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AN ESTIMATION OF QUASI-ARITHMETIC MEAN BY ARITHMETIC MEAN AND ITS APPLICATIONS

MASARU TOMINAGA

ABSTRACT. The quasi-arithmetic mean inequality says that if f is an increasing strictly convex function on an interval I, then $f^{-1}(\langle f(A)x,x\rangle) \geq \langle Ax,x\rangle$ for all unit vectors x in a Hilbert space H and a selfadjoint operator A on H, whose spectrum is contained in I. In this paper, we consider reverse inequalities of the quasi-arithmetic mean inequality. For each $\lambda > 0$ we observe an upper bound of a difference

$$f^{-1}(\langle f(A)x,x\rangle) - \lambda\langle Ax,x\rangle.$$

We find a condition on vectors x which attain the optimal bounds.

Replacing a given function $f(t)$ by a power, the logarithmic and the exponential function, we show these reverse quasi-arithmetic mean inequalities and equality conditions, in which the obtained constants are expressed by a generalized Kantorovich constant, the Specht ratio and the logarithmic mean.

1. INTRODUCTION

Let f be a strictly increasing continuous function on an interval I. Then

$$f^{-1}\Big(\frac{1}{n}\sum_{i=1}^{n} f(a_i)\Big) \tag{1.1}$$

is called the quasi-arithmetic mean of $a = (a_1,\ldots,a_n) \in I^n(\subset \mathbb{R}^n)$ by f (cf. [15]). Typical examples are arithmetic, geometric and harmonic means which correspond to functions $f(t) = t$, $\log t$ and $-\frac{1}{t}$, respectively.

Throughout this paper, an operator means a bounded linear operator on a Hilbert space H. For each unit vector $x \in H$, we consider

$$f^{-1}(\langle f(A)x,x\rangle) \tag{1.2}$$

for all selfadjoint operators A whose spectra are contained in I, as an operator version of the quasi-arithmetic mean (1.1). Incidentally, $\langle Ax,x\rangle$ is regarded as the arithmetic mean. Indeed, (1.1) is obtained by putting $A = \operatorname{diag}(a_1,\ldots,a_n)$ and $x = \frac{1}{\sqrt{n}}\begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix}$ in (1.2), and obviously $\langle Ax,x\rangle = \frac{1}{n}\sum a_i$. If we choose the logarithmic function $f(t) = \log t$, then its quasi-arithmetic mean $\exp\langle(\log A)x,x\rangle$ for a fixed unit

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vector $x \in H$ is called the determinant of a positive invertible operator A in [3], [5]. It is known that if A satisfies $0 < m \leq A \leq M$, then

$$\exp\langle(\log A)x, x\rangle \leq \langle Ax, x\rangle \leq S(h)\exp\langle(\log A)x, x\rangle \tag{1.3}$$

where $h := \frac{M}{m}$ and $S(h)$ is the *Specht ratio* [22] defined by

$$S(h) := \frac{h^{\frac{1}{h-1}}}{e\log h^{\frac{1}{h-1}}} \quad \text{for } h > 0 \text{ with } h \neq 1 \qquad \text{and} \qquad \text{S(1):=1.} \tag{1.4}$$

The Specht ratio $S(h)$ is an upper bound of the arithmetic mean by the geometric one (cf. [2], [5], [24]). In general, for every (strictly increasing) convex (resp. concave) f, the quasi-arithmetic mean inequality

$$f^{-1}(\langle f(A)x, x\rangle) \geq \langle Ax, x\rangle \quad \left(\text{resp. } f^{-1}(\langle f(A)x, x\rangle) \leq \langle Ax, x\rangle\right) \tag{1.5}$$

holds for all unit vectors $x \in H$ which is equivalent to the Jensen inequality

$$\langle f(A)x, x\rangle \geq f(\langle Ax, x\rangle) \quad (\text{resp. } \langle f(A)x, x\rangle \leq f(\langle Ax, x\rangle)).$$

On the other hand, the following reverse inequality of (1.5) for $f(t) = t^p$ is given by Furuta [8]: For $p > 1$ (resp. $0 < p < 1$)

$$\begin{gathered} \langle Ax, x\rangle^p \leq \langle A^p x, x\rangle \leq K(h,p)\langle Ax, x\rangle^p \\ (\text{resp. } \langle Ax, x\rangle^p \geq \langle A^p x, x\rangle \geq K(h,p)\langle Ax, x\rangle^p) \end{gathered} \tag{1.6}$$

where $K(h,p)$ is a *generalized Kantorovich constant* (cf. [2], [9], [10]) defined by

$$K(h,p) := \frac{1}{h-1}\frac{h^p - h}{p-1}\left(\frac{p-1}{h^p-h}\frac{h^p-1}{p}\right)^p \tag{1.7}$$

for all $h > 0$ and $p \in \mathbb{R}$.

In this paper, we give bounds of a difference

$$f^{-1}(\langle f(A)x, x\rangle) - \lambda\langle Ax, x\rangle \tag{1.8}$$

for each $\lambda > 0$. Precisely, we determine the optimal constant $F(\lambda) = F(m, M, f, \lambda)$ (resp. $G(\lambda) = G(m, M, f, \lambda)$) such that

(1.9)
$$f^{-1}(\langle f(A)x, x\rangle) - \lambda\langle Ax, x\rangle \leq F(\lambda) \quad \left(\text{resp. } f^{-1}(\langle f(A)x, x\rangle) - \lambda\langle Ax, x\rangle \geq G(\lambda)\right)$$

for strictly increasing convex (resp. concave) functions f on $[m, M]$ without differentiability. Next we obtain useful estimations by applying it to functions t^p, $\log t$, e^t and $t\log t$. In particular, we have bounds of the ratio $f^{-1}(\langle f(A)x, x\rangle)/\langle Ax, x\rangle$ and the difference $f^{-1}(\langle f(A)x, x\rangle) - \langle Ax, x\rangle$ which include (1.3), (1.6) and other known results. Furthermore, we investigate a specific vector x which gives the equality for the obtained inequality (1.9).

2. Main results

In this section, we consider an optimal estimation of a difference $f^{-1}(\langle f(A)x,x\rangle)-\lambda\langle Ax,x\rangle$ for each $\lambda>0$ under some conditions.

For the sake of convenience, we define two constants α_f and β_f for all real valued function $f(t)$ on the interval $[m,M]$ with $m<M$ as follows:

$$\alpha_f:=\frac{f(M)-f(m)}{M-m}\quad\text{and}\quad\beta_f:=\frac{Mf(m)-mf(M)}{M-m}.$$

In particular, if $f(t)$ is strictly increasing, then $\alpha_f>0$.

We first estimate the quasi-arithmetic mean $f^{-1}(\langle f(A)x,x\rangle)$ by the arithmetic mean $\langle Ax,x\rangle$ for an increasing function f. We use the Mond-Pečarić method [12].

Lemma 2.1. *Let A be a selfadjoint operator on a Hilbert space H such that $m\leq A\leq M$ for some scalars $m<M$. Let $f(t)$ be a strictly increasing continuous function on the interval $[m,M]$. If there exists $t_0\in[m,M]$ where it satisfies an inequality $f(t)\leq\alpha_f(t-t_0)+f(t_0)$, then for each $\lambda>0$*

$$f^{-1}(\langle f(A)x,x\rangle)-\lambda\langle Ax,x\rangle\leq\max_{m\leq t\leq M}\left\{f^{-1}(\alpha_f(t-t_0)+f(t_0))-\lambda t\right\}\tag{2.1}$$

holds for all unit vectors $x\in H$.

