi identies, we have

$$H_{ji}=F^{kh}K_{kjih}. \tag{1.10}$$

A vector field v^i is called analytic, if it satisfies [8]

(1.10) $$\underset{v}{\pounds} F_j^{\cdot i} = v^r \nabla_r F_j^{\cdot i} - F_j^{\cdot r} \nabla_r v^i + F_r^{\cdot i} \nabla_j v^r = 0,$$

where $\underset{v}{\pounds}$ denotes the operator of Lie derivation with sespect to v^i.

A pure tensor $T_{i_1 i_2 \cdots i_p}{}^{j_1 j_2 \cdots j_q}$ is called analytic, if it satisfies [8]

(1.11) $$\Phi_l T_{(i)}{}^{(j)} \equiv F_l^{\cdot r} \nabla_r T_{(i)}{}^{(j)} - \nabla_l (F_{i_1}^{\cdot r} T_{r i_2 \cdots i_p}{}^{(j)}) + \sum_{k=1}^{p} (\nabla_{i_k} F_l^{\cdot r}) T_{i_1 \cdots \underset{(k)}{r} \cdots i_p}{}^{(j)}$$
$$+ \sum_{k=1}^{q} (\nabla_l F_r^{\cdot j_k} - \nabla_r F_l^{\cdot j_k}) T_{(i)}{}^{j_1 \cdots \overset{(k)}{r} \cdots j_q} = 0$$

where we have put

$$T_{(i)}{}^{(j)} = T_{i_1 i_2 \cdots i_p}{}^{j_1 j_2 \cdots j_q}.$$

§2. Semi-Kählerian spaces

We shall consider a semi-Kählerian space, then it holds that

(2.1) $$\nabla_j F_{ih} - F_j^{\cdot b} F_i^{\cdot a} \nabla_b F_{ah} = 0,$$

(2.2) $$\nabla_r F_i^{\cdot r} = 0.$$

Operating ∇_h to (2.1), we have

$$\nabla_h \nabla_j F_i^{\cdot h} - F_j^{\cdot b} (\nabla_h F_i^{\cdot a})(\nabla_b F_a^{\cdot h}) - F_i^{\cdot a} (\nabla_h F_j^{\cdot b})(\nabla_b F_a^{\cdot h}) - F_j^{\cdot b} F_i^{\cdot a} \nabla_h \nabla_b F_a^{\cdot h} = 0.$$

It is easily verified that in the left hand side of the above equation the second term is zero and the third term is symmetric with respect to j and i.

Hence we have

(2.3) $$O_{ji} (\nabla_h \nabla_j F_i^{\cdot h}) = O_{ij} (\nabla_h \nabla_i F_j^{\cdot h}).$$

On the other hand, applying the Ricci's identity to $F_i^{\cdot h}$, we get

$$\nabla_h \nabla_j F_i^{\cdot h} - \nabla_j \nabla_h F_i^{\cdot h} = K_{hjr}{}^h F_i^{\cdot r} - K_{hji}{}^r F_r^{\cdot h}.$$

By virtue of (2.2) and (1.9), we have

(2.4) $$\nabla_h \nabla_j F_i^{\cdot h} = \widetilde{K}_{ij} - H_{ij}.$$

Substituting (2.4) into (2.3), we have [5]

(2.5) $$O_{ji} H_{ji} = 0$$

Next, using (2.2), we have

$$0=\nabla_h[\nabla_j(F_i^{\cdot h}F^{ji})]=F^{ji}\nabla_h\nabla_jF_i^{\cdot h}+(\nabla_jF_i^{\cdot h})(\nabla_hF^{ji})$$

Substituting (2.4) into the last equation, we get

$$(\nabla_jF_i^{\cdot h})(\nabla_hF^{ji})=K-H. \tag{2.6}$$

On the other hand, if we transvect ∇^hF^{ji} to (2.1), we obtain

$$(\nabla_jF_{ih})(\nabla^hF^{ji})=0. \tag{2.7}$$

Hence we have

$$K-H=0. \tag{2.8}$$

In the next place, we shall consider some analytic tensors.

