Nihonkai Math. J.
Vol.17(2006), 187-197

The C-numerical range of a 3×3 normal matrix

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Abstract In this note we study the shape of the C-numerical range of a 3×3 normal matrix.

1. Introduction and Results

In this decade many authors obtained new results in numerical ranges, numerical radii of linear operators and their related topics (cf. [2, 3, 6, 7, 9, 11]). In the paper [5] the author studied a special case of the C-numerical ranges. Recent work [4] provides us a new method to treat the C-numerical ranges. We will prove "weak convexity" of the C-numerical ranges in some sense.

Suppose that $C = \mathrm{diag}(c_1, c_2, c_3)$ and $T = \mathrm{diag}(\alpha_1, \alpha_2, \alpha_3)$ are complex 3×3 diagonal matrices. We consider a compact subset $W_C(T)$ of the Gaussian plane $\mathbf{C}$ defined by

$$W_C(T) = \{\mathrm{tr}(C\,UTU^*) : U \in M_3(\mathbf{C}),\ U^*U = UU^* = I_3\}. \tag{1.1}$$

However this range is not necessarily convex, this range is star-shaped with respect to the point

$$(1/3)(c_1 + c_2 + c_3)(\alpha_1 + \alpha_2 + \alpha_3) \in W_C(T). \tag{1.2}$$

We consider the following 6 special points of $W_C(T)$:

$$\sigma_1 = c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3,\ \sigma_2 = c_1\alpha_2 + c_2\alpha_3 + c_3\alpha_1,\ \sigma_3 = c_1\alpha_3 + c_2\alpha_1 + c_3\alpha_2, \tag{1.3}$$

$$\sigma_4 = c_1\alpha_1 + c_2\alpha_3 + c_3\alpha_2,\ \sigma_5 = c_1\alpha_3 + c_2\alpha_2 + c_3\alpha_1,\ \sigma_6 = c_1\alpha_2 + c_2\alpha_1 + c_3\alpha_3. \tag{1.4}$$

These are called σ-points of the range $W_C(T)$. The 9 line segments

$$[\sigma_j, \sigma_k] = \{(1-t)\sigma_j + t\sigma_k : 0 \leq t \leq 1\} \tag{1.5}$$

($j = 1,2,3,\ k = 4,5,6$) are contained in the range $W_C(T)$. Au-Yeung and Poon gave these results in [1]. We remark that the direction of these line segments :

$$\sigma_6 - \sigma_1 = (c_1 - c_2)(\alpha_2 - \alpha_1),\ \sigma_4 - \sigma_3 = (c_1 - c_2)(\alpha_1 - \alpha_3),$$
$$\sigma_5 - \sigma_2 = (c_1 - c_2)(\alpha_3 - \alpha_2),\ \sigma_4 - \sigma_2 = (c_1 - c_3)(\alpha_1 - \alpha_2),$$
$$\sigma_5 - \sigma_1 = (c_1 - c_3)(\alpha_3 - \alpha_1),\ \sigma_6 - \sigma_3 = (c_1 - c_3)(\alpha_2 - \alpha_3),$$
$$\sigma_5 - \sigma_3 = (c_2 - c_3)(\alpha_2 - \alpha_1),\ \sigma_6 - \sigma_2 = (c_2 - c_3)(\alpha_1 - \alpha_3),$$
$$\sigma_4 - \sigma_1 = (c_2 - c_3)(\alpha_3 - \alpha_2).$$

The following is our new result.

Theorem 1. **1***Suppose that* $C = \mathrm{diag}(c_1, c_2, c_3)$ *and* $T = \mathrm{diag}(\alpha_1, \alpha_2, \alpha_3)$ *are complex* 3×3 *diagonal matrices. Suppose that two elements* z_1, z_2 *of* $W_C(T)$ *satisfy the equation*

$$z_2 - z_1 = s\,(c_i - c_j)(\alpha_p - \alpha_q), \tag{1.6}$$

for some $s \in \mathbf{R}$ *with* $s \neq 0$ *and* $1 \leq i \neq j \leq 3$, $1 \leq p \neq q \leq 3$. *Then the line segment* $[z_1, z_2]$ *is contained in the range* $W_C(T)$.

