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KÄHLER STRUCTURES ON THE COTANGENT BUNDLES OVER SOME STATISTICAL MANIFOLDS

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ABSTRACT. We prove that a Kähler structure induced on the cotangent bundle over a statistical manifold by using the dual connection has constant holomorphic sectional curvature under some conditions. Some examples which stem from information geometry are provided.

1. INTRODUCTION

Let (M, ∇, h) be a statistical manifold which is regarded as a generalization of statistical models. It is known that an almost Kähler structure (J, g) can be introduced on the tangent bundle TM of a statistical manifold M. We denote by T the difference tensor between the Levi-Civita connection $\widehat{\nabla}$ of the Riemannian metric h and the torsion free affine connection ∇. The tensor T plays an important role in statistical manifolds.

In the preceding papers[Sa1, Sa2], we have studied a family of almost Kähler structures on the tangent bundles of some statistical models induced from the α-connection. We have shown that the tangent bundles of some statistical models possess Kähler structures with constant holomorphic sectional curvature. The key fact to lead these results is the following theorem originally due to Shima.

Theorem 1.1 ([Sh, Sa2]). *Let (M, ∇, h) be a statistical manifold. If the affine connection ∇ is flat, and the difference tensor T satisfies the equation*

$$(*) \qquad (\nabla_X T)_Y Z = \lambda\{h(X,Y)Z + h(X,Z)Y\}^1$$

for some constant λ, then the tangent bundle (TM, J, g) is a Kähler manifold of constant holomorphic sectional curvature -2λ.

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[1]According to Shima[Sh], Hessian manifold satisfying this condition is said to be of constant Hessian curvature 2λ.

In this paper, we shall consider an almost Kähler structure on the cotangent bundle T^*M of a statistical manifold M. We denote by (J^*, g^*) the almost Kähler structure on T^*M, induced by *the dual connection* ∇^* of ∇ with respect to h. We shall prove the following dual version theorem.

Theorem A. *Let* (M, ∇, h) *be a statistical manifold. If the affine connection* ∇ *is flat, and the difference tensor* T *satisfies the condition* $(*)$ *in Theorem* 1.1, *then the cotangent bundle* (T^*M, J^*, g^*) *is a Kähler manifold of constant holomorphic sectional curvature* -2λ.

As a consequence, we see that the statement of Theorem A is also applicable to the statistical manifolds given in [Sa2]. We shall describe briefly these examples in the last section.

Throughout this paper, we assume that all manifolds are connected and smooth and further that all quantities on manifolds are smooth, unless otherwise specified.

2. Statistical manifolds

Let (M, ∇, h) be a statistical manifold, that is, ∇ is a torsion free affine connection, h is a Riemannian metric on M and ∇h is symmetric. Let ∇^* be the dual connection of ∇ with respect to h, given by

$$Xh(Y,Z) = h(\nabla_X Y, Z) + h(Y, \nabla^*_X Z), \tag{2.1}$$

for any vector fields X, Y, Z on M. Since ∇h is symmetric, the dual connection ∇^* is also torsion free.

For the Levi-Civita connection $\widehat{\nabla}$ of h, we define the *difference tensor fields* T and T^* by

$$T_X Y := \widehat{\nabla}_X Y - \nabla_X Y, \tag{2.2}$$

$$T^*_X Y := \widehat{\nabla}_X Y - \nabla^*_X Y, \tag{2.3}$$

for any vector fields X, Y on M. We have

$$T(X,Y,Z) := h(T_X Y, Z) = \frac{1}{2}(\nabla_X h)(Y,Z), \tag{2.4}$$

which shows that T is symmetric in X, Y, Z. We also have the following identities

$$\nabla^*_X Y = \widehat{\nabla}_X Y + T_X Y, \tag{2.5}$$

$$\nabla^*_X Y - \nabla_X Y = 2T_X Y, \tag{2.6}$$

$$2\widehat{\nabla}_X Y = \nabla_X Y + \nabla^*_X Y, \tag{2.7}$$

$$T^*_X Y = -T_X Y, \tag{2.8}$$

$$\begin{aligned}(\nabla_X T)(Y, Z) &+ (\nabla^*_X T^*)(Y, Z)\\ &= -2(T_X T_Y Z - T_Y T_X Z - T_Z T_X Y),\end{aligned} \tag{2.9}$$

where $T(X, Y) = T_X Y, T^*(X, Y) = T^*_X Y$.