THEOREM 2.1. *In a semi-Kählerian space, if a tensor $H_j^{\cdot i}$ is analytic, then $H(=K)$ is an absolute constant.*

Proof. From (2.5) $H_j^{\cdot i}$ is a pure tensor. Applying analytic operation Φ_l to $H_j^{\cdot i}$, we get

$$\Phi_lH_j^{\cdot i}\equiv F_l^{\cdot r}\nabla_rH_j^{\cdot i}-F_r^{\cdot i}\nabla_lH_j^{\cdot r}+H_r^{\cdot i}\nabla_jF_l^{\cdot r}-H_j^{\cdot r}\nabla_rF_l^{\cdot i}=0.$$

By contraction with respect to j and i, we have

$$F_r^{\cdot i}\nabla_lH_i^{\cdot r}=0.$$

On the other hand,

$$\nabla_lH=\nabla_l(F_r^{\cdot i}H_{\cdot i}^{r})=-F_r^{\cdot i}\nabla_lH_i^{\cdot r}-H_i^{\cdot r}\nabla_lF_r^{\cdot i}=-F_r^{\cdot i}\nabla_lH_i^{\cdot r}=0.$$

N.B. This theorem is valid for an almost Hermitian space with a pure tensor $H_j^{\cdot i}$, for instance a Kählerian space and a K-space, but in a K-space $H\neq K$.

THEOREM 2.2. *In an Hermitian space, $\widetilde{H}_j^{\cdot i}$ is analytic, if and only if $H_j^{\cdot i}$ is analytic.*

Proof. Let $H_j^{\cdot i}$ be analytic, then by virtue of (1.3), and the purity of $\widetilde{H}_j^{\cdot i}$, we can easily get

$$\Phi_lF_j^{\cdot i}=N_{lj}^{\ \ i}=0,\quad \Phi_l\widetilde{H}_j^{\cdot i}=\Phi_l(F_j^{\cdot r}H_{\cdot r}^{i})=H_{\cdot r}^{i}\Phi_lF_j^{\cdot r}+F_j^{\cdot r}\Phi_lH_{\cdot r}^{i}=0.$$

The converse is obvious.

In a semi-Kählerian space, it is unknown that the Ricci tensor K_{ji} is pure or hybrid. But $O_{j}{}^{i}K_{j}{}^{i}$ is pure, then we have

THEOREM 2.3. *In an Hermitian space $O_{j}{}^{i}\widetilde{K}_{j}{}^{i}$ is analytic if and only if $O_{j}{}^{i}K_{j}{}^{i}$ is analytic, and in this case K is an absolute constant.*

In fact, let $O_j{}^iK_j{}^i$ be analytic, then we have

$$O_j{}^i\widetilde{K}_j{}^i = O_j{}^i(F_j^{\cdot r}K_r{}^i) = F_j^{\cdot r}(O_r{}^iK_r{}^i),$$

$$\Phi_l\,(O_j{}^i\widetilde{K}_j{}^i) = F_j^{\cdot r}\Phi_l(O_r{}^iK_r{}^i) = 0,$$

$$\Phi_l(O_j{}^iK_j{}^i) = F_l^{\cdot r}\nabla_r(O_j{}^iK_j{}^i) - F_r^{\cdot i}\nabla_l(O_j{}^rK_j{}^r) + (O_r{}^iK_r{}^i)\nabla_jF_l^{\cdot r} - (O_j{}^rK_j{}^r)\nabla_rF_l^{\cdot r} = 0.$$

Transvecting the last equation with respect to j and i, we get

$$\nabla_rK = 0.$$

§3. Analytic holomorphically projective transformations

If we put $\underset{v}{\pounds}\{{}^{\,h}_{ji}\} = t_{ji}{}^h$, then the following identities are known [11]:

$$t_{ji}{}^h \equiv \underset{v}{\pounds}\{{}^{\,h}_{ji}\} = \nabla_j\nabla_iv^h + K_{rji}{}^hv^r, \tag{3.1}$$

$$\underset{v}{\pounds}\nabla_jF_i^{\cdot h} - \nabla_j\underset{v}{\pounds}F_i^{\cdot h} = F_i^{\cdot r}t_{jr}{}^h - F_r^{\cdot h}t_{ji}{}^r, \tag{3.2}$$

$$\underset{v}{\pounds}K_{kji}{}^h = \nabla_kt_{ji}{}^h - \nabla_jt_{ki}{}^h. \tag{3.3}$$