Proof. The inequality $f(t)\leq\alpha t+\beta$ implies $f(A)\leq\alpha A+\beta$. Since f^{-1} is increasing and $m\leq\langle Ax,x\rangle\leq M$, it follows that

$$f^{-1}(\langle f(A)x,x\rangle)-\lambda\langle Ax,x\rangle\leq f^{-1}(\alpha\langle Ax,x\rangle+\beta)-\lambda\langle Ax,x\rangle\leq\max_{m\leq t\leq M}\{f^{-1}(\alpha t+\beta)-\lambda t\}.$$

Moreover, if we put $\alpha=\alpha_f$ and $\beta=-\alpha_f t_0+f(t_0)$, then (2.1) holds. □

Here we put

$$f^{(-)}(t):=\lim_{h\to-0}\frac{f(t+h)-f(t)}{h}\quad\text{and}\quad f^{(+)}(t):=\lim_{h\to+0}\frac{f(t+h)-f(t)}{h}$$

if exist, and I_f means the interval spanned by $\frac{\alpha_f}{f^{(-)}(M)}$ and $\frac{\alpha_f}{f^{(+)}(m)}$. (In particular, if $f^{(-)}(t),f^{(+)}(t)=\pm\infty$ for $t\in[m,M]$, then we put $\frac{\alpha_f}{f^{(-)}(t)},\frac{\alpha_f}{f^{(+)}(t)}=0$.) If $f(t)$ is convex, then $f^{(-)}(t)\leq f^{(+)}(t)$. Moreover we suppose that $f(t)$ is increasing. Then for each $\lambda\in I_f$, there exists a unique $\mu_\lambda\in[m,M]$ such that $f^{(-)}(\mu_\lambda)\leq\frac{\alpha_f}{\lambda}\leq f^{(+)}(\mu_\lambda)$.

In our previous note [20], we gave reverse inequalities related to the Jensen one under an assumption that $f(t)$ is twice differentiable. But we have the following theorem without differentiability.

Theorem 2.2. *Let A be a selfadjoint operator on a Hilbert space H such that $m\leq A\leq M$ for some scalars $m<M$. Let $f(t)$ be a real valued continuous increasing and strictly convex (resp. strictly concave) function on $[m,M]$ with $f^{(+)}(m)\neq0$*

(resp. $f^{(-)}(M) \neq 0$). Then for each $\lambda > 0$

$$f^{-1}(\langle f(A)x, x\rangle) - \lambda\langle Ax, x\rangle \le F(m, M, f, \lambda)$$
$$\left(resp.\ f^{-1}(\langle f(A)x, x\rangle) - \lambda\langle Ax, x\rangle \ge G(m, M, f, \lambda)\right) \tag{2.2}$$

holds for all unit vectors $x \in H$ where

$$F(m, M, f, \lambda) := \begin{cases} (1-\lambda)M & \text{if } 0 < \lambda < \frac{\alpha_f}{f^{(-)}(M)} \\ \mu_\lambda - \frac{f(\mu_\lambda)-\beta_f}{\alpha_f}\lambda & \text{if } \lambda \in I_f \\ (1-\lambda)m & \text{if } \lambda > \frac{\alpha_f}{f^{(+)}(m)} \end{cases}$$
$$\left(resp.\quad G(m, M, f, \lambda) := \begin{cases} (1-\lambda)m & \text{if } 0 < \lambda < \frac{\alpha_f}{f^{(+)}(m)} \\ \mu_\lambda - \frac{f(\mu_\lambda)-\beta_f}{\alpha_f}\lambda & \text{if } \lambda \in I_f \\ (1-\lambda)M & \text{if } \lambda > \frac{\alpha_f}{f^{(-)}(M)} \end{cases} \right). \tag{2.3}$$

Furthermore the equivalent condition for which the equality of (2.2) holds is as follows:
(i) *If $0 < \lambda < \frac{\alpha_f}{f^{(-)}(M)}$ (resp. $0 < \lambda < \frac{\alpha_f}{f^{(+)}(m)}$), then x is an eigenvector for M (resp. m).*
(ii) *If $\lambda \in I_f$, then*

$$x = x_f(\lambda) := \sqrt{1 - \frac{f(\mu_\lambda) - f(m)}{f(M) - f(m)}}\, e_m + \sqrt{\frac{f(\mu_\lambda) - f(m)}{f(M) - f(m)}}\, e_M \tag{2.4}$$

where e_m and e_M are corresponding unit eigenvectors to m and M, respectively.
(iii) *If $\lambda > \frac{\alpha_f}{f^{(+)}(m)}$ (resp. $\lambda > \frac{\alpha_f}{f^{(-)}(M)}$), then x is an eigenvector for m (resp. M).*

Proof. We only prove the case that $f(t)$ is convex. Then we have $f(t) \le \alpha_f(t - m) + f(m) = \alpha_f t + \beta_f$ for all $t \in [m, M]$. Since $f(t)$ is strictly increasing on $[m, M]$, $f^{-1}(t)$ can be defined on $[f(m), f(M)]$ and strictly increasing. If $t_0 = m$ in (2.1), then we have

$$f^{-1}(\langle f(A)x, x\rangle) - \lambda\langle Ax, x\rangle \le \max_{m \le t \le M}\{f^{-1}(\alpha_f t + \beta_f) - \lambda t\}$$

holds for all unit vectors $x \in H$. We put

$$h_\lambda(t) := f^{-1}(\alpha_f t + \beta_f) - \lambda t \quad \text{for } t \in [m, M].$$

Then we see the strict concavity of $h_\lambda(t)$ by the strict convexity of $f(t)$ and $h_\lambda^{(\pm)}(t) = \frac{\alpha_f}{f^{(\pm)}(f^{-1}(\alpha_f t + \beta_f))} - \lambda$, respectively. We divide into three cases

$$\text{(i) } 0 < \lambda < \frac{\alpha_f}{f^{(-)}(M)}, \quad \text{(ii) } \lambda \in I_f := \left[\frac{\alpha_f}{f^{(-)}(M)}, \frac{\alpha_f}{f^{(+)}(m)}\right] \quad \text{and} \quad \text{(iii) } \lambda > \frac{\alpha_f}{f^{(+)}(m)}.$$

Firstly we suppose (ii), i.e., $\lambda \in I_f$. Since $h_\lambda(t)$ is strictly concave and $h_\lambda^{(-)}(M) \le 0 \le h_\lambda^{(+)}(m)$, there exists a unique $t_\lambda \in [m, M]$ such that $h_\lambda^{(+)}(t_\lambda) \le 0 \le h_\lambda^{(-)}(t_\lambda)$ and so $\max_{m \le t \le M} h_\lambda(t) = h_\lambda(t_\lambda)$. (If either $h_\lambda^{(+)}(m)$ or $h_\lambda^{(-)}(M) = 0$, then the maximum is given by $t = t_\lambda = m$ or M, respectively.) Let $\mu_\lambda := f^{-1}(\alpha_f t_\lambda + \beta_f) \in [m, M]$. Then we have $t_\lambda = \frac{f(\mu_\lambda)-\beta_f}{\alpha_f}$ and

$$\max_{m\le t\le M} h_\lambda(t) = h_\lambda(t_\lambda) = f^{-1}(\alpha_f t_\lambda + \beta_f) - \lambda t_\lambda = \mu_\lambda - \frac{f(\mu_\lambda)-\beta_f}{\alpha_f}\lambda.$$