In the case the range $W_C(T)$ is convex, the assertion of the above theorem follows immediately from the convexity of the range $W_C(T)$. If c_1, c_2, c_3 are colinear, then the range $W_C(T)$ is convex by Westwick's theorem ([10]). If $\alpha_1, \alpha_2, \alpha_3$ are colinear, then the range $W_C(T) = W_T(C)$ is convex. So we may assume that $(c_i - c_j)(\alpha_i - \alpha_j) \neq 0$ for $1 \leq i < j \leq 3$ and c_i's lie on a circle and α_j's lie on a circle. A 3×3 real matrix $A = (a_{ij})$ is called *doubly stochastic* if (a_{ij}) satisfies

$$a_{ij} \geq 0, \tag{1.7}$$

for $i, j = 1,2,3$ and

$$\sum_{i=1}^{3} a_{iq} = 1, \quad \sum_{j=1}^{3} a_{pj} = 1, \tag{1.8}$$

for $p, q = 1,2,3$. A 3×3 doubly stochastic matrix A is called *orthostochastic* if there exists a 3×3 unitary matrix $U = (u_{ij})$ with $a_{ij} = |u_{ij}|^2$ $(i, j = 1,2,3)$. In [5] it was shown that if $A = (a_{ij})$ is a boundary point of the set of 3×3 orthostochastic matrices, then the point (a_{ij}) satisfies the equation

$$F(a_{ij} : i, j = 1,2,3)$$

$$
\begin{aligned}
&= a_{11}^2 a_{12}^2 + a_{21}^2 a_{22}^2 + a_{31}^2 a_{32}^2 - 2a_{11}a_{12}a_{21}a_{22} - 2a_{11}a_{12}a_{31}a_{32} - 2a_{21}a_{22}a_{31}a_{32} \\
&= a_{11}^2 a_{13}^2 + a_{21}^2 a_{23}^2 + a_{31}^2 a_{33}^2 - 2a_{11}a_{13}a_{21}a_{23} - 2a_{11}a_{13}a_{31}a_{33} - 2a_{21}a_{23}a_{31}a_{33} \\
&= a_{12}^2 a_{13}^2 + a_{22}^2 a_{23}^2 + a_{32}^2 a_{33}^2 - 2a_{12}a_{13}a_{22}a_{23} - 2a_{12}a_{13}a_{32}a_{33} - 2a_{22}a_{23}a_{32}a_{33} \\
&= 0.
\end{aligned}
$$

We call that a general point z of $W_C(T)$ is represented by

$$z = \sum_{i=1}^{3} \sum_{j=1}^{3} c_i \, \alpha_j \, a_{ij}, \tag{1.9}$$

where $A = (a_{ij})$ is a boundary point of the set of all 3×3 orthostochastic matrices, and hence the polynomial F vanishes at (a_{ij}). Conversely the point z with the expression (1.9) by an orthostochastic matrix (a_{ij}) belongs to $W_C(T)$. We prove Theorem 1.1 by using this relation.

We define a subset K of the unit circle by

$$K = \{z \in \mathbf{C} : |z| = 1, z_1, z_2 \in W_C(T) \,\text{and}\, z_2 - z_1 = t\, z \,\text{for some}\, t \in \mathbf{R} \;\; \text{imply}\, [z_1, z_2] \in W_C(T)\}. \tag{1.10}$$

This set K is symmetric with respect to the origin. Theorem 1.1 implies that this set contains a set

$$\{\epsilon \frac{\sigma_j - \sigma_k}{|\sigma_j - \sigma_k|} : j = 1, 2, 3, k = 4, 5, 6, \epsilon = \pm 1\}. \tag{1.11}$$

Under the condition that neither the points c_1, c_2, c_3 nor the points $\alpha_1, \alpha_2, \alpha_3$ are colinear, the range $W_C(A)$ contains an interior point. Under this condition the range $W_C(T)$ is convex if and only if K coincides with the unit circle. We can show that the set K coincides with the set (1.11) in a case. So we may assert that Theorem 1.1 is best possible in some sense. In the case $C = \mathrm{diag}(c_1, c_2, c_3)$, $T = \mathrm{diag}(\alpha_1, \alpha_2, \alpha_3)$ with $|\alpha_1| = |\alpha_2| = |\alpha_3| = 1$, $|c_1| = |c_2| = |c_3| = 1$, $\alpha_1\alpha_2\alpha_3 = c_1c_2c_3 = 1$, the boundary of the range $W_C(A)$ consists of some line segments $\ell_{j,k} \subset [\sigma_j, \sigma_k]$ ($j = 1, 2, 3$, $k = 4, 5, 6$) and arcs Γ_j of the deltoid

$$\Gamma = \{2\exp(i\theta) + \exp(-2i\theta) : 0 \le \theta \le 2\pi\}. \tag{1.12}$$