A vector field v^i is called an infinitesimal holomorphically projective transformation or briefly an H. P. Transformation, if it satisfies

$$t_{ji}{}^h = \underset{v}{\pounds}\{{}^{\,h}_{ji}\} = \rho_j\delta_i{}^h + \rho_i\delta_j{}^h - \tilde{\rho}_jF_i^{\cdot h} - \tilde{\rho}_iF_j^{\cdot h} \tag{3.4}$$

where ρ_i is a certain vector and $\tilde{\rho}_i = F_i^{\cdot r}\rho_r$. We call ρ_i the associated vector of the *H. P.* transformation. Contracting (3.4) with respect to i and h, we get $\rho_j = \{1/2(n+1)\}\nabla_j\nabla_rv^r$. Hence ρ_i is a gradient vector. Thus it holds that

$$\nabla_j\rho_i = \nabla_i\rho_j. \tag{3.5}$$

Now, in an almost Hermitian space, we shall introduce a curve which satisfies the following differential equations [3]

$$\frac{d^2x^h}{dt^2} + \{{}^{\,h}_{ji}\}\frac{dx^j}{dt}\frac{dx^i}{dt} = \alpha\frac{dx^h}{dt} + \beta F_j^{\cdot h}\frac{dx^j}{dt} \tag{3.6}$$

where α and β are certain functions of the parameter t. Such a curve is called a holomorphically flat curve and has the property that the tangent holomorphic plane deplaced parallelly along the curve remains holomorphically tangent to the curve.

Let v^i be an infinitesimal transformation and we assume that an infinitesimal point transformation $'x^i = x^i + \varepsilon v^i$ transforms any holomorphically flat curve into such a curve.

A necessary and sufficient condition for a vector field v^i to be such a transformation is that [2]

(3.7) $$\dot{x}^j \underset{v}{\pounds} F_j^{\cdot i} = a\dot{x}^i + bF_j^{\cdot i}\dot{x}^j$$

(3.8) $$\dot{x}^j \dot{x}^i t_{ji}{}^h = c\dot{x}^h + dF_j^{\cdot h}\dot{x}^j$$

are hold for any direction $\dot{x}^i = dx^i/dt$, where a, b, c, and d are some functions of x^i and $\dot{x}^i$.

The following Lemmas are known [2].

LEMMA 1. *In an almost complex space, let $a_j{}^i$ be a hybrid tensor, if it satisfies*

$$a_r{}^i u^r = \alpha u^i + \beta F_r^{\cdot i} u^r$$

for any vector u^i, where α and β are real valued functions of u^i, then $a_j{}^i$ must be zero tensor.

LEMMA 3. *Let $t_{ji}{}^h$ be a symmetric tensor with respect to j and i. If it satisfies*

$$t_{ji}{}^h u^j u^i = \alpha u^h + \beta F_j^{\cdot h} u^j$$

for any vector u^i, then $t_{ji}{}^h$ takes the following form

$$t_{ji}{}^h = \rho_j \delta_i^h + \rho_i \delta_j^h + \sigma_j F_i^{\cdot h} + \sigma_i F_j^{\cdot h}$$

where α and β are real valued functions of u^i and ρ_i and σ_i are certain vectors.

Now, let v^i be an *H. P.* transformation, then from (3.7) and Lemma 1, we have

(3.9) $$\underset{v}{\pounds} F_j^{\cdot i} = 0.$$

Next, from (3.8) and Lemm 3, we have

(3.10) $$t_{ji}{}^h = \rho_j \delta_i^h + \rho_i \delta_j^h + \sigma_j F_i^{\cdot h} + \sigma_i F_j^{\cdot h}.$$

If we substitute (3.9) into (3.2), then we get

$$\underset{v}{\pounds} \nabla_j F_i^{\cdot h} = t_{jr}{}^h F_i^{\cdot r} - t_{ji}{}^r F_r^{\cdot h}.$$

Contracting with j and h and using (2.2), we have

$$t_{jr}{}^j F_i^{\cdot r} - t_{ji}{}^r F_r^{\cdot j} = 0.$$

Substituting (3.10) into the last equation, we have $\sigma_j=-\tilde{\rho}_j$. Hence

$$t_{ji}{}^h=\rho_j\delta_i{}^h+\rho_i\delta_j{}^h-\tilde{\rho}_jF_i^{\cdot h}-\tilde{\rho}_iF_j^{\cdot h}. \tag{3.11}$$

Therefore v^i is analytic and at the same time an *H. P.* transformation. The converse is evident. Thus we have the following.