Moreover we suppose that

$$f^{-1}(\langle f(A)x, x\rangle) - \lambda\langle Ax, x\rangle = f^{-1}(\alpha_f\langle Ax, x\rangle + \beta_f) - \lambda\langle Ax, x\rangle = h_\lambda(t_\lambda),$$

for some unit vector $x = x_{f,\lambda}$. Then $\langle f(A)x, x\rangle = \alpha_f\langle Ax, x\rangle + \beta_f$ and $\langle Ax, x\rangle = t_\lambda$ by the strict concavity of h_λ. Since $f(t) < \alpha_f t + \beta_f$ for $t \in (m, M)$, the former holds if and only if x is a linear combination of eigenvectors e_m and e_M corresponding to m and M, respectively. We may write $x = \sqrt{1-s^2}e_m + se_M$ for some $s \in [0, 1]$. So we have

$$\frac{f(\mu_\lambda)-\beta_f}{\alpha_f} = t_\lambda = \langle Ax, x\rangle = (1-s^2)m + s^2M = (M-m)s^2 + m,$$

or $s^2 = \frac{f(\mu_\lambda)-\beta_f-m\alpha_f}{(M-m)\alpha_f} = \frac{f(\mu_\lambda)-f(m)}{f(M)-f(m)}$ and hence x is of form (2.4). On the other hand, if $x = x_f(\lambda)$ in (2.4), then we have

$$\begin{aligned} &f^{-1}(\langle f(A)x_f(\lambda), x_f(\lambda)\rangle) - \lambda\langle Ax_f(\lambda), x_f(\lambda)\rangle \\ &= f^{-1}\left((f(M)-f(m))\frac{f(\mu_\lambda)-f(m)}{f(M)-f(m)} + f(m)\right) - \lambda\left((M-m)\frac{f(\mu_\lambda)-f(m)}{f(M)-f(m)} + m\right) \\ &= \mu_\lambda - \frac{f(\mu_\lambda)-f(m)+m\alpha_f}{\alpha_f}\lambda = \mu_\lambda - \frac{f(\mu_\lambda)-\beta_f}{\alpha_f}\lambda. \end{aligned}$$

We suppose (i), i.e., $0 < \lambda < \frac{\alpha_f}{f^{(-)}(M)}$. Since $h_\lambda(t)$ is strictly concave,

$$\begin{aligned} &\min\{h_\lambda^{(+)}(m),\ h_\lambda^{(-)}(M),\ h_\lambda^{(\pm)}(t) : m < t < M\} \\ &= h_\lambda^{(-)}(M) = \frac{\alpha_f}{f^{(-)}(f^{-1}(\alpha_f M + \beta_f))} - \lambda = \frac{\alpha_f}{f^{(-)}(M)} - \lambda > 0. \end{aligned}$$

Hence $h_\lambda(t)$ is strictly increasing in $[m, M]$. Then we have

$$\max_{m\le t\le M} h_\lambda(t) = h_\lambda(M) = f^{-1}(\alpha_f M + \beta_f) - \lambda M = f^{-1}(f(M)) - \lambda M = (1-\lambda)M.$$

Moreover we suppose that

$$f^{-1}(\langle f(A)x, x\rangle) - \lambda\langle Ax, x\rangle = f^{-1}(\alpha_f\langle Ax, x\rangle + \beta_f) - \lambda\langle Ax, x\rangle = (1-\lambda)M,$$

for some unit vector x. Since $h_\lambda(\langle Ax, x\rangle) = h_\lambda(M)$, we see that $\langle Ax, x\rangle = M$ and so x is an eigenvector for M.

We suppose (iii), i.e., $\lambda > \frac{\alpha_f}{f^{(+)}(m)}$. Then we see $\max\{h_\lambda^{(+)}(m),\ h_\lambda^{(-)}(M),\ h_\lambda^{(\pm)}(t) : m < t < M\} < 0$. So the maximum of $h_\lambda(t)$ attains at $t = m$, and we have

$$\max_{m\le t\le M} h_\lambda(t) = h_\lambda(m) = (1-\lambda)m.$$

Moreover we suppose that

$$f^{-1}(\langle f(A)x, x\rangle) - \lambda\langle Ax, x\rangle = (1-\lambda)m$$

if and only if x is an eigenvector for m by a similar way to (i). $\square$

Under the same assumptions in Theorem 2.2, we denote the secant through two points $(m, f(m))$ and $(M, f(M))$ as $y = \alpha_f t + \beta_f$. Suppose $mM \geq 0$. Since $f(t)$ is strictly convex, there exists a unique tangent line of the graph $y = f(t)$ through $(0, \beta_f)$

$$y = f(\nu_f) + f^0(\nu_f)(t - \nu_f)$$

where $(\nu_f, f(\nu_f))$ is the point of tangency and $f^0(\nu) := \frac{f(\nu)-\beta_f}{\nu}$. Since we see that $\nu_f \in [m, M]$, the equation $\beta_f = f(\nu) - f^0(\nu)\nu$ has a unique solution $\nu = \nu_f$.

By using the above fact we give explicit estimations of the ratio and difference of $f^{-1}(\langle f(A)x, x\rangle)$ by $\langle Ax, x\rangle$.

Corollary 2.3. *Suppose the hypothesis of Theorem 2.2.*
(i) *Suppose $mM \geq 0$. Then the ratio inequality*

$$f^{-1}(\langle f(A)x, x\rangle) \leq \lambda_f\langle Ax, x\rangle \quad \left(\textit{resp. } f^{-1}(\langle f(A)x, x\rangle) \geq \lambda_f\langle Ax, x\rangle\right) \tag{2.5}$$

holds for all unit vectors $x \in H$ where $\lambda_f := \frac{\alpha_f}{f^0(\nu_f)} \in I_f$ is a unique solution of $F(\lambda) := F(m, M, f, \lambda) = 0$ (resp. $G(\lambda) := G(m, M, f, \lambda) = 0$). Moreover the equality of (2.5) holds if and only if both m and M are eigenvalues of A and $x = x_f(\lambda_f)$ where $x = x_f(\cdot)$ is defined in Theorem 2.2.

(ii) *The difference inequality*

$$\begin{gathered} f^{-1}(\langle f(A)x, x\rangle) - \langle Ax, x\rangle \leq \mu_1 - \frac{f(\mu_1) - \beta_f}{\alpha_f} \\ \left(\textit{resp. } f^{-1}(\langle f(A)x, x\rangle) - \langle Ax, x\rangle \geq \mu_1 - \frac{f(\mu_1) - \beta_f}{\alpha_f}\right) \end{gathered} \tag{2.6}$$

holds for all unit vectors $x \in H$ where $\mu = \mu_1 \in (m, M)$ is a unique solution of the equation $f^0(\mu) = \alpha_f$. Moreover the equality of (2.6) holds if and only if both m and M are eigenvalues of A and and $x = x_f(1)$ where $x = x_f(\cdot)$ is defined in Theorem 2.2.