(cf. [5]). If $C = T = \mathrm{diag}(1, -1/2 + \sqrt{3}i/2, -1/2 - \sqrt{3}i/2)$, then the boundary of $W_C(T)$ coincides with the deltoid Γ. In this case $\sigma_1 = \sigma_2 = \sigma_3 = 0$ and $\sigma_4 = 3$, $\sigma_5 = 3\,(-1/2 - \sqrt{3}i/2)$, $\sigma_6 = 3\,(-1/2 + \sqrt{3}i/2)$ and the set K coincides with

$$\{\exp(i\,\frac{2k\pi}{6}) : k = 0, 1, 2, 3, 4, 5\}.$$

This follows from Theorem 1.1 and the strict concaveness of the arc:

$$\{2\exp(i\theta)+\exp(-2i\theta) : 0 \leq \theta \leq \frac{2\pi}{3}\}.$$

2. Proof of the theorem

In this section we shall prove Theorem 1.1. By using the relations

$$\begin{aligned} a_{13} &= 1-a_{11}-a_{12}, \quad a_{23}=1-a_{21}-a_{22}, \\ a_{31} &= 1-a_{11}-a_{21}, \quad a_{32}=1-a_{12}-a_{22}, \\ a_{33} &= a_{11}+a_{12}+a_{21}+a_{22}-1, \end{aligned}$$

We rewrite the equation of a boubdary point of the set of the 3×3 orthostochastic matrices as the following :

$$\begin{aligned} F(a_{11},a_{12},a_{21},a_{22}) &= a_{11}^2a_{22}^2+a_{12}^2a_{21}^2-2a_{11}a_{12}a_{21}a_{22} \\ -2a_{11}a_{22}(a_{11}+a_{22}) &- 2a_{12}a_{21}(a_{12}+a_{21})-2(a_{11}a_{12}a_{21}+a_{11}a_{12}a_{22}+a_{11}a_{21}a_{22} \\ +a_{12}a_{21}a_{22}) &+ a_{11}^2+a_{12}^2+a_{21}^2+a_{22}^2+2(a_{11}a_{12}+a_{11}a_{21}+a_{12}a_{22}+a_{21}a_{22} \\ +2a_{11}a_{22}+2a_{12}a_{21}) &- 2(a_{11}+a_{12}+a_{21}+a_{22})+1=0. \end{aligned} \tag{2.1}$$

This equation is solved with respect to a_{11} as the following :

$$\begin{aligned} (a_{22}-1)^2a_{11} &= a_{12}a_{21}a_{22}+(1-a_{12}-a_{22})(1-a_{21}-a_{22}) \\ &+2\epsilon\sqrt{a_{12}a_{21}a_{22}}\sqrt{(1-a_{12}-a_{22})(1-a_{21}-a_{22})}, \end{aligned} \tag{2.2}$$

($\epsilon=\pm 1$) on the set

$$\begin{aligned} \{(a_{12},a_{21},a_{22}) : 0 \leq a_{12}, 0 \leq a_{21}, 0 \leq a_{22}, a_{22} \neq 1 \\ a_{12}+a_{22} \leq 1, a_{21}+a_{22} \leq 1\}. \end{aligned} \tag{2.3}$$

If $a_{22}\to 1$, and hence $a_{12}\to 0$, $a_{21}\to 0$, then a_{11} may converges to an arbitrary point of $[0,1]$.