We denote by K, K^* and $\widehat{K}$ the curvature tensors of ∇, ∇^* and $\widehat{\nabla}$, respectively. Then we have from (2.1)

$$h(K(X, Y)Z, W) = -h(K^*(X, Y)W, Z),$$

which implies that ∇-flatness and ∇^*-flatness are equivalent.

If ∇ and ∇^* are flat connections, we have

$$\widehat{K}(X, Y)Z = -\,(T_X T_Y Z - T_Y T_X Z), \tag{2.10}$$

$$(\widehat{\nabla}_X T)(Y, Z) = (\widehat{\nabla}_Y T)(X, Z), \tag{2.11}$$

$$2\widehat{K}(X, Y)Z = (\nabla_X T)(Y, Z) - (\nabla_Y T)(X, Z). \tag{2.12}$$

3. The cotangent bundle of a statistical manifold

Let (M, ∇, h) be an n-dimensional statistical manifold and $\pi : T^*M \to M$ be its cotangent bundle. As usually, we adopt (q^i, p_i) as a coordinate of $p = p_i dx^i \in T^*_x M$, where (x^i) denotes a coordinate system on a neighborhood U of x in M and $q^i = x^i \circ \pi$. Then a tangent vector $X = \xi^i \frac{\partial}{\partial q^i} + \eta_i \frac{\partial}{\partial p_i} \in T_p(T^*M)$ can be decomposed into the horizontal and vertical parts X^h, X^v with respect to the dual connection ∇^* of ∇:

$$X^h = \xi^i \frac{\partial}{\partial q^i} + \xi^k \Gamma^*_{ik} \frac{\partial}{\partial p_i}, \quad X^v = (\eta_i - \xi^k \Gamma^*_{ik}) \frac{\partial}{\partial p_i},$$

where $\Gamma^*_{ik} = p_r \Gamma^{*\,r}_{ik}$ and $\Gamma^{*\,i}_{jk}$ are coefficients of ∇^*. By abuse of notation, we often write Γ^*_{ik} instead of $\Gamma^*_{ik} \circ \pi$, and so on.

For a local coordinate system (q^i, p_i) on $\pi^{-1}(U) \subset T^*M$, we put for $i = 1, 2, \ldots, n$,

$$
(3.1) \qquad \begin{aligned}
X_i &:= \Big(\frac{\partial}{\partial q^i}\Big)^h = \frac{\partial}{\partial q^i} + \Gamma^*_{ik} \frac{\partial}{\partial p_k}, \\
Y^i &:= \Big(\frac{\partial}{\partial p_i}\Big)^v = \frac{\partial}{\partial p_i}.
\end{aligned}
$$

By using this local frame $\{X_i, Y^i\}$, we introduce a Riemannian metric g^* and an almost complex structure J^* on T^*M by

$$
(3.2) \qquad g^*(X_i, X_j) = h_{ij},\ g^*(X_i, Y^j) = g^*(Y^i, X_j) = 0,\ g^*(Y^i, Y^j) = h^{ij}
$$

$$
(3.3) \qquad J^* X_i = h_{ij} Y^j, \quad J^* Y^i = -h^{ij} X_j.
$$

where $h_{ij} = h(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})$ and h^{ij} are the components of the inverse matrix of (h_{ij}).

