Nihonkai Math. J.
Vol.17(2006), 199-204

LINEAR OPERATORS PRESERVING LEFT UNITARILY INVARIANT NORMS ON MATRICES

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ABSTRACT. In this note, we characterize linear operators on $M_n(\mathbb{C})$ preserving all left unitarily invariant norms and study linear operators preserving some left unitarily invariant norm.

1. INTRODUCTION

In operator theory, unitarily invariant norms and related topics have been studied intensively ([2]); a norm $N(\cdot)$ on $M_n(\mathbb{C})$ is said to be unitarily invariant (abbreviated to ui) if

$$N(UAV) = N(A)$$

for all $A \in M_n(\mathbb{C})$ and all unitary $U, V \in M_n(\mathbb{C})$. Linear operators which preserve unitarily invariant norms are also studied well; a linear operator T on $M_n(\mathbb{C})$ is said to preserve a norm $N(\cdot)$ on $M_n(\mathbb{C})$ if

$$N(T(A)) = N(A) \quad (\forall A \in M_n(\mathbb{C})).$$

A. R. Sourour [13] gives a characterization of linear operators preserving a unitarily invariant norm which is not a scalar multiple of the Frobenius norm. More precisely, see Proposition 2.1 and Remark 2.2 below.

In this note, we would like to have a corresponding result to Proposition 2.1 for left unitarily invariant norms on $M_n(\mathbb{C})$. In [4], we call a norm $\mu(\cdot)$ on $M_n(\mathbb{C})$ left unitarily invariant (abbreviated to lui) if

$$\mu(UA) = \mu(A)$$

for all $A \in M_n(\mathbb{C})$ and all unitary $U \in M_n(\mathbb{C})$, and study basic properties of left unitarily invariant norms. We think that we can recognize unitarily invariant norms deeper by studying left unitarily invariant ones; we continue our research on lui norms in this note. We show that for left unitarily invariant norms two conditions corresponding to (i) and (ii) of Proposition 2.1 are not equivalent. In fact, we have a characterization of linear operators preserving all left unitarily invariant norms

2000 Mathematics Subject Classification 47A30, 47A63
Key Words: Left unitarily invariant norm; Linear operator preserving norms; Unitarily invariant norm.

in Theorem 2.3, and another one of linear operators preserving some left unitarily invariant norms in Proposition 3.1.

2. Linear operators preserving all LUI norms

Let $M_n(\mathbb{C})$ be the set of all complex n-square matrices. Let us recall the following characterization of linear operators on $M_n(\mathbb{C})$ preserving unitarily invariant norms:

Proposition 2.1. *Let T be a linear operator on $M_n(\mathbb{C})$. Then the following are equivalent:*
(i) T satisfies

$$N(T(A)) = N(A) \quad (\forall A \in M_n(\mathbb{C}))$$

for all unitarily invariant norms $N(\cdot)$ on $M_n(\mathbb{C})$.
(ii) T satisfies

$$N(T(A)) = N(A) \quad (\forall A \in M_n(\mathbb{C}))$$

for some unitarily invariant norm $N(\cdot)$ on $M_n(\mathbb{C})$ that is not a scalar multiple of the Frobenius norm.
(iii) There exist unitary matrices $U, V \in M_n(\mathbb{C})$ such that

$$T(A) = UAV \quad (\forall A \in M_n(\mathbb{C})),$$

or

$$T(A) = UA^tV \quad (\forall A \in M_n(\mathbb{C})),$$

where A^t denotes the transpose of A.

Remark 2.2. This is proved by A. R. Sourour [13, Theorem 2], C. K. Li and N. K. Tsing [7, Theorem 3.1], by K. Morita [10] for the Hilbert norm, by B. Russo [11, 12] for the trace norm, by J. Arazy [1] for $p-$norms, by R. Grone and M. Marcus [5] for the Ky-Fan norms. See also [9, 6, 8, 3].