Proof. We only prove the case that $f(t)$ is convex. It follows from $0 < f^{(+)}(m) \leq \alpha_f \leq f^{(-)}(M)$ that we may consider the case $\lambda \in I_f$ to estimate $f^{-1}(\langle f(A)x, x\rangle)$ by $\langle Ax, x\rangle$ in (2.3). Indeed, we see $\lambda \neq 1$ and $F(\lambda) \neq 0$ in other two cases.

We put $\lambda_f := \frac{\alpha_f}{f^0(\nu_f)} \in I_f$. It implies that there exists a unique solution $\mu = \mu_{\lambda_f} \in [m, M]$ such that $f^{(-)}(\mu) \leq \frac{\alpha_f}{\lambda_f} \leq f^{(+)}(\mu)$. Moreover the inequality $f^{(-)}(\mu) \leq f^0(\nu_f) \leq f^{(+)}(\mu)$ has a unique solution $\mu = \nu_f$. So it implies that $\mu_{\lambda_f} = \nu_f$ and hence

$$F(m, M, f, \lambda_f) = \mu_{\lambda_f} - \frac{f(\mu_{\lambda_f}) - \beta_f}{\alpha_f}\lambda_f = \nu_f - \frac{f(\nu_f) - \beta_f}{f^0(\nu_f)} = 0.$$

Hence (2.5) is ensured by (2.2). Next the difference inequality (2.6) is obtained by $\lambda = 1$ in (2.2), in which we remark $1 \in I_f$ and $\mu_1 \in (m, M)$ by the mean value theorem. Furthermore, equality conditions of inequalities (2.5) and (2.6) are obtained by (2.4). $\square$

3. Applications for the power function

We consider applications of Theorem 2.2 to the power function $f(t) = t^p$ $(t > 0)$ for $p > 0$. Here we cite the Hölder-McCarthy inequality [18]: For $p \geq 1$ (resp. $0 \leq p \leq 1$)

$$\langle A^p x, x\rangle \geq \langle Ax, x\rangle^p \quad (\text{resp. } \langle A^p x, x\rangle \leq \langle Ax, x\rangle^p) \tag{3.1}$$

(cf. [6], [8], [17]). We give a complementary inequality of (3.1), that is, the following estimation of a difference $\langle A^p x, x\rangle^{1/p} - \lambda\langle Ax, x\rangle$ for each $\lambda > 0$. As a matter of fact, we concentrate the case of $\lambda \in I_p = I_{t^p}$ spanned by $\frac{h^p-1}{ph^{p-1}(h-1)}$ and $\frac{h^p-1}{p(h-1)}$.

Theorem 3.1. *Let A be a positive operator on a Hilbert space H such that $0 < m \leq A \leq M$ for some scalars $m < M$ and $h := \frac{M}{m}(> 1)$. If $p > 1$ (resp. $0 < p < 1$), then for every $\lambda \in I_p$*

$$\begin{gathered}\langle A^p x, x\rangle^{\frac{1}{p}} - \lambda\langle Ax, x\rangle \leq m\frac{h^p - h}{h^p - 1}\left\{\left(\frac{K(h,p)}{\lambda}\right)^{\frac{1}{p-1}} - \lambda\right\} \\ \left(\textit{resp. } \langle A^p x, x\rangle^{\frac{1}{p}} - \lambda\langle Ax, x\rangle \geq m\frac{h^p - h}{h^p - 1}\left\{\left(\frac{K(h,p)}{\lambda}\right)^{\frac{1}{p-1}} - \lambda\right\}\right)\end{gathered} \tag{3.2}$$

holds for all unit vectors $x \in H$ where $K(h,p)$ is defined by (1.7).

Moreover the equality of (3.2) holds if and only if both m and M are eigenvalues of A and

(3.3)

$$x = x_p(\lambda)$$
$$:= \sqrt{\frac{1}{h^p-1}\left\{h^p - \left(\frac{h^p-1}{p\lambda(h-1)}\right)^{\frac{p}{p-1}}\right\}}\, e_m + \sqrt{\frac{1}{h^p-1}\left\{\left(\frac{h^p-1}{p\lambda(h-1)}\right)^{\frac{p}{p-1}} - 1\right\}}\, e_M$$

where e_m and e_M are corresponding unit eigenvectors to m and M, respectively.

Proof. We suppose $p > 1$ and replace $f(t)$ by t^p for all $t > 0$ in Theorem 2.2. Then we have $\alpha_f = \frac{M^p - m^p}{M-m}$ $\left(= m^{p-1}\frac{h^p-1}{h-1}\right)$, $\beta_f = \frac{Mm^p - mM^p}{M-m}$ $\left(= m^p\frac{h-h^p}{h-1}\right)$ and $I_p := I_f = \left[\frac{h^p-1}{ph^{p-1}(h-1)}, \frac{h^p-1}{p(h-1)}\right]$. For each $\lambda \in I_p$ it follows that the equation $(\mu^p)' = m^{p-1}\frac{h^p-1}{\lambda(h-1)}$

has a unique solution $\mu=\mu_\lambda:=m\left(\frac{h^p-1}{p\lambda(h-1)}\right)^{\frac{1}{p-1}}$ $(\in[m,M])$, and

$$\begin{aligned}\mu_\lambda-\frac{f(\mu_\lambda)-\beta_f}{\alpha_f}\lambda&=m\left(\frac{h^p-1}{p\lambda(h-1)}\right)^{\frac{1}{p-1}}-\lambda m\frac{h-1}{h^p-1}\left(\frac{h^p-1}{p\lambda(h-1)}\right)^{\frac{p}{p-1}}+m\frac{h-h^p}{h^p-1}\lambda\\&=\frac{p-1}{p}m\left(\frac{h^p-1}{p\lambda(h-1)}\right)^{\frac{1}{p-1}}+m\frac{h-h^p}{h^p-1}\lambda\\&=m\frac{h^p-h}{h^p-1}\left[\left\{\left(\frac{h^p-1}{h^p-h}\frac{p-1}{p}\right)^p\left(\frac{h^p-h}{\lambda(h-1)(p-1)}\right)\right\}^{\frac{1}{p-1}}-\lambda\right]\\&=m\frac{h^p-h}{h^p-1}\left\{\left(\frac{K(h,p)}{\lambda}\right)^{\frac{1}{p-1}}-\lambda\right\},\end{aligned}$$

which gives (3.2).

Moreover, since we have

$$\frac{f(\mu_\lambda)-f(m)}{f(M)-f(m)}=\frac{m^p\left(\frac{h^p-1}{p\lambda(h-1)}\right)^{\frac{p}{p-1}}-m^p}{M^p-m^p}=\frac{1}{h^p-1}\left\{\left(\frac{h^p-1}{p\lambda(h-1)}\right)^{\frac{p}{p-1}}-1\right\},$$

the equality condition is given in (3.3) by (2.4).

We can apply a similar discussion to the above in the case of $0<p<1$. □

Comparing with our previous result [23, Corollary 2], the equivalent condition (3.3) that the equality of (3.2) holds, is added. The following ratio inequality and difference one were shown by Furuta [8] and Izumino [16], respectively. They are obtained as a corollary of Theorem 3.1 by the facts that $\lambda=K(h,p)^{\frac{1}{p}}(\in I_p)$ is a unique solution of the equation $\left(\frac{K(h,p)}{\lambda}\right)^{\frac{1}{p-1}}-\lambda=0$ and $1\in I_p$.