It is obvious that (J^*, g^*) is almost Hermitian, and its Kähler form Ω^* is given by

$$
\Omega^*(X_i, X_j) = 0, \qquad \Omega^*(X_i, Y^j) = -\delta_i^j, \qquad \Omega^*(Y^i, Y^j) = 0.
$$

Note that the coframe of $\{X_i, Y^i\}$ is $\{dq^i, \delta p_i = dp_i - \Gamma^*_{ik} dq^k\}$, and the dual connection ∇^* of ∇ is torsion free, we have

$$
\begin{aligned}
\Omega^* &= -\delta_i^j dq^i \wedge \delta p_j = -dq^i \wedge (dp_i - \Gamma^*_{ik} dq^k) \\
&= -dq^i \wedge dp_i + \Gamma^*_{ik} dq^i \wedge dq^k = -dq^i \wedge dp_i.
\end{aligned}
$$

It follows that

$$
d\Omega^* = 0.
$$

Thus we have the following

Proposition 3.1. *Let (M, ∇, h) be a statistical manifold. Then the almost Hermitian structure (J^*, g^*) on T^*M induced by the dual connection ∇^* is almost Kählerian.*

Lemma 3.2. *For a local frame $\{X_i, Y^i\}$, we have*

$$
(3.4) \qquad \begin{aligned}
[X_i, X_j] &= p_l K^*_{ijk}{}^l Y^k, \\
[X_i, Y^j] &= -\Gamma^*_{ik}{}^j Y^k, \\
[Y^i, Y^j] &= 0,
\end{aligned}
$$

where $K^*_{ijk}{}^l$ are the components of the curvature tensor K^* of M with respect to the dual connection ∇^*.

Proof. By definition,

$$
[X_i, X_j] = \Big[\frac{\partial}{\partial q^i} + \Gamma^*_{ih} \frac{\partial}{\partial p_h}, \frac{\partial}{\partial q^j} + \Gamma^*_{jk} \frac{\partial}{\partial p_k}\Big]
$$

$$
\begin{aligned}
&=\Big(\frac{\partial}{\partial q^i}\varGamma^*_{jk} + \varGamma^*_{ih}\frac{\partial}{\partial p_h}\varGamma^*_{jk}\Big)\frac{\partial}{\partial p_k} - \Big(\frac{\partial}{\partial q^j}\varGamma^*_{ih} + \varGamma^*_{jk}\frac{\partial}{\partial p_k}\varGamma^*_{ih}\Big)\frac{\partial}{\partial p_h}\\
&=p_l\Big(\frac{\partial}{\partial x^i}\varGamma^{*\,l}_{jk} + \varGamma^{*\,l}_{ih}\varGamma^{*\,h}_{jk}\Big)\frac{\partial}{\partial p_k} - p_l\Big(\frac{\partial}{\partial x^j}\varGamma^{*\,l}_{ih} + \varGamma^{*\,l}_{jk}\varGamma^{*\,k}_{ih}\Big)\frac{\partial}{\partial p_h}\\
&=p_l K^{*\ \,l}_{ijk}\, Y^k.
\end{aligned}
$$

Thus we obtain the first equation of (3.4). The second and the last equations of (3.4) are obvious. □

By the above Lemma 3.2, the Nijenhuis tensor N^* of J^* is given by

$$
\begin{aligned}
N^*(X_i, X_j) &= -(K^{*\ \,l}_{ijk}\, p_l)\, Y^k,\\
N^*(X_i, Y^j) &= -(h^{jr}h^{ks}K^{*\ \,l}_{irs}\, p_l)\, X_k,\\
N^*(Y^i, Y^j) &= (h^{ir}h^{js}K^{*\ \,l}_{rsk}\, p_l)\, Y^k.
\end{aligned} \tag{3.5}
$$

Therefore we have the following

Proposition 3.3. *Let (M, ∇, h) be a statistical manifold with flat affine connection ∇. Then the almost Hermitian structure (J^*, g^*) on T^*M induced by ∇^* is Kählerian.*