THEOREM 3.1. *In an almost Hermitian space with the relation* $\nabla_rF_i^{\cdot r}=0$, *in order that an infinitesimal H. P. transformation carried any holomorphically flat curve into such a curve, it is necessary and sufficient that it is an analytic H. P. transformation,*

In a semi-Kählerian space, let v^i be an analytic *H. P.* transformation. If we substitute (3.11) into (3.3), we have

$$\begin{aligned}\underset{v}{\pounds}K_{kji}{}^h=&\delta_j{}^h\nabla_k\rho_i-\delta_k{}^h\nabla_j\rho_i-F_j^{\cdot h}\nabla_k\tilde{\rho}_i+F_k^{\cdot h}\nabla_j\tilde{\rho}_i-F_i^{\cdot h}(\nabla_k\tilde{\rho}_j-\nabla_j\tilde{\rho}_k)\\&-\tilde{\rho}_j\nabla_kF_i^{\cdot h}+\tilde{\rho}_k\nabla_jF_i^{\cdot h}+\tilde{\rho}_i(\nabla_jF_k^{\cdot h}-\nabla_kF_j^{\cdot h}).\end{aligned} \tag{3.12}$$

Transvecting with $F_h^{\cdot k}$ and making use of (3.9), (2.2) and (1.9), we have

$$\underset{v}{\pounds}H_{ji}=-2F_j^{\cdot r}\nabla_r\rho_i+2nF_i^{\cdot r}\nabla_r\rho_j+(2n+1)(\nabla_jF_i^{\cdot h})\rho_r-(\nabla_iF_j^{\cdot r})\rho_r. \tag{3.13}$$

Taking the alternating part with respect to j and i, we get

$$\underset{v}{\pounds}H_{ji}=-(n+1)[(F_j^{\cdot r}\nabla_r\rho_i-F_i^{\cdot r}\nabla_r\rho_j)-(\nabla_jF_i^{\cdot r}-\nabla_iF_j^{\cdot r})\rho_r], \tag{3.14}$$

and

$$(n-1)(F_j^{\cdot r}\nabla_r\rho_i+F_i^{\cdot r}\nabla_r\rho_j)+n(\nabla_jF_i^{\cdot r}+\nabla_iF_j^{\cdot r})\rho_r=0.$$

This is equivalent to

$$2(n-1)O_{ji}(F_j^{\cdot r}\nabla_r\rho_i)+n(\nabla_jF_i^{\cdot r}+\nabla_iF_j^{\cdot r})\rho_r=0. \tag{3.15}$$

If we operate O_{ji} to (3.15), then by virtue of (1.7) and (1.8), we have

$$F_j^{\cdot r}\nabla_r\rho_i+F_i^{\cdot r}\nabla_r\rho_j=0. \tag{3.16}$$

Therefore

$$(\nabla_jF_i^{\cdot r}+\nabla_iF_j^{\cdot r})\rho_r=0. \tag{3.17}$$

From the last two equations, we find

$$\nabla_j\tilde{\rho}_i+\nabla_i\tilde{\rho}_j=0.$$

THEOREM 3.2. *In a semi-Kählerian space, if* ρ_i *is the associated vector of an* *analytic H. P. transformation, then* $\tilde{\rho}_i$ *is a Killing vector.*

From (3.13), (3.16) and (3.17), for an analytic *H. P.* transformation v^i, we have

$$\underset{v}{\pounds} H_{ji} = 2(n+1)\nabla_j \tilde{\rho}_i.$$

Operating O_{ji} to the last equation and taking account of (2.5) and (3.9), we get

$$O_{ji}\underset{v}{\pounds} H_{ji} = \underset{v}{\pounds} O_{ji} H_{ji} = 0 = 2(n+1) O_{ji}\nabla_j \tilde{\rho}_i.$$