The equation (2.2) implies that the solution a_{11} satisfies

$$a_{11} \geq 0, \tag{2.4}$$

on the set (2.3). In fact we have

$$\begin{aligned}&\{a_{11}a_{12}a_{21} + (a_{12} + a_{22} - 1)(a_{21} + a_{22} - 1)\}^2 \\ &-4a_{12}a_{21}a_{22}(1 - a_{12} - a_{22})(1 - a_{21} - a_{22}) \\ &= (a_{22} - 1)^2(a_{12}a_{21} - a_{12} - a_{21} - a_{22} + 1)^2.\end{aligned}$$

The solution (2.2) on (2.3) satisfies

$$1 - a_{11} - a_{12} \geq 0. \tag{2.5}$$

In fact we have

$$\begin{aligned}(a_{22} - 1)^2(1 - a_{11} - a_{12}) \geq\ & a_{21}(1 - a_{12} - a_{22}) + a_{12}a_{22}(1 - a_{21} - a_{22}) \\ & -2\sqrt{a_{12}a_{21}a_{22}}\sqrt{(1 - a_{12} - a_{22})(1 - a_{21} - a_{22})},\end{aligned}$$

where

$$a_{21}(1 - a_{12} - a_{22}) + a_{12}a_{22}(1 - a_{21} - a_{22}) \geq 0,$$

and

$$\begin{aligned}&\{a_{21}(1 - a_{12} - a_{22}) + a_{12}a_{22}(1 - a_{21} - a_{22})\}^2 - 4a_{12}a_{21}a_{22}(1 - a_{12} - a_{22})(1 - a_{21} - a_{22}) \\ &= (1 - a_{22})^2(a_{12}a_{21} + a_{12}a_{22} - a_{21})^2 \geq 0.\end{aligned}$$

Similarly the solution (2.2) on (2.3) satisfies

$$1 - a_{11} - a_{21} \geq 0. \tag{2.6}$$

The solution (2.2) on (2.3) satisfies

$$a_{11} + a_{12} + a_{21} + a_{22} - 1 \geq 0. \tag{2.7}$$

In fact we have

$$\begin{aligned}(a_{22} - 1)^2(a_{11} + a_{12} + a_{21} + a_{22} - 1) \geq\ & a_{12}a_{21} + a_{22}(1 - a_{12} - a_{22})(1 - a_{21} - a_{22}) \\ & -2\sqrt{a_{12}a_{21}a_{22}}\sqrt{(1 - a_{12} - a_{22})(1 - a_{21} - a_{22})},\end{aligned}$$

where

$$a_{12}a_{21} + a_{22}(1 - a_{12} - a_{22})(1 - a_{21} - a_{22}) \geq 0,$$

and

$$\begin{aligned}
&\{a_{12}a_{21} + a_{22}(1 - a_{12} - a_{22})(1 - a_{21} - a_{22})\}^2 \\
&\quad -4a_{12}a_{21}a_{22}(1 - a_{12} - a_{22})(1 - a_{21} - a_{22}) \\
&= (a_{22} - 1)^2(-a_{22} + a_{12}a_{21} + a_{12}a_{22} + a_{21}a_{22} + a_{22}^2)^2 \geq 0.
\end{aligned}$$

We also remark that the content of the radical in (2.2) satisfies

$$a_{12}a_{21}a_{22}(1 - a_{12} - a_{22})(1 - a_{21} - a_{22}) \geq 0. \tag{2.8}$$

The restriction of the 2-valued function a_{11} to sub convex domain of (2.3) also satisfies automatically the linear inequalities (2.4), (2.5), (2.6), (2.7) and the inequality (2.8). We shall prove the following lemma.

Lemma 2.1*Suppose that* $C = \mathrm{diag}(c_1, c_2, c_3)$, $T = \mathrm{diag}(a_1, a_2, a_3)$ *where the diagonal entries* a_j*'s and* c_j*'s are arbitrary complex numbers. Suppose that* σ_j *(*$j = 1, 2, 3, 4, 5, 6$*) are points of the range* $W_C(T)$ *defined by (1.3), (1.4) and* $\phi \in [0, 2\pi]$ *is an arbitrary angle. Then the equation*

$$\begin{aligned}
\{\Re(z\exp(-i\phi)) : z \in W_C(T)\} = \{&\Re(z\exp(-i\phi)) : z \in [\sigma_j, \sigma_k] \\
&(j = 1, 2, 3, k = 4, 5, 6)\}
\end{aligned} \tag{2.9}$$

holds.

Proof of Lemma 2.1. We consider even permutation matrices:

$$P_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad P_3 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

and odd permutation matrices:

$$P_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad P_5 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad P_6 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The point σ_j corresponds to P_j by the relation

$$(c_1, c_2, c_3)P_j(\alpha_1, \alpha_2, \alpha_3)^T = \sigma_j$$

($1 \leq j \leq 6$). The above 6 matrices are orthostochastic matrices : $P_j = P_j \circ P_j$ ($1 \leq j \leq 6$) where $\circ$ denotes the Hadamard product. By Birkhoff's theorem ([8], p.