Let D^* denote the Levi-Civita connection of the Riemannian metric g^* on T^*M. By Lemma 3.2, we have for a local frame $\{X_i, Y^i\}$,

$$
\begin{aligned}
D^*_{X_i}X_j &= \widehat{\varGamma}^k_{ij}\, X_k + \frac{1}{2}K^{*\ \,l}_{ijk}\, p_l\, Y^k,\\
D^*_{X_i}Y^j &= \frac{1}{2}h^{rj}h^{sk}K^{*\ \,l}_{sir}\, p_l\, X_k - \widehat{\varGamma}^j_{ik}\, Y^k,\\
D^*_{Y^i}X_j &= -\frac{1}{2}h^{ir}h^{sk}K^{*\ \,l}_{jsr}\, p_l\, X_k + \frac{1}{2}h_{rk}\nabla^*_j h^{ri}\, Y^k,\\
D^*_{Y^i}Y^j &= -\frac{1}{2}h^{kr}\nabla^*_r h^{ij}\, X_k,
\end{aligned} \tag{3.6}
$$

where $\widehat{\varGamma}_{ij}{}^k$ are coefficients of the Levi-Civita connection $\widehat{\nabla}$ in (M, h). Especially, in the case of $K(= K^*) = 0$, taking account of (2.3),(2.4) we have

$$
\begin{aligned}
D^*_{X_i}X_j &= \widehat{\varGamma}^k_{ij}\, X_k,\\
D^*_{X_i}Y^j &= -\widehat{\varGamma}^j_{ik}\, Y^k,\\
D^*_{Y^i}X_j &= -T^{*i}{}_{jk}\, Y^k,\\
D^*_{Y^i}Y^j &= T^{*ijk}\, X_k.
\end{aligned} \tag{3.7}
$$

where $T^*{}_{ij}{}^k$ are components of T^* with respect to a coordinate system (x^i). From these, it is easy to see that $D^*J^* = 0$. So, we see again that (J^*, g^*) is a Kähler structure.

4. Proof of Theorem A

First of all, we shall calculate the curvature tensor on the tangent bundle T^*M over a statistical manifold M with flat affine connection ∇. Let $\{X_i, Y^i\}$ be a local frame defined by (3.1). We obtain the following

Proposition 4.1. *Let (M, ∇, h) be a statistical manifold with flat affine connection ∇. Then the curvature tensor R^* of the cotangent bundle (T^*M, J^*, g^*) is given by*

$$
\begin{aligned}
&R^*(X_i, X_j)X_k = \widehat{K}_{ijk}{}^{l}\, X_l,\\
&R^*(X_i, X_j)Y^k = -\widehat{K}_{ijl}{}^{k}\, Y^l,\\
&R^*(X_i, Y^j)X_k = \{-\nabla^*_i T^{*j}_{kl} + T^{*r}_{il} T^{*\,j}_{kr} + T^{*\,r}_{ik} T^{*j}_{rl}\}\, Y^l,\\
&R^*(X_i, Y^j)Y^k = h^{kr} h^{ls}\{\nabla^*_i T^{*\,j}_{rs} - T^{*\,t}_{is} T^{*j}_{rt} - T^{*\,t}_{ir} T^{*j}_{st}\}\, X_l,\\
&R^*(Y^i, Y^j)X_k = -\{T^{*\,j}_{kr} T^{*irl} - T^{*\,i}_{kr} T^{*jrl}\}\, X_l,\\
&R^*(Y^i, Y^j)Y^k = -\{T^{*jkr} T^{*\,i}_{rl} - T^{*ikr} T^{*j}_{rl}\}\, Y^l,
\end{aligned}
\tag{4.1}
$$

with respect to a local frame $\{X_i, Y^i\}$.

Proof. We show only the third and fourth equations of (4.1). The rest of equations are obtained similarly.