Corollary 3.2. *Assume that the conditions of Theorem 3.1 hold and $p>1$ (resp. $0<p<1$).*

(i) *The ratio inequality*

$$\langle A^px,x\rangle^{\frac{1}{p}}\le K(h,p)^{\frac{1}{p}}\langle Ax,x\rangle\quad\left(\text{resp. }\langle A^px,x\rangle^{\frac{1}{p}}\ge K(h,p)^{\frac{1}{p}}\langle Ax,x\rangle\right)\tag{3.4}$$

holds for all unit vectors $x\in H$ where $K(h,p)$ is defined by (1.7). Moreover the equality of (3.4) holds if and only if both m and M are eigenvalues of A and

$$\begin{aligned}x&=x_p(K(h,p)^{\frac{1}{p}})\\&=\sqrt{\frac{1}{h^p-1}\left(h^p-\frac{1}{h-1}\frac{h^p-h}{p-1}\right)}e_m+\sqrt{\frac{1}{h^p-1}\left(\frac{1}{h-1}\frac{h^p-h}{p-1}-1\right)}e_M.\end{aligned}$$

(ii) *The difference inequality*

$$\langle A^p x, x\rangle^{\frac{1}{p}} - \langle Ax, x\rangle \leq m\frac{h^p - h}{h^p - 1}\left(K(h,p)^{\frac{1}{p-1}} - 1\right) \tag{3.5}$$
$$\left(\textit{resp.}\ \langle A^p x, x\rangle^{\frac{1}{p}} - \langle Ax, x\rangle \geq m\frac{h^p - h}{h^p - 1}\left(K(h,p)^{\frac{1}{p-1}} - 1\right)\right)$$

holds for all unit vectors $x \in H$ *where* $K(h,p)$ *is defined by* (1.7). *Moreover the equality of* (3.5) *holds if and only if both* m *and* M *are eigenvalues of* A *and*

$$x = x_p(1)$$
$$= \sqrt{\frac{1}{h^p - 1}\left\{h^p - \left(\frac{h^p - 1}{p(h-1)}\right)^{\frac{p}{p-1}}\right\}}e_m + \sqrt{\frac{1}{h^p - 1}\left\{\left(\frac{h^p - 1}{p(h-1)}\right)^{\frac{p}{p-1}} - 1\right\}}e_M.$$

We have the following result which is given in [25] as a modification of [20, Corollaries 7 and 15]:

Corollary 3.3. (Yamazaki [25]) *Let* A *be a positive operator on* H *such that* $0 < m \leq A \leq M$ *for some scalars* $m < M$ *and* $h = \frac{M}{m} (> 1)$. *Then*
(i) *The ratio inequality*

$$\langle A^p x, x\rangle^{\frac{1}{p}} \leq K(h^r, \frac{p}{r})^{\frac{1}{p}} \langle A^r x, x\rangle^{\frac{1}{r}} \tag{3.6}$$

holds for $p > r > 0$ *and all unit vectors* $x \in H$.

(ii) *The difference inequality*

$$\langle A^p x, x\rangle - \langle Ax, x\rangle^p \leq m^p \frac{h^p - h}{h - 1}\left\{K(h,p)^{\frac{1}{p-1}} - 1\right\} \tag{3.7}$$

holds for $p > 1$ *and all unit vectors* $x \in H$.

Proof. In (3.4) we replace A and p by A^r and $\frac{p}{r}$, respectively. Then we have (3.6).

In (3.5) we replace A and p by A^p and $\frac{1}{p}$, respectively. Then it follows from the inversion formula [2, Lemma 4] that

$$\begin{aligned}\langle A^{p\cdot\frac{1}{p}} x, x\rangle^{\frac{1}{1/p}} - \langle A^p x, x\rangle &\geq m^p \frac{h^{p\cdot\frac{1}{p}} - h^p}{h^{p\cdot\frac{1}{p}} - 1}\left(K(h^p, \frac{1}{p})^{\frac{1}{\frac{1}{p}-1}} - 1\right)\\ &= -m^p \frac{h^p - h}{h - 1}\left(K(h^p, \frac{1}{p})^{\frac{p}{1-p}} - 1\right)\\ &= -m^p \frac{h^p - h}{h - 1}\left(K(h,p)^{\frac{1}{p-1}} - 1\right).\end{aligned}$$

Therefore we have (3.7). □

4. Applications for logarithmic and exponential functions

We consider applications to the logarithmic function $f(t) = \log t$ in Theorem 2.2. Here we note that for a unit vector $x \in H$, the *determinant* $\exp\langle(\log A)x, x\rangle$ for all positive invertible operators A is defined by J.I. Fujii and Y. Seo [3]. The determinant is considered as a continuous (weighted) geometric mean (with the weighted x). The following inequality is a variational expression of the arithmetic - geometric mean inequality:

$$\exp\langle(\log A)x, x\rangle \le \langle Ax, x\rangle. \tag{4.1}$$

We give a complementary inequality of (4.1), that is, a lower bound of a difference $\exp\langle(\log A)x, x\rangle - \lambda\langle Ax, x\rangle$ for each $\lambda > 0$. For the sake of convenient, we use the *logarithmic mean*

$$L(m, M) := \frac{M - m}{\log M - \log m} \quad \text{and} \quad L(m, m) := m \tag{4.2}$$

for all $0 < m, M$.

Theorem 4.1. *Let A be a positive invertible operator on a Hilbert space H such that $0 < m \le A \le M$ for some scalars $m < M$ and $h := \frac{M}{m}(> 1)$. Then for each $\lambda \in I_{\log} := \left[\frac{m}{L(m,M)}, \frac{M}{L(m,M)}\right]$*

$$\exp\langle(\log A)x, x\rangle - \lambda\langle Ax, x\rangle \ge -\lambda L(m, M)\log(\lambda S(h)) \tag{4.3}$$

holds for all unit vectors $x \in H$ where $L(m, M)$ and $S(h)$ are defined by (4.2) *and* (1.4), *respectively.*

Moreover the equality of (4.3) *holds if and only if both m and M are eigenvalues of A and*

$$x = x_{\log}(\lambda) := \sqrt{1 - \frac{\log(\lambda L(1, h))}{\log h}}\, e_m + \sqrt{\frac{\log(\lambda L(1, h))}{\log h}}\, e_M \tag{4.4}$$

where e_m and e_M are corresponding unit eigenvectors to m and M, respectively.

Proof. We replace $f(t)$ by $\log t$ in Theorem 2.2. Then we have $\alpha_f = L(m, M)^{-1}$, $\beta_f = \frac{M\log m - m\log M}{M - m}$ and $I_f = I_{\log}$. For every $\lambda \in I_{\log}$ the equation $(\log\mu)' = (\lambda L(m, M))^{-1}$ has a unique solution $\mu = \mu_\lambda := \lambda L(m, M)(\in [m, M])$. Hence we have

$$\begin{aligned}
\mu_\lambda - \frac{f(\mu_\lambda) - \beta_f}{\alpha_f}\lambda &= \lambda L(m, M) - \lambda L(m, M)\left(\log(\lambda L(m, M)) - \frac{M\log m - m\log M}{M - m}\right) \\
&= \lambda L(m, M)\log\frac{e(\log M - \log m)m^{\frac{M}{M-m}}}{\lambda(M - m)M^{\frac{m}{M-m}}} \\
&= \lambda L(m, M)\log\frac{e\log h^{\frac{1}{h-1}}}{\lambda h^{\frac{1}{h-1}}} = -\lambda L(m, M)\log(\lambda S(h))
\end{aligned}$$

which gives (4.3).