200), the convex hull of these 6 matrices coincides the convex set of all 3×3 doubly stochastic matrices. We call representation (1.9) of a general point of $W_C(T)$ and the fact that the 9 line segments (1.5) is contained in $W_C(T)$. By using these facts, we obtain the inclusion

$$\cup\{[\sigma_j, \sigma_k] : j = 1,2,3, k = 4,5,6\} \subset W_C(T) \subset \mathrm{conv}(W_C(T))$$
$$= \{(c_1, c_2, c_3) S(\alpha_1, \alpha_2, \alpha_3)^T : S \text{ is a doubly stochastic matrix}\}$$
$$= \mathrm{conv}([\sigma_j, \sigma_k] : j = 1,2,3, k = 1,2,3).$$

The projection $\pi : z(\in \mathbf{C}) \to \Re(z\exp(-i\phi))(\in \mathbf{R})$ satisfies $\pi(\Gamma) = \pi(\mathrm{conv}(\Gamma))$ for every compact connected set $\Gamma \subset \mathbf{C}$. Thus the relation (2.9) follows from the above inclusion.

Proof of Theorem 1.1. We assume that two points z_1, z_2 of the range $W_C(T)$ satisfy the equation

$$z_2 - z_1 = s\,(c_1 - c_3)(\alpha_1 - \alpha_3),$$

for some $s \in \mathbf{R}$, $s \neq 0$ with $c_1 \neq c_3$, $\alpha_1 \neq \alpha_3$. By using a translation, we may assume that $c_3 = \alpha_3 = 0$. Under this assumption, a general point z of $W_C(T)$ is represented by

$$z = c_1\alpha_1 a_{11} + c_1\alpha_2 a_{12} + c_2\alpha_1 a_{21} + c_2\alpha_2 a_{22},$$

where (a_{ij}) is a doubly stochastic matrix satisfying the equation $F(a_{11}, a_{12}, a_{21}, a_{22}) = 0$. We choose angle $\phi \in [0, 2\pi]$ so that

$$\Re(c_1\alpha_1 \exp(-i\phi)) = 0.$$

We consider an affine constraint

$$\Re([c_1\alpha_2 a_{12} + c_2\alpha_1 a_{21} + c_2\alpha_2 a_{22}]\exp(-i\phi)) = \Re(z_1 \exp(-i\phi)), \tag{2.10}$$

on the hypersurface $F(a_{11}, a_{12}, a_{21}, a_{22}) = 0$ under the condition

$$0 \leq a_{12},\ 0 \leq a_{21},\ 0 \leq a_{22}, a_{12} + a_{22} \leq 1, a_{21} + a_{22} \leq 1.$$

The affine constrant is reduced to a trivial condition if and only if the equations

$$\Re(c_1\alpha_2 \exp(-i\phi)) = 0, \quad \Re(c_2\alpha_1 \exp(-i\phi)) = 0, \quad \Re(c_2\alpha_2 \exp(-i\phi)) = 0$$

hold. If these equations hold, then the range $W_C(T)$ lies on a straight line, and the connectedness of the group $U(3)$ guarantees that $[z_1, z_2] \subset W_C(T)$. So we assume that at least one of

$$\Re(c_1 a_2 \exp(-i\phi)), \quad \Re(c_2 a_1 \exp(-i\phi)), \quad \Re(c_2 a_2 \exp(-i\phi))$$

is non-zero. By Lemma 2.1 there exists a point $z_0 \in [\sigma_j, \sigma_k]$ satisfying

$$\Re(z_0 \exp(-i\phi)) = \Re(z_1 \exp(-i\phi)),$$

for some $j \in \{1, 2, 3\}, k \in \{4, 5, 6\}$. This implies that there exists a point $(a_{11}, a_{12}, a_{21}, a_{22})$ satisfying the affine constraint and

$$a_{12}a_{21}a_{22}(1 - a_{12} - a_{22})(1 - a_{21} - a_{22}) = 0.$$

Thus the two parts of the graph of 2-valued function a_{11} on the domain (2.3) with the constraint (2.10) is connected. As a continuous image of this connected set, the set