By virtue of (3.7) and (2.3),

$$
\begin{aligned}
D^*_{X_i} D^*_{Y^j} X_k &= -\, D^*_{X_i}(T^{*j}_{kl}\, Y^l) = -(\frac{\partial}{\partial x^i} T^{*j}_{kl} - T^{*\,j}_{kr} \widehat{\Gamma}_{il}{}^{r})\, Y^l,\\
&= -\,(\frac{\partial}{\partial x^i} T^{*\,j}_{kl} - T^{*\,j}_{kr} \Gamma^{*r}_{il} - T^{*\,j}_{kr} T^{*r}_{il})\, Y^l,\\
D^*_{Y^j} D^*_{X_i} X_k &= D^*_{Y^j}(\widehat{\Gamma}_{ik}{}^{r}\, X_r) = -\widehat{\Gamma}_{ik}{}^{r} T^{*j}_{rl}\, Y^l,\\
&= -\,(\Gamma^{*\,r}_{ik} T^{*j}_{rl} + T^{*\,r}_{ik} T^{*j}_{rl})\, Y^l,
\end{aligned}
$$

and

$$
D^*_{[X_i, Y^j]} X_k = -\, \Gamma^{*j}_{ir} D^*_{Y^r} X_k = \Gamma^{*j}_{ir} T^{*\,r}_{kl}\, Y^l.
$$

So, we have

$$
R^*(X_i, Y^j)X_k = \{-\nabla^*_i T^{*j}_{kl} + T^{*r}_{il} T^{*\,j}_{kr} + T^{*\,r}_{ik} T^{*j}_{rl}\}\, Y^l.
$$

From

$$
R^*(X_i, Y^j)Y^k = h^{kr} J^* R^*(X_i, Y^j)X_r,
$$

we have immediately the fourth equation. $\square$

We now proceed to complete the proof of Theorem A. From the assumption $(*)$ of the theorem and (2.12), we see that (M, h) has constant sectional curvature $k = -\frac{\lambda}{2}$, that is, taking account of (2.10), we have

$$\widehat{K}_{ijk}{}^{l} = -(T_{ir}{}^{l}T_{jk}{}^{r} - T_{jr}{}^{l}T_{ik}{}^{r}) = k(h_{jk}\delta_i^l - h_{ik}\delta_j^l).$$

Furthermore, by (2.9), we have

$$\nabla_i^* T_{jk}^*{}^{l} = -\lambda(h_{ij}\delta_k^l + h_{jk}\delta_i^l) + 2T_{ij}{}^{r}T_{kr}{}^{l}.$$

Consequently, for a local frame $\{X_i, Y^i\}$, the curvature tensor R^* in Proposition 4.1 can be written as

$$\begin{aligned}
R^*(X_i, X_j)X_k &= k(h_{jk}\delta_i^l - h_{ik}\delta_j^l)\, X_l,\\
R^*(X_i, X_j)Y^k &= -k(h_{jl}\delta_i^k - h_{il}\delta_j^k)\, Y^l,\\
R^*(X_i, Y^j)X_k &= -k(h_{ik}\delta_l^j + h_{il}\delta_k^j + 2h_{kl}\delta_i^j)\, Y^l,\\
R^*(X_i, Y^j)Y^k &= k(h^{jl}\delta_i^k + h^{jk}\delta_i^l + 2h^{kl}\delta_i^j)\, X_l,\\
R^*(Y^i, Y^j)X_k &= k(h^{il}\delta_k^j - h^{jl}\delta_k^i)\, X_l,\\
R^*(Y^i, Y^j)Y^k &= k(h^{jk}\delta_l^i - h^{ik}\delta_l^j)\, Y^l.
\end{aligned} \tag{4.2}$$

For any tangent vector $X = \xi^i X_i + \eta_i Y^i$ on T^*M, we note that $J^*X = -h^{ij}\eta_j X_i + h_{ij}\xi^j Y^i$ and $g^*(X, X) = h_{ij}\xi^i\xi^j + h^{ij}\eta_i\eta_j$ by (3.2) and (3.3). Making use of (4.2), we obtain by straightforward computation

$$g^*(R^*(X, J^*X)X, J^*X) = -4k\, g^*(X, X)^2, \tag{4.3}$$

for any vector field X on T^*M. The equality (4.3) shows that the holomorphic sectional curvature of (T^*M, J^*, g^*) is constant $4k = -2\lambda$. □

5. Examples

In this section, we shall consider some statistical manifolds of constant curvature arising from the statistical models in information geometry [A, AN]. For details, we shall refer to our paper [Sa2].