Thus from (3.16) and the last equation, we have

THEOREM 3.3. *In a semi-Kählerian space, if* ρ_i *is the associated vector of an analytic H. P. transformation, then* $\nabla_j\rho_i$ *and* $\nabla_j\tilde{\rho}_i$ *are both hybrid with respect to j and i.*

From (3.14)(3.16) and (3.17), we get

$$\underset{v}{\pounds} H_{ji} = -2(n+1)[F_j^{\cdot r}\nabla_r \rho_i - (\nabla_j F_i^{\cdot r})\rho_r].$$

From which we have

$$\underset{v}{\pounds} \tilde{H}_{ji} = -2(n+1)[\nabla_j \rho_i + (\nabla_j F_i^{\cdot r})\tilde{\rho}_r]. \tag{3.18}$$

Next, if we contract (3.12) with respect to h and k, then we have

$$\underset{v}{\pounds} K_{ji} = -2n\nabla_j\rho_i - (F_j^{\cdot r}\nabla_r\tilde{\rho}_i + F_i^{\cdot r}\nabla_r\tilde{\rho}_j).$$

By virtue of the theorem 3.3, it holds that

$$F_j^{\cdot r}\nabla_r\tilde{\rho}_i - F_i^{\cdot r}\nabla_r\tilde{\rho}_j = 0.$$

Therefore we have

$$\underset{v}{\pounds} K_{ji} = -2[(n+1)\nabla_j\rho_i + (\nabla_j F_i^{\cdot h})\tilde{\rho}_r]. \tag{3.19}$$

Eilminating $(\nabla_j F_i^{\cdot r})\tilde{\rho}_r$ from (3.18) and (3.19), we obtain

$$\underset{v}{\pounds}[\tilde{H}_{ji} - (n+1)K_{ji}] = 2n(n+1)\nabla_j\rho_i. \tag{3.20}$$

§4. Certain Einstein semi-Kählerian spaces

We shall call a semi-Kählerian space with a Ricci tensor proportional to g_{ji} an Einstein semi-Kählerian space, that is,

$$K_{ji} = \frac{K}{2n} g_{ji} \tag{4.1}$$

is valid. We suppose that $K \neq 0$. It is well known that K is an absolute constant.

Moreover in this space, if we assume that H_{ji} be proportional to F_{ji}, i.e;

$$H_{ji}=\lambda F_{ji}. \tag{4.2}$$

Then we have

$$\tilde{H}_{ji}=\frac{H}{2n}g_{ji}. \tag{4.3}$$

On the other hand, in §2, we have seen that in a semi-Kählerian space

$$K=H$$

is valid.

Thus the assumption (4.2) is equivalent to

$$\tilde{H}_{ji}=K_{ji}. \tag{4.4}$$

Afterward, we shall consider an Einstein semi-Kählerian space satisfying (4.4).

N.B. An Hermitian space satisfying (4.4) is not a Kählerian space. S. Koto has called it a S. K. II space [5].

Now, let v^i be an analytic *H. P.* transformation, then (3.20) holds. From (4.4) we have

$$\underset{v}{\pounds} K_{ji}=-(n+1)\nabla_j\rho_i. \tag{4.5}$$

From (4.1), (3.5) and the relation $\underset{v}{\pounds} g_{ji}=\nabla_j\rho_i+\nabla_i\rho_j$, we obtain

$$\nabla_j(v_i-\frac{1}{k}\rho_i)+\nabla_i(v_j-\frac{1}{k}\rho_j)=0$$

where we have put $\qquad k=-K/n(n+1).$

If we define p_i by

$$p_i=v_i-\frac{1}{k}\rho_i,$$

then p_i is a Killing vector. Next, if we put $q_i=\frac{1}{k}\tilde{\rho}_i$, then q_i is also a Killing vector by virtue of Theorem 3.2.

Thus we obtain the following.

THEOREM 4.1 *In an Einstein semi-Kählerian space satisfying* $\tilde{H}_{ji}=K_{ji}$ *an analytic H. P. transformation* v^i *is uniquely decomposed into the form*

$$v^i=p^i+F_r^{\cdot i}q^r \tag{4.6}$$

where p^i *and* q^i *are both Killing vectors.*

N.B. Theorem 4.1 is a particular case of the Matsushima's theorem in a compact Kählerian space. [10]. For a *K*-space cf. Tachibana, S. [9].