$$\begin{aligned} &\{z \in W_C(T) : \Re(z \exp(-i\phi)) = \Re(z_1 \exp(-i\phi))\} \\ &= \{z \in W_C(T) : z = z_1 + s c_1 \alpha_1 \text{ for some } s \in \mathbf{R}\}, \end{aligned} \tag{2.11}$$

is connected. The assertion of Theorem 1.1 for the case

$$(c_i - c_j)(\alpha_p - \alpha_q) = (c_1 - c_3)(\alpha_1 - \alpha_3)$$

follows from (2.11). By changing the roles of (c_1, c_2, c_3) or $(\alpha_1, \alpha_2, \alpha_3)$, we can prove the assertion of Theorem 1.1 for the other cases.

3. Example

We give an example to illustrate Theorem 1.1. Let

$$c_1 = \frac{63}{65} - \frac{16}{65}i, \quad c_2 = -\frac{3}{5} + \frac{4}{5}i, \quad c_3 = -\frac{5}{13} - \frac{12}{13}i,$$

$$\alpha_1 = \frac{40}{41} + \frac{9}{41}i, \quad \alpha_2 = -\frac{528}{697} + \frac{455}{697}i, \quad \alpha_3 = -\frac{15}{17} - \frac{8}{17}i.$$

Then the numbers c_j's and α_j's lie on the unit circle $|z| = 1$ and satisfy the condition $c_1 \, c_2 \, c_3 = 1$, $\alpha_1 \, \alpha_2 \, \alpha_3 = 1$. Set $C = \mathrm{diag}(c_1, c_2, c_3)$, $T = \mathrm{diag}(\alpha_1, \alpha_2, \alpha_3)$. The 6 σ-points of this system are given by

$$\sigma_1 = \frac{7583}{9061} - \frac{1342}{45305}i, \sigma_2 = \frac{7237}{45305} - \frac{5340}{9061}i, \sigma_3 = -\frac{37969}{45305} + \frac{38874}{45305}i$$

$$\sigma_4 = \frac{126829}{45305} - \frac{124}{45305} i, \sigma_5 = -\frac{54881}{45305} - \frac{20130}{9061} i, \sigma_6 = -\frac{12953}{9061} + \frac{11606}{45305} i.$$

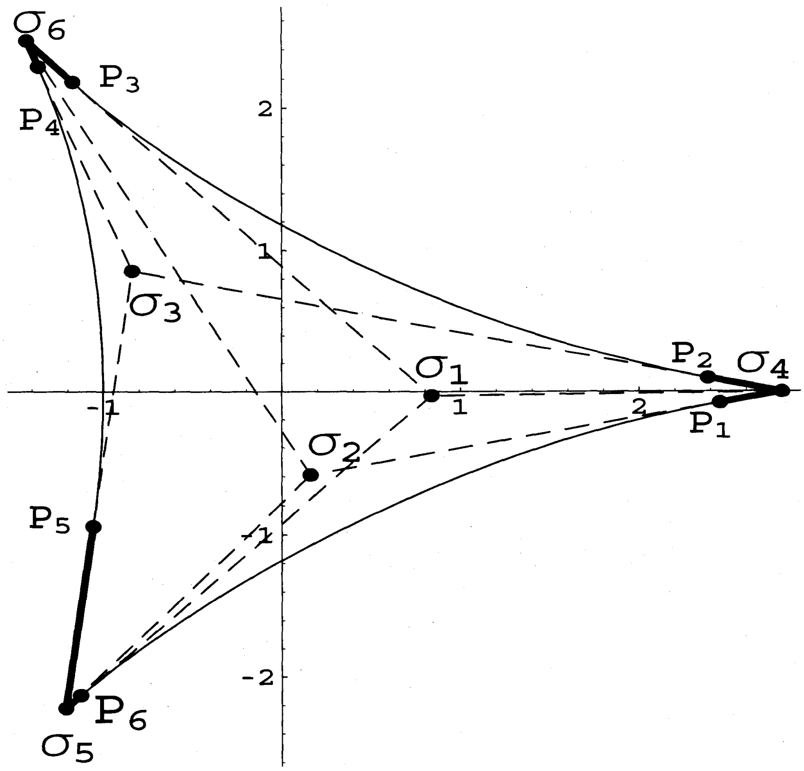

Figure 1:

In this example 6 points of the boundary of $W_C(T)$ appear as the point of tangency of the deltoid Γ defined by (1.13) and some line segement $[\sigma_j, \sigma_k]$. Those are given by the following:

$$P_1 = \frac{17723}{7225} - \frac{576}{7225} i, \quad P_2 = \frac{195906479}{82101721} + \frac{7805242}{82101721} i,$$
$$P_3 = -\frac{57241}{48841} + \frac{106480}{48841} i, \quad P_4 = -\frac{57313}{42025} + \frac{96026}{42025} i,$$
$$P_5 = -\frac{1289697}{1221025} - \frac{1149984}{1221025} i, \quad P_6 = \frac{17723}{7225} - 5767225\, i,$$

where $P_1, \ldots, P_6$ lie on respective line segments $[\sigma_2, \sigma_4]$,$[\sigma_3, \sigma_4]$, $[\sigma_1, \sigma_6]$, $[\sigma_3, \sigma_6]$, $[\sigma_3, \sigma_5]$, $[\sigma_1, \sigma_5]$. Figure 1 shows the boundary of the range $W_C(T)$ and the σ-points and the points P_j's. We consider the set K defined by (1.11). In this situation, a unit complex number $z = \exp(i\,\theta)$ with $-\pi/2 \leq \theta \leq \pi/2$ belongs to K if and only if the slope $m = \tan\theta$ satisfies one of the inequalities

$$m_1 = -\frac{19}{7} \leq m \leq m_2 = -\frac{11}{10}, m_3 = -\frac{31}{131} \leq m \leq m_4 = \frac{2}{9},$$

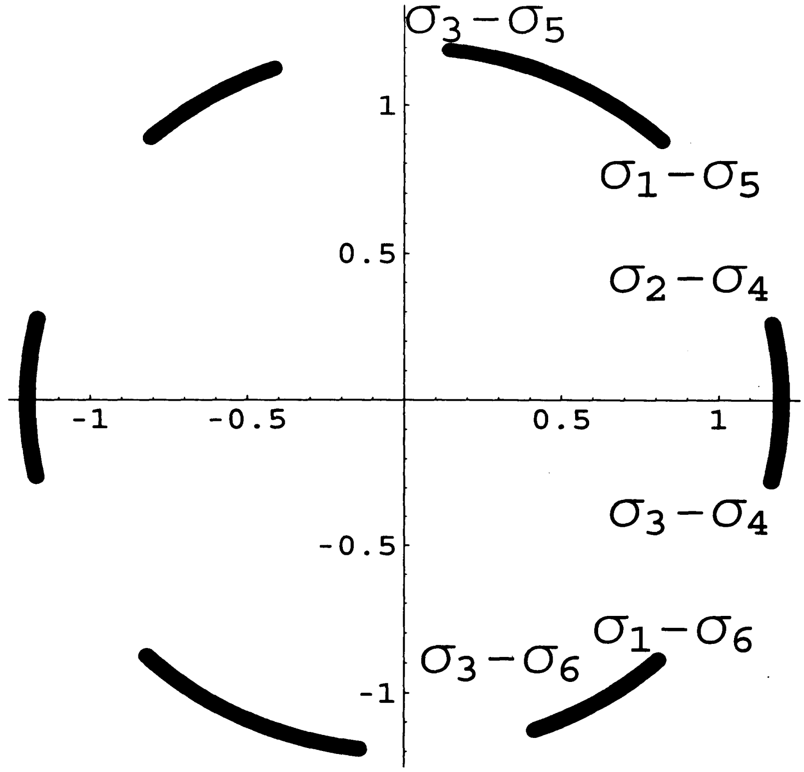

Figure 2:

$$m_5 = \frac{61}{57} \leq m \leq m_6 = \frac{33}{4},$$

where $m_1, \ldots, m_6$ are respeective slopes of the line segments $[\sigma_3, \sigma_6]$, $[\sigma_1, \sigma_6]$, $[\sigma_3, \sigma_4]$, $[\sigma_2, \sigma_4]$, $[\sigma_1, \sigma_5]$, $[\sigma_3, \sigma_5]$. Figure 2 shows 6/5-times K.

Acknowledgement The author would like to express his thanks to the refree for his (or her) careful reading of the manuscript and some useful suggestions.

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Received September 1, 2005
Revised October 21, 2006