5.1. **The half space $\mathbb{R}_+^n$.** Let $\mathbb{R}_+^n = \{x = (x^1, \ldots, x^n) \in \mathbb{R}^n | x^1 > 0\}$ be the n-dimensional half space. On $\mathbb{R}_+^n$, we shall consider the Poincaré metric $h = (h_{ij})$,

$$h_{ij} = \frac{1}{(x^1)^2}\delta_{ij}. \tag{5.1}$$

Taking account of the α-connection in the statistical model of normal distributions, we shall introduce a family of torsion free affine connections on $(\mathbb{R}^n_+, h)$ by

$$\nabla^{(\alpha)} = \widehat{\nabla} - T^{(\alpha)} \tag{5.2}$$

for $\alpha \in \mathbb{R}$, where $T^{(\alpha)} = (T_{ij}^{(\alpha)k})$ is a (1,2)-tensor defined by

$$T_{ij}^{(\alpha)k} = \frac{\alpha}{x^1}\left(\delta_{i1}\delta_j^k + \delta_{j1}\delta_i^k + \delta_1^k\delta_{ij} - \delta_{i1}\delta_{j1}\delta_1^k\right). \tag{5.3}$$

Then $(\mathbb{R}^n_+, \nabla^{(\alpha)}, h)$ becomes a statistical manifold, and the dual connection of $\nabla^{(\alpha)}$ with respect to h is $\nabla^{(-\alpha)}$. We see that $\nabla^{(\alpha)}$ is flat when $\alpha = \pm 1$. Further we have

$$\begin{aligned}
\nabla_i^{(\alpha)} T_{jk}^{(\alpha)l} =& \frac{2\alpha^2}{(x^1)^2}(\delta_{i1}\delta_{jk}\delta_1^l - \delta_{i1}\delta_{j1}\delta_{k1}\delta_1^l) \\
&+ \frac{\alpha(1+\alpha)}{(x^1)^2}(\delta_{i1}\delta_{j1}\delta_k^l + \delta_{i1}\delta_{k1}\delta_j^l + \delta_{j1}\delta_{k1}\delta_i^l + \delta_{j1}\delta_{ik}\delta_1^l \\
&\qquad + \delta_{k1}\delta_{ij}\delta_1^l - \delta_{jk}\delta_i^l - 2\delta_{i1}\delta_{j1}\delta_{k1}\delta_1^l) \\
&- \frac{\alpha(1-\alpha)}{(x^1)^2}(\delta_{ij}\delta_k^l + \delta_{ik}\delta_j^l - \delta_{i1}\delta_{jk}\delta_1^l + \delta_{i1}\delta_{j1}\delta_{k1}\delta_1^l).
\end{aligned}$$

In the case of $\alpha = -1$, we obtain

$$\begin{aligned}
\nabla_i^{(-1)} T_{jk}^{(-1)l} =& \frac{2}{(x^1)^2}(\delta_{ij}\delta_k^l + \delta_{ik}\delta_j^l) \\
=& 2(h_{ij}\delta_k^l + h_{ik}\delta_j^l)
\end{aligned}$$

which shows that the condition $(*)$ in Theorem A is fulfilled with $\lambda = 2$.

When $\alpha = 1$, we see, however, the condition $(*)$ is not satisfied.