From (4.6) we have

$$\underset{v}{\pounds}\{{}^{h}_{ji}\}=\underset{p}{\pounds}\{{}^{h}_{ji}\}-\underset{q}{\pounds}\{{}^{h}_{ji}\}=\frac{1}{k}\underset{\rho}{\pounds}\{{}^{h}_{ji}\}.$$

Substituting (3.1) and (3.11) into the last equation, we obtain

$$\nabla_j\nabla_i\rho^h+K_{rji}{}^h\rho^r=k(\rho_j\delta_i{}^h+\rho_i\delta_j{}^h-\bar{\rho}_jF_i^{\cdot h}-\bar{\rho}_iF_j^{\cdot h}). \tag{4.7}$$

From which it follows

THEOREM 4.2 *In an Einstein semi-Kählerian space satisfying $\widetilde{H}_{ji}=K_{ji}$, the associated vector of an analytic H. P. transformation is an H. P. transformation.*

§5. Conformally flat semi-Kählerian spaces

Now we suppose a semi-Kählerian space to be conformally flat, then the curvature tensor takes the following form [11]

$$2(n-1)K_{kjih}=g_{kh}K_{ji}-g_{jh}K_{ki}+g_{ji}K_{kh}-K/(2n-1)(g_{ji}g_{kh}-g_{ki}g_{jh}).$$

Transvecting with F^{kh}, we get

$$2(n-1)H_{ji}=\widetilde{K}_{ji}-\widetilde{K}_{ij}-\{1/(2n-1)\}KF_{ji}. \tag{5.1}$$

From which we have

$$(2n-1)H-K=0.$$

Taking account of (2.8), we obtain for $n>1$

$$K=H=0.$$

THEOREM 5.1 *If a semi-Kählerian space is conformally flat, then the space has a vanishing scalar curvature.*

THEOREM 5.2. *In a conformally flat semi-Kählerian space, the tensor H_{ji} is effective.*

In this space, if we suppose that $\widetilde{H}_{ji}=K_{ji}$, then $H_{ji}=\widetilde{K}_{ji}$ holds, and from (5.1) we have

$$(n-1)H_{ji}=\frac{1}{2}(\widetilde{K}_{ji}-\widetilde{K}_{ij})={}^*O_{ji}\widetilde{K}_{ji}={}^*O_{ji}H_{ji}=H_{ji}.$$

$$H_{ji}=\widetilde{K}_{ji}=0.$$

$$K_{ji}=0.$$

THEOREM 5.3. [5] *If a semi-Kählerian space satisfying $\widetilde{H}_{ji}=K_{ji}$ is conformally flat, then it is of zero curvature.*

THEOREM 5.4. *If a semi-Kählerian space is of constant curvature, then it is of* *zero curvature.*

§6. Semi-Kählerian spaces conformal to a Kählerian spaces

The following theorem is known [11], [12].

THEOREM *A necessary and sufficient condition that 2n-dimensional Hermitian space be conformal to a Kählerian space is that*

for $2n>4$ $$C_{jih}\equiv F_{jih}-1/2(n-1)(F_{ji}F_h+F_{ih}F_j+F_{hj}F_i)=0$$

and for $2n=4$ $$C_{ji}\equiv 2\nabla_{[j}F_{i]}=0$$

Where $F_j=F_{jih}F^{ih}$, $F_{jih}=3\nabla_{[j}F_{ih]}$.

Now we suppose that a semi-Kählerian space be conformal to a Kählerian space, then the above theorem is valid. From (2.2) we get

$$F_j=0$$

therefore we have for $n>2$

$$F_{jih}=0.$$

THEOREM 6.1. *In order that a semi-Kählerian space be conformal to a Kählerian space, it is necessary sufficient that the tensor* F_{ji} *be harmonic.*

THEOREM 6.2. *A necessary and sufficient condition that a semi-Kählerian space be conformal to a Kählerian space is it coincides with a Kählerian space.*

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Department of Mathematics, Niigata University