Moreover, since we have

$$\frac{f(\mu_\lambda)-f(m)}{f(M)-f(m)}=\frac{\log(\lambda L(m,M))-\log m}{\log M-\log m}=\frac{1}{\log h}\log\left(\lambda\frac{h-1}{\log h}\right)=\frac{1}{\log h}\log\left(\lambda L(1,h)\right),$$

the equality condition is given in (4.4) by (2.4). □

The following ratio inequality and difference one with equality conditions were shown in [5] and [3], respectively. They are obtained as a corollary of Theorem 4.1 by the facts that $\lambda = S(h)^{-1}(\in I_{\log})$ is a unique solution of the equation $-L(m,M)\log(\lambda S(h))=0$ and $1\in I_{\log}$. The constant $L(m,M)\log S(h)$ which is represented in the difference inequality is called the *Mond-Shisha difference* (cf. [3], [4], [12], [21]).

Corollary 4.2. *Assume that the conditions of Theorem 4.1 hold.*
(i) *The ratio inequality*

$$\exp\langle(\log A)x,x\rangle \geq S(h)^{-1}\langle Ax,x\rangle \tag{4.5}$$

holds for all unit vectors $x\in H$. Moreover the equality of (4.5) *holds if and only if both m and M are eigenvalues of A and*

$$x=x_{\log}(S(h)^{-1})=\sqrt{\frac{h}{h-1}-\frac{1}{\log h}}e_m+\sqrt{\frac{1}{\log h}-\frac{1}{h-1}}e_M.$$

(ii) *The difference inequality*

$$\exp\langle(\log A)x,x\rangle-\langle Ax,x\rangle \geq -L(m,M)\log S(h) \tag{4.6}$$

holds for all unit vectors $x\in H$. Moreover the equality of (4.6) *holds if and only if both m and M are eigenvalues of A and*

$$x=x_{\log}(1)=\sqrt{1-\frac{\log L(1,h)}{\log h}}e_m+\sqrt{\frac{\log L(1,h)}{\log h}}e_M.$$

We give an upper bound of a difference $\log\langle e^A x,x\rangle-\lambda\langle Ax,x\rangle$ for each $\lambda>0$. Ando, Hiai and Petz used this type inequality in [1] and [14]. We need a 2-variable extension of the Specht ratio defined by

$$S(h,\lambda):=\frac{h^{\frac{\lambda}{h-1}}}{e\log h^{\frac{\lambda}{h-1}}}\quad\text{for } h>0,\ \lambda>0 \qquad\text{and}\qquad S(1,\lambda):=\frac{e^\lambda}{\lambda e}. \tag{4.7}$$

Theorem 4.3. *Let A be a selfadjoint operator on a Hilbert space H such that $m\leq A\leq M$ for some scalars $m<M$. Then for each $\lambda\in I_{\exp}:=\left[\frac{e^M-e^m}{e^M(M-m)},\frac{e^M-e^m}{e^m(M-m)}\right]$,*

$$\log\langle e^A x,x\rangle-\lambda\langle Ax,x\rangle \leq \log\{e^{(1-\lambda)m}S(e^{M-m},\lambda)\} \tag{4.8}$$

holds for all unit vectors $x\in H$ where $S(h,\lambda)$ is defined by (4.7).

Moreover the equality of (4.8) *holds if and only if both* m *and* M *are eigenvalues of* A *and*

$$(4.9)\quad x = x_{\exp}(\lambda) := \sqrt{\frac{e^{M-m}}{e^{M-m}-1} - \frac{1}{\lambda(M-m)}}\, e_m + \sqrt{\frac{1}{\lambda(M-m)} - \frac{1}{e^{M-m}-1}}\, e_M$$

where e_m *and* e_M *are corresponding unit eigenvectors to* m *and* M*, respectively.*

Proof. If we replace $f(t)$ by e^t in Theorem 2.2, then we have $\alpha_f = \frac{e^M-e^m}{M-m}$, $\beta_f = \frac{Me^m-me^M}{M-m}$ and $I_f = I_{\exp}$. For every $\lambda \in I_{\exp}$ the equation $(e^\mu)' = \frac{e^M-e^m}{\lambda(M-m)}$ has a unique solution $\mu = \mu_\lambda := \log\frac{e^M-e^m}{\lambda(M-m)}$ $(\in [m, M])$. Hence putting $h := e^{M-m}$ we have

$$\mu_\lambda - \frac{f(\mu_\lambda)-\beta_f}{\alpha_f}\lambda = \log\frac{e^M-e^m}{\lambda(M-m)} - 1 + \frac{Me^m-me^M}{e^M-e^m}\lambda$$

$$= \log\frac{(h-1)e^m}{\lambda e\log h} + \left(\frac{M-m}{h-1} - m\right)\lambda = \log\frac{e^{m(1-\lambda)}h^{\frac{\lambda}{h-1}}}{e\log h^{\frac{\lambda}{h-1}}} = \log\{e^{(1-\lambda)m}S(h,\lambda)\},$$

which gives (4.8).

Moreover, since we have

$$\frac{f(\mu_\lambda)-f(m)}{f(M)-f(m)} = \frac{\frac{e^M-e^m}{\lambda(M-m)} - e^m}{e^M-e^m} = \frac{1}{\lambda(M-m)} - \frac{1}{e^{M-m}-1},$$

the equality condition is given in (4.9) by (2.4). □

Next we see that for $mM \geq 0$ the equation $\log\{e^{(1-\lambda)m}S(e^{M-m},\lambda)\} = 0$ has a unique solution $\lambda = \lambda_{\exp}$ $(\in I_{\exp})$, and moreover $1 \in I_{\exp}$. So we have the following corollary:

Corollary 4.4. *Assume that the conditions of Theorem 4.3 hold.*
(i) *Let* $mM \geq 0$. *Then the equation* $e^{(1-\lambda)m}S(e^{M-m},\lambda) = 1$ *has a unique solution* $\lambda = \lambda_{\exp}(\in I_{\exp})$, *and the ratio inequality*

$$(4.10)\quad \log\langle e^A x, x\rangle \leq \lambda_{\exp}\langle Ax, x\rangle$$

holds for all unit vectors $x \in H$. *Moreover the equality of* (4.10) *holds if and only if both* m *and* M *are eigenvalues of* A *and*

$$x = x_{\exp}(\lambda_{\exp}) = \sqrt{\frac{e^{M-m}}{e^{M-m}-1} - \frac{1}{\lambda_{\exp}(M-m)}}\, e_m + \sqrt{\frac{1}{\lambda_{\exp}(M-m)} - \frac{1}{e^{M-m}-1}}\, e_M.$$

(ii) *The difference inequality*

$$(4.11)\quad \log\langle e^A x, x\rangle - \langle Ax, x\rangle \leq \log S(e^{M-m})$$

holds for all unit vectors $x \in H$. Moreover the equality of (4.11) *holds if and only if both m and M are eigenvalues of A and*

$$x = x_{\exp}(1) = \sqrt{\frac{e^{M-m}}{e^{M-m}-1} - \frac{1}{(M-m)}}\,e_m + \sqrt{\frac{1}{(M-m)} - \frac{1}{e^{M-m}-1}}\,e_M.$$