Hence, by virtue of Theorem A, we have

Theorem 5.1. *Let $(\mathbb{R}^n_+, \nabla^{(\alpha)}, h)$ be a statistical manifold where the metric h and the affine connection $\nabla^{(\alpha)}$ are given in* (5.1) *and* (5.2)*. The almost Kähler structure $(J^{(-\alpha)}, g^{(-\alpha)})$ on the cotangent bundle $T^*\mathbb{R}^n_+$ induced by the connection $\nabla^{(-\alpha)}$ is Kählerian if and only if $\alpha = \pm 1$, and if $\alpha = -1$, then $(T^*\mathbb{R}^n_+, J^{(1)}, g^{(1)})$ has constant holomorphic sectional curvature -4.*

5.2. **The open subset S^n_+ of S^n.** Let $S^n_+ = \{x = (x^0, x^1, \ldots, x^n) \in \mathbb{R}^{n+1} | (x^0)^2 + (x^1)^2 + \cdots + (x^n)^2 = c^2, x^i > 0 \text{ for } i = 0, \ldots, n\}$ be the "positive part "of n-dimensional sphere S^n in $\mathbb{R}^{n+1}$ with radius $c > 0$. With respect to a local coordinate system $(x^1, \ldots, x^n)$ on S^n_+, the induced Riemannian metric $h = (h_{ij})$ is given by

$$h_{ij} = \delta_{ij} + \frac{x^i x^j}{(x^0)^2}. \tag{5.4}$$

By considering the α-connection in the statistical model of discrete distributions, we shall define a family of affine connections on S^n_+ by

$$\nabla^{(\alpha)} = \widehat{\nabla} - T^{(\alpha)} \tag{5.5}$$

for $\alpha \in \mathbb{R}$, where $T^{(\alpha)} = (T^{(\alpha)k}_{ij})$ is a (1,2)-tensor given by

$$T^{(\alpha)k}_{ij} = \frac{\alpha}{c^4}\Big(\delta_{ij}\delta^k_i \frac{c^2}{x^i} - \delta_{ij}x^k - \frac{x^i x^j x^k}{(x^0)^2}\Big). \tag{5.6}$$

Then it is easy to see that $(S^n_+, \nabla^{(\alpha)}, h)$ is a statistical manifold and $\nabla^{(\alpha)}$ is flat when $\alpha = \pm c^2$.

We have

$$\begin{aligned}\nabla^{(\alpha)}_i T^{(\alpha)l}_{jk} =& \frac{\alpha(\alpha - c^2)}{c^4}\delta_{ij}\delta_{ik}\delta^l_i \frac{1}{(x^i)^2} \\ &+ \frac{\alpha(\alpha - c^2)}{c^6}\Big\{h_{jk}\delta^l_i - \delta_{ij}\delta_{ik}\frac{x^l}{x^i} + \frac{x^i x^j x^k x^l}{(x^0)^4}\Big\} \\ &- \frac{\alpha(\alpha + c^2)}{c^6}\big(h_{ij}\delta^l_k + h_{ik}\delta^l_j\big).\end{aligned} \tag{5.7}$$

In the case of $\alpha = c^2$, (5.7) reduces to

$$\nabla^{(c^2)}_i T^{(c^2)l}_{jk} = -\frac{2}{c^2}\big(h_{ij}\delta^l_k + h_{ik}\delta^l_j\big)$$

which shows that the condition $(*)$ in Theorem A is satisfied with $\lambda = -\frac{2}{c^2}$. Thus, applying Theorem A, we have the following

Theorem 5.2. *Let $(S^n_+, \nabla^{(\alpha)}, h)$ be a statistical manifold where the metric h and the affine connection $\nabla^{(\alpha)}$ are given in* (5.4) *and* (5.5). *The almost Kähler structure $(J^{(-\alpha)}, g^{(-\alpha)})$ on the cotangent bundle $T^*S^n_+$ induced by the connection $\nabla^{(-\alpha)}$ is Kählerian if and only if $\alpha = \pm c^2$, and if $\alpha = c^2$, then $(T^*S^n_+, J^{(-c^2)}, g^{(-c^2)})$ has constant holomorphic sectional curvature $\frac{4}{c^2}$.*