Here is our main theorem:

Theorem 2.3. *Let T be a linear operator on $M_n(\mathbb{C})$. Then the following are equivalent:*
(i) T satisfies

$$\mu(T(A)) = \mu(A) \quad (\forall A \in M_n(\mathbb{C}))$$

for all left unitarily invariant norms $\mu(\cdot)$ on $M_n(\mathbb{C})$.
(ii) There exists a unitary matrix $U \in M_n(\mathbb{C})$ such that

$$T(A) = UA \quad (\forall A \in M_n(\mathbb{C})).$$

This follows immediately from Proposition 2.4 and Theorem 2.5.

Proposition 2.4. *Let $A, B \in M_n(\mathbb{C})$. Then the following are equivalent:*
(i) $\mu(A) \leqq \mu(B)$ *for all left unitarily invariant norms μ.*
(ii) $\|Ax\|_2 \leqq \|Bx\|_2 \quad (\forall x \in \mathbb{C}^n)$.

Here, $\|\cdot\|_2$ *is the Euclidean norm (or* l_2 *norm)* : $\|x\|_2 := \sqrt{\sum_{i=1}^{n} |x_i|^2}$ *for* $x = (x_i) \in \mathbb{C}^n$.

The condition (ii) in Proposition 2.4 is implicitly given in the proof of [4, Theorem 2.3]; therefore, the detail is left to the reader.

Theorem 2.5. *Let* T *be a linear operator on* $M_n(\mathbb{C})$ *and let* $\|\cdot\|_2$ *be the Euclidean norm on* $\mathbb{C}^n$*. Then the following are equivalent:*
(i) T *satisfies*

$$\|T(A)x\|_2 = \|Ax\|_2 \quad (\forall x \in \mathbb{C}^n, \forall A \in M_n(\mathbb{C})).$$

(ii) $T(I)$ *is unitary and*

$$T(A) = T(I)A \quad (\forall A \in M_n(\mathbb{C})).$$

Proof. (ii) $\Longrightarrow$ (i): clear.
(i) $\Longrightarrow$ (ii): by assumption,

$$\|T(I)x\|_2 = \|x\|_2 \quad (\forall x \in \mathbb{C}^n).$$

Hence, $T(I)$ is unitary. We may assume that $T(I) = I$ by considering the linear operator $T(I)^*T : M_n(\mathbb{C}) \to M_n(\mathbb{C})$. For simplicity, we give a proof for the case $n = 2$ because similar argument works for $n \geqq 3$.

Let $E_{11}, E_{12}, E_{21}, E_{22}$ be the standard basis for $M_2(\mathbb{C})$:

$$E_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, E_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, E_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, E_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Our claim is that $T(E_{ij}) = E_{ij} \quad (\forall i, j = 1, 2)$: T is the identity operator on $M_2(\mathbb{C})$. Since

$$\|T(E_{11})\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\|_2 = \|E_{11}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\|_2 = \|\begin{pmatrix} x_1 \\ 0 \end{pmatrix}\|_2$$

for all $x_1, x_2 \in \mathbb{C}$, $T(E_{11})$ is represented as

$$T(E_{11}) = \begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix}$$

for some $a, c \in \mathbb{C}$ with $|a|^2 + |c|^2 = 1$. Similarly,

$$T(E_{22}) = \begin{pmatrix} 0 & b \\ 0 & d \end{pmatrix}$$

for some $b, d \in \mathbb{C}$ with $|b|^2 + |d|^2 = 1$. The linearity of T and the assumption $T(I) = I$ imply that

$$I = T(I) = T(E_{11}) + T(E_{22}) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Therefore, $a = d = 1, b = c = 0$: $T(E_{11}) = E_{11}, T(E_{22}) = E_{22}$. Similar argument for E_{12}, E_{21} yields that

$$T(E_{12}) = \begin{pmatrix} 0 & q \\ 0 & s \end{pmatrix}, T(E_{21}) = \begin{pmatrix} p & 0 \\ r & 0 \end{pmatrix},$$

for some $p, q, r, s \in \mathbb{C}$ with $|p|^2 + |r|^2 = |q|^2 + |s|^2 = 1$. Since

$$\begin{aligned} \left\| \begin{pmatrix} 1 & q \\ 0 & s \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right\|_2 &= \left\| \{T(E_{11}) + T(E_{12})\} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right\|_2 \\ &= \left\| (E_{11} + E_{12}) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right\|_2 = \left\| \begin{pmatrix} x_1 + x_2 \\ 0 \end{pmatrix} \right\|_2 \end{aligned}$$

for all $x_1, x_2 \in \mathbb{C}$, $q = 1$ and $s = 0$: $T(E_{12}) = E_{12}$. Similarly, it follows that $T(E_{21}) = E_{21}$. Therefore, the proof is complete. □