Remark 4.5. *Taking $m = 0$ in* (4.10), *we have $\lambda = \lambda_{\exp} := \frac{e^M-1}{M}$ as the solution of the equation $\log\{e^{(1-\lambda)m}S(e^{M-m}, \lambda)\} = 0$. Hence the following ratio inequality related to* (4.10) *holds*:

$$\log\langle e^A x, x\rangle \le \frac{e^M - 1}{M}\langle Ax, x\rangle.$$

Replacing A and λ with $\log A^p$ and $\frac{q}{p}$, respectively in Theorem 4.3, we have the following result by Furuta and Giga [11]:

Corollary 4.6. (Furuta and Giga [11, Theorem 5.1]) *Let A be a positive invertible operator on H such that $0 < m \le A \le M$ for some scalars $m < M$ and $h = \frac{M}{m}(> 1)$. Then*

$$S_h(p,q)\exp\langle(\log A^q)x, x\rangle \ge \langle A^p x, x\rangle \ (\ge \exp\langle(\log A^q)x, x\rangle) \tag{4.12}$$

hold for all unit vectors x and all $p, q > 0$ with $q \le \frac{h^p-1}{\log h} \le qh^p$ where

$$S_h(p,q) := m^{p-q}\frac{h^{\frac{q}{h^p-1}}}{e\log h^{\frac{q}{h^p-1}}}. \tag{4.13}$$

The constant $S_h(p,q)\ \left(= m^{p-q}S(h^p, \frac{q}{p})\right)$ has been already appeared in [19].

Concluding this section, we give reverse inequalities of the inequality

$$\log\langle Ax, x\rangle \ge \langle(\log A)x, x\rangle \tag{4.14}$$

for all unit vectors x and a positive invertible operator A.

Corollary 4.7. *Let A be a positive invertible operator on H such that $0 < m \le A \le M$ for some scalars $m < M$ and $h = \frac{M}{m}(> 1)$. Then for each $\lambda \in I_{\log} := \left[\frac{L(m,M)}{M}, \frac{L(m,M)}{m}\right]$.*

$$\log\langle Ax, x\rangle - \lambda\langle(\log A)x, x\rangle \le \log\{m^{1-\lambda}S(h,\lambda)\} \tag{4.15}$$

holds for all unit vectors $x \in H$, where $S(h, \lambda)$ is defined in (4.7). *Moreover the equality of* (4.15) *holds if and only if both m and M are eigenvalues of A and*

$$x = \sqrt{\frac{h}{h-1} - \frac{1}{\lambda\log h}}\,e_m + \sqrt{\frac{1}{\lambda\log h} - \frac{1}{h-1}}\,e_M \tag{4.16}$$

where e_m and e_M are corresponding unit eigenvectors to m and M, respectively.

In particular, the following inequalities hold:

(i) *The ratio inequality*

$$\log\langle Ax, x\rangle \leq \lambda_{\log}\langle(\log A)x, x\rangle \tag{4.17}$$

holds for $m \geq 1$ *or* $0 < M \leq 1$ *and all unit vectors* $x \in H$ *where* $\lambda = \lambda_{\log}(\in I_{\log})$ *is a unique solution of the equation* $m^{1-\lambda}S(h,\lambda) = 1$. *Moreover the equality of* (4.17) *holds if and only if both* m *and* M *are eigenvalues of* A *and*

$$x = \sqrt{\frac{h}{h-1} - \frac{1}{\lambda_{\log}\log h}}e_m + \sqrt{\frac{1}{\lambda_{\log}\log h} - \frac{1}{h-1}}e_M \tag{4.18}$$

where e_m *and* e_M *are corresponding unit eigenvectors to* m *and* M, *respectively.*

(ii) *The difference inequality*

$$\log\langle Ax, x\rangle - \langle(\log A)x, x\rangle \leq \log S(h). \tag{4.19}$$

holds for all unit vectors $x \in H$. *Moreover the equality of* (4.19) *holds if and only if both* m *and* M *are eigenvalues of* A *and*

$$x = \sqrt{\frac{h}{h-1} - \frac{1}{\log h}}e_m + \sqrt{\frac{1}{\log h} - \frac{1}{h-1}}e_M \tag{4.20}$$

where e_m *and* e_M *are corresponding unit eigenvectors to* m *and* M, *respectively.*

Remark 4.8. *We obtain* (4.5) *by taking exponential in both sides of* (4.19). *Incidentally* (4.19) *has been given in* [7].

5. Furuta-Pečarić inequality

From the convexity of $t\log t$, the Jensen inequality implies

$$\langle(A\log A)x, x\rangle \geq \langle Ax, x\rangle\log\langle Ax, x\rangle$$

for all unit vectors x and positive invertible operators A. In [13], Furuta and Pečarić showed the following difference inequality as a reverse inequality of it:

Theorem FP. (Furuta and Pečarić [13]) *Let* A *be a positive invertible operator on a Hilbert space* H *such that* $0 < m \leq A \leq M$ *for some scalars* $m < M$ *and* $h := \frac{M}{m}(> 1)$. *Then the difference inequality*

$$\langle Ax, x\rangle\log\langle Ax, x\rangle - \langle(A\log A)x, x\rangle \geq -\frac{M}{L(1,h)}(S(h) - 1) \tag{5.1}$$

holds for all unit vectors $x \in H$ *where* $S(h)$ *is defined by* (1.4).

From the viewpoint of the quasi-arithmetic mean inequality (1.5), we give an estimation of a difference $\langle Ax, x\rangle\log\langle Ax, x\rangle - \lambda\langle(A\log A)x, x\rangle$ which is an extension of Theorem FP. Moreover we discuss the case the equality holds for the obtained inequality. For the sake of convenience, we recall (4.7); $S(h,\lambda) := \frac{h^{\frac{\lambda}{h-1}}}{e\log h^{\frac{\lambda}{h-1}}}$ $(h > 0, \lambda > 0)$ and $S(h,1) = S(h)$ in particular.

Theorem 5.1. *Let A be a positive invertible operator on a Hilbert space H such that $0 < m \leq A \leq M$ for some scalars $m < M$, $h = \frac{M}{m}(> 1)$ and $I_0 := \left[\frac{L(1,h)}{h}, \frac{L(1,h)}{h}\log(eh)\right]$. Then for each $\lambda \in I_0$*

$$\begin{aligned}&\langle Ax, x\rangle \log\langle Ax, x\rangle - \lambda\langle (A\log A)x, x\rangle \\ &\quad \geq (1-\lambda)(\log m)\langle Ax, x\rangle - \frac{\lambda M}{L(1,h)}(h^{\lambda-1}S(h,\lambda) - 1)\end{aligned} \tag{5.2}$$

holds for all unit vectors $x \in H$.

Moreover the equality of (5.2) holds if and only if both m and M are eigenvalues of A and

$$x = x_0(\lambda) := \sqrt{1 - \frac{h^{\frac{\lambda h}{h-1}}/e - m}{h-1}}\, e_m + \sqrt{\frac{h^{\frac{\lambda h}{h-1}}/e - m}{h-1}}\, e_M \tag{5.3}$$

where e_m and e_M are corresponding unit eigenvectors to m and M, respectively.

Remark 5.2. *Since $1 \in I_0$ in Theorem 5.1, we obtain Theorem FP by putting $\lambda = 1$ in (5.2), and that the equality of (5.1) holds if and only if both m and M are eigenvalues of A and*

$$x = \sqrt{1 - \frac{h^{\frac{h}{h-1}}/e - m}{h-1}}\, e_m + \sqrt{\frac{h^{\frac{h}{h-1}}/e - m}{h-1}}\, e_M \tag{5.4}$$

where e_m and e_M are corresponding unit eigenvectors to m and M, respectively.