5.3. **The open subset H^n_+ of H^n.** Let $H^n = \{x = (x^0, x^1, \ldots, x^n) \in \mathbb{R}^{n+1} | (x^0)^2 - (x^1)^2 - \cdots - (x^n)^2 = c^2, x^0 > 0\}$ be the n-dimensional hyperbolic space in $\mathbb{R}^{n+1}$ equipped with Minkowski metric. We denote by $H^n_+ = \{x \in H^n | x^i > 0$ for $i = 0, \ldots, n\}$ the "positive part "of H^n. With respect to a local coordinate system $(x^1, \ldots, x^n)$ on H^n_+, the induced Riemannian metric $h = (h_{ij})$ is given by

$$h_{ij} = \delta_{ij} - \frac{x^i x^j}{(x^0)^2}. \tag{5.8}$$

We introduce a family of torsion free affine connections by

$$\nabla^{(\alpha)} = \widehat{\nabla} - T^{(\alpha)} \tag{5.9}$$

for $\alpha \in \mathbb{R}$, where $T^{(\alpha)} = (T_{ij}^{(\alpha)k})$ is given by

$$T_{ij}^{(\alpha)k} = \frac{\alpha}{c^4}\Big(\delta_{ij}\delta_i^k \frac{c^2}{x^i} + \delta_{ij}x^k - \frac{x^i x^j x^k}{(x^0)^2}\Big). \tag{5.10}$$

Then it is easy to see that $(H_+^n, \nabla^{(\alpha)}, h)$ is a statistical manifold and $\nabla^{(\alpha)}$ is flat when $\alpha = \pm c^2$. We have

$$\begin{aligned}\nabla_i^{(\alpha)} T_{jk}^{(\alpha)l} =& \frac{\alpha(\alpha - c^2)}{c^4}\delta_{ij}\delta_{ik}\delta_i^l \frac{1}{(x^i)^2} \\ &- \frac{\alpha(\alpha - c^2)}{c^6}\Big\{h_{jk}\delta_i^l - \delta_{ij}\delta_{ik}\frac{x^l}{x^i} + \frac{x^i x^j x^k x^l}{(x^0)^4}\Big\} \\ &+ \frac{\alpha(\alpha + c^2)}{c^6}\big(h_{ij}\delta_k^l + h_{ik}\delta_j^l\big),\end{aligned} \tag{5.11}$$

and when $\alpha = c^2$, (5.11) reduces to

$$\nabla_i^{(c^2)} T_{jk}^{(c^2)l} = \frac{2}{c^2}\big(h_{ij}\delta_k^l + h_{ik}\delta_j^l\big).$$

Thus, we have the following

Theorem 5.3. *Let $(H_+^n, \nabla^{(\alpha)}, h)$ be a statistical manifold where h and $\nabla^{(\alpha)}$ are given in* (5.8) *and* (5.9). *The almost Kähler structure $(J^{(-\alpha)}, g^{(-\alpha)})$ on the cotangent bundle $T^*H_+^n$ induced by the connection $\nabla^{(-\alpha)}$ is Kählerian if and only if $\alpha = \pm c^2$, and if $\alpha = c^2$, then $(T^*H_+^n, J^{(-c^2)}, g^{(-c^2)})$ has constant holomorphic sectional curvature $-\frac{4}{c^2}$.*

References

[A] S. Amari, *Differential-Geometrical Methods in Statistics*, Lecture Notes in Statistics 28. Springer-Verlag, 1985.

[AN] S. Amari and H. Nagaoka, *Methods of Information Geometry*, American Mathematical Society, Oxford University Press, 2000.

[Sa1] T. Sato, On a family of almost Kähler structures on the tangent bundles over some statistical models, in: *Topics in Almost Hermitian Geometry and Related Fields*(eds. Y. Matsushita et al.), World Scientific, 2005, 215–225.

[Sa2] T. Sato, On a family of almost Kähler structures on the tangent bundles over some statistical models II, to appear in JP Journal of Geometry and Topology, 2007,

[Sh] H. Shima, *Hessian Geometry*, Shokabo, Tokyo, 2001(in Japanese).

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