As a corollary, we have:

Corollary 2.6. *Let T be a linear operator on $M_n(\mathbb{C})$. If $T(A)A^{-1}$ is unitary for each invertible matrix $A \in M_n(\mathbb{C})$, then there is a unitary matrix $U \in M_n(\mathbb{C})$ such that*

$$T(A) = UA$$

for all $A \in M_n(\mathbb{C})$.

3. Linear operators preserving some LUI norm

Let $N(\cdot)$ be a unitarily invariant norm on $M_n(\mathbb{C})$ and let $C \in M_n(\mathbb{C})$ be an invertible matrix. Then we have a left unitarily invariant norm $N_C(\cdot)$ on $M_n(\mathbb{C})$ defined as

$$N_C(A) := N(AC) \quad (A \in M_n(\mathbb{C})).$$

This norm is unitarily invariant if and only if C is a scalar multiple of a unitary matrix [4, Proposition 2.2].

Proposition 3.1. *Let $N(\cdot)$ be a unitarily invariant norm on $M_n(\mathbb{C})$ which is not a scalar multiple of the Frobenius norm, $C \in M_n(\mathbb{C})$ an invertible matrix, and $N_C(\cdot)$ the left unitarily invariant norm on $M_n(\mathbb{C})$ defined above. Then a linear operator $T : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ preserves the norm $N_C(\cdot)$, that is,*

$$N_C(T(A)) = N_C(A) \quad (\forall A \in M_n(\mathbb{C})),$$

if and only if there are unitary matrices $U, V \in M_n(\mathbb{C})$ such that

$$T(A) = UACVC^{-1} \quad (\forall A \in M_n(\mathbb{C})),$$

or

$$T(A) = U(AC)^t VC^{-1} \quad (\forall A \in M_n(\mathbb{C})).$$

This is an immediate consequence of Proposition 2.1.

Example 3.2. Let $A \in M_n(\mathbb{C})$ be represented by column vectors $a_1, a_2, \ldots, a_n \in \mathbb{C}^n$ as

$$A = (a_1\ a_2\ \ldots\ a_n).$$

Let $\|\cdot\|_0$ be a norm on $M_n(\mathbb{C})$ defined by

$$\|A\|_0 := \max\{\|a_i\|_\infty : 1 \leqq i \leqq n\} \quad (A = (a_1\ a_2\ \ldots\ a_n) \in M_n(\mathbb{C})),$$

where $\|\cdot\|_\infty$ denotes the max norm (or l_∞ norm) on $\mathbb{C}^n$. The norm $\|\cdot\|_0$ on $M_n(\mathbb{C})$ is not left (and right) unitarily invariant. To this norm, we have a left unitarily invariant norm $\mu(\cdot)$ defined as

$$\mu(A) := \max\{\|UA\|_0 : U \text{ is unitary}\} \quad (A \in M_n(\mathbb{C})).$$

This is not (right) unitarily invariant. It is easy to see that

$$\mu(A) = \max\{\|a_i\|_2 : 1 \leqq i \leqq n\} \quad (A = (a_1\ a_2\ \ldots\ a_n) \in M_n(\mathbb{C})),$$

where $\|\cdot\|_2$ is the Euclidean norm on $\mathbb{C}^n$.

For instance, when T is a linear operator given by a permutation for $a_1, a_2, \ldots, a_n$, then T preserves μ, but is not represented as left multiplication of a unitary matrix in general.

We remark that Proposition 3.1 and Example 3.2 also explain the difference between Proposition 2.1 for unitarily invariant norms and Theorem 2.3 for left unitarily invariant ones.

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Received August 10, 2006
Revised November 1, 2006