For the proof of Theorem 5.1, we show the following lemma which represents a lower bound of a difference $\langle Ax, x\rangle \log\langle Ax, x\rangle - \lambda\langle (A\log A)x, x\rangle$ under $1 \leq m \leq A \leq M$:

Lemma 5.3. *Let A be a positive invertible operator on H such that $1 \leq m \leq A \leq M$, $h = \frac{M}{m}(> 1)$ and $I_1 := \left[\frac{\log(me)}{\log(Mh^{\frac{1}{h-1}})}, \frac{\log(Me)}{\log(Mh^{\frac{1}{h-1}})}\right]$. Then for each $\lambda \in I_1$*

$$\langle Ax, x\rangle \log\langle Ax, x\rangle - \lambda\langle (A\log A)x, x\rangle \geq -\frac{\lambda M}{L(1,h)}(M^{\lambda-1}S(h,\lambda) - 1) \tag{5.5}$$

holds for all unit vectors $x \in H$.

Moreover the equality of (5.5) holds if and only if both m and M are eigenvalues of A and

$$x = x_1(\lambda) := \sqrt{1 - \frac{(Mh^{\frac{1}{h-1}})^{\lambda}/e - m}{M-m}}\, e_m + \sqrt{\frac{(Mh^{\frac{1}{h-1}})^{\lambda}/e - m}{M-m}}\, e_M \tag{5.6}$$

where e_m and e_M are corresponding unit eigenvectors to m and M, respectively.

Proof. We put $f_1(t) := t\log t$ for all $t \geq 1$. Moreover in Theorem 2.2, we replace $f(t)$ and A by $f_1^{-1}(t)$ and $A\log A$, respectively. Then we see that $(f_1^{-1})'(f_1(t)) = (f_1'(t))^{-1}$

and $f_1'(t) = \log te$. So we have $\alpha_f = \{\log(Mh^{\frac{1}{h-1}})\}^{-1}$, $\beta_f = \frac{M\log h}{\log\frac{M^h}{m}}$ and $I_f = I_1$. For every $\lambda \in I_1$ the inequality (2.2) is represented as follows:

$$\langle Ax, x\rangle \log\langle Ax, x\rangle - \lambda\langle A\log Ax, x\rangle \geq \mu_\lambda \log\mu_\lambda - \frac{\mu_\lambda - \beta_f}{\alpha_f}\lambda$$

where $\mu = \mu_\lambda \in [m, M]$ is a unique solution of the equation $(f_1^{-1})'(\mu\log\mu) = \frac{\alpha_f}{\lambda}$. More precisely, we investigate the right side of the above inequality. The equation $(f_1^{-1})'(\mu\log\mu) = \frac{\alpha_f}{\lambda}$ ensures $\log\mu e = \frac{\lambda}{\alpha_f}$ and so

$$\mu = \mu_\lambda := e^{\frac{\lambda}{\alpha_f}-1} = \frac{(Mh^{\frac{1}{h-1}})^\lambda}{e} \ (\in [m, M]).$$

Hence we have

$$\begin{aligned}\mu_\lambda\log\mu_\lambda - \frac{\mu_\lambda - \beta_f}{\alpha_f}\lambda &= e^{\frac{\lambda}{\alpha_f}-1}\left(\frac{\lambda}{\alpha_f} - 1\right) - \frac{e^{\frac{\lambda}{\alpha_f}-1} - \beta_f}{\alpha_f}\lambda = -e^{\frac{\lambda}{\alpha_f}-1} + \frac{\beta_f}{\alpha_f}\lambda \\ &= -\frac{(Mh^{\frac{1}{h-1}})^\lambda}{e} + \frac{\log(Mh^{\frac{1}{h-1}})\cdot M\log h}{\log\frac{M^h}{m}}\lambda \\ &= -\frac{\lambda M}{L(1,h)}\left(\frac{(h-1)M^\lambda h^{\frac{\lambda}{h-1}}}{\lambda Me\log h} - 1\right) \\ &= -\frac{\lambda M}{L(1,h)}\left(M^{\lambda-1}S(h,\lambda) - 1\right),\end{aligned}$$

which gives (5.5).

The equality condition is given in (5.6) by (2.4). □

Proof of Theorem 5.1. In Lemma 5.3, we replace A by $\frac{A}{m}$, that is, $1 \leq A \leq h$, and the interval I_1 is corresponding to I_0. So by (5.5) we have for $\lambda \in I_0$

$$\begin{aligned}&\left\langle \frac{A}{m}x, x\right\rangle \log\left\langle \frac{A}{m}x, x\right\rangle - \lambda\left\langle\left(\frac{A}{m}\log\frac{A}{m}\right)x, x\right\rangle \\ &= \frac{1}{m}\{\langle Ax, x\rangle(\log\langle Ax, x\rangle - \log m) - \lambda(\langle(A\log A)x, x\rangle - (\log m)\langle Ax, x\rangle\} \\ &= \frac{1}{m}\{\langle Ax, x\rangle\log\langle Ax, x\rangle - \lambda\langle(A\log A)x, x\rangle - (1-\lambda)(\log m)\langle Ax, x\rangle\} \\ &\geq -\frac{\lambda h}{L(1,h)}(h^{\lambda-1}S(h,\lambda) - 1).\end{aligned}$$

Hence we have the desired inequality (5.2). □

Next we note that the equation $-\frac{\lambda M}{L(1,h)}(M^{\lambda-1}S(h,\lambda)-1) = 0$ has a unique solution $\lambda = \lambda_1$ ($\in I_1$). So we have the following corollary:

Corollary 5.4. *Under the same assumption of Lemma 5.3, the ratio inequality*

$$\langle Ax, x\rangle \log\langle Ax, x\rangle \geq \lambda_1 \langle (A \log A)x, x\rangle \tag{5.7}$$

holds for all unit vectors $x \in H$ *where* $\lambda = \lambda_1 (\in I_1)$ *is a unique solution of the equation* $-\frac{\lambda M}{L(1,h)}(M^{\lambda-1}S(h,\lambda) - 1) = 0$. *Moreover the equality of* (5.7) *holds if and only if both* m *and* M *are eigenvalues of* A *and*

$$x = x_1(\lambda_1) := \sqrt{1 - \frac{(Mh^{\frac{1}{h-1}})^{\lambda_1}/e - m}{M-m}}\, e_m + \sqrt{\frac{(Mh^{\frac{1}{h-1}})^{\lambda_1}/e - m}{M-m}}\, e_M \tag{5.8}$$

where e_m *and* e_M *are corresponding unit eigenvectors to* m *and* M, *respectively.*

If $m = 1$ in Corollary 5.4, then we have $\lambda_1 = \frac{L(1,M)}{M}$. So the following reverse of the Jensen inequality is obtained:

$$\langle Ax, x\rangle \log\langle Ax, x\rangle \geq \frac{L(1,M)}{M}\langle (A \log A)x, x\rangle \quad \text{for all unit vectors } x \in H. \tag{5.9}$$

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TOYAMA NATIONAL COLLEGE OF TECHNOLOGY, 13, HONGO-MACHI, TOYAMA-SHI, 939-8630, JAPAN
E-mail address: mtommy@toyama-nct.ac.jp

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