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REMARKS ON SOME ALMOST HERMITIAN STRUCTURE ON THE TANGENT BUNDLE

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Dedicated to Professor Kouei Sekigawa on his retirement

ABSTRACT. In [5], M. Tahara and Y. Watanabe constructed a family of almost Hermitian structures (J, G) on the tangent bundle TM of a Riemannian manifold and constructed a family of Hermitian and Kähler structure on the tangent bundle on a space form. It is well-known that there are sixteen classes of almost Hermitian manifolds ([3]). In this paper, we give the conditions for (J, G) such that TM belongs to each of these sixteen classes.

1. Canonical almost Kähler structure on TM

Let $M = (M, g)$ be an n-dimensional Riemannian manifold and $\pi : TM \to M$ the tangent bundle of M. It is well-known that TM is a $2n$-dimensional manifold. At each point $u \in TM$, the n-dimensional subspace

$$V_u = \ker(d\pi)_u$$

of $T_u(TM)$, the tangent space of TM at u, is called the vertical subspace. If $(\tilde{x}^i)$ be a local coordinates about $\pi(u) \in M$, then $(x^i, \xi^i) = (\tilde{x}^i \circ \pi, d\tilde{x}^i)$ is a local coordinates about u and V_u is of the form

$$V_u = \left\{ \sum_{i=1}^{n} A^i \frac{\partial}{\partial \xi^i} \;\middle|\; A^i \in \mathbb{R} \right\}.$$

Thus, the vertical subspace V_u is naturally identified with $T_{\pi(u)}M$, the tangent space of M at $\pi(u)$, via

$$\iota : V_u \to T_{\pi(u)}M; \qquad \iota\left(\sum_{i=1}^{n} A^i \frac{\partial}{\partial \xi^i} \right) = A^i \frac{\partial}{\partial \tilde{x}^i}.$$

We denote by H_u the horizontal subspace of $T_u(TM)$ with respect to the Riemannian connection $\tilde{\nabla}$ of g. Then, we have a direct sum decomposition $T_u(TM) = H_u \oplus V_u$,

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which defines smooth distributions on TM. With respect to the local coordinates (x^i, ξ^i) about u, the horizontal subspace H_u is of the form

$$H_u = \left\{ \sum_{i=1}^{n} A^i \left(\frac{\partial}{\partial \tilde{x}^i} - \sum_{j,k=1}^{n} \Gamma_{ij}^k(\pi(u))\xi^j(u)\frac{\partial}{\partial \xi^k} \right) \,\middle|\, A^i \in \mathbb{R} \right\},$$

where Γ_{ij}^k is the connection coefficient of $\tilde{\nabla}$. For each $X \in T_p(M)$ and $u \in TM$ with $\pi(u) = p \in M$, there exists unique vector $X_u^H \in H_u$ (resp. $X_u^V \in V_u$), called the horizontal lift (resp. vertical lift) of X, such that $d\pi(X_u^H) = X$ (resp. $\iota(X_u^V) = X$).

The connection map $K : TTM \to TM$ is defined by $K(A) = \iota(A^V)$, where A^V is the vertical component of A. The map K is a homomorphism of the two vector bundles $TTM \to TM$ (the tangent bundle of TM) and $d\pi : TTM \to TM$. Moreover, $K|_{V_u} = \iota$, $H_u = \ker(K|_{T_u(TM)})$ and if we regard a vector field $X \in \mathfrak{X}(M)$ as a C^∞-map $X : M \to TM$, we have

$$K(dX(u)) = \tilde{\nabla}_u X$$

for $u \in TM$.

Proposition 1.1. *Let $X, Y \in \mathfrak{X}(M)$. For each $u \in TM$, we have*

$$[X^V, Y^V]_u = 0, \tag{1.1}$$

$$[X^H, Y^V]_u = (\tilde{\nabla}_X Y)_u^V, \tag{1.2}$$

$$d\pi([X^H, Y^H]_u) = [X, Y]_{\pi(u)}, \tag{1.3}$$

$$K([X^H, Y^H]_u) = -R(X_{\pi(u)}, Y_{\pi(u)})u, \tag{1.4}$$

where R is the curvature tensor of M defined by $R(X, Y) = [\tilde{\nabla}_X, \tilde{\nabla}_Y] - \tilde{\nabla}_{[X,Y]}$.

The canonical symplectic structure ω_0 on TM is defined by

$$\omega_0(A, B) = g(K(A), d\pi(B)) - g(K(B), d\pi(A))$$

for $A, B \in \mathfrak{X}(TM)$. With respect to the local coordinates (x^i, ξ^i) of TM, ω_0 is given by

$$\omega_0 = -\sum_{i,j=1}^{n} dx^i \wedge d((g_{ij} \circ \pi)\xi^j).$$

By means of the metric g, we can identify TM with the cotangent bundle T^*M of M. Then, ω_0 can be regarded as the canonical symplectic structure on T^*M.

The Sasaki-metric G_0 is a Riemannian metric on TM defined by

$$G_0(A, B) = g(d\pi(A), d\pi(B)) + g(K(A), K(B)),$$

for $A, B \in \mathfrak{X}(TM)$, or equivalently,

$$G_0(X_u^H, Y_u^H) = G_0(X_u^V, Y_u^V) = g(X_{\pi(u)}, Y_{\pi(u)}), \quad G_0(X_u^H, Y_u^V) = 0,$$

for $X, Y \in \mathfrak{X}(M)$ and $u \in TM$. Then $\pi : (TM, G_0) \to (M, g)$ is a Riemannian submersion. On one hand, the canonical almost complex structure J_0 on TM is defined by

$$J_0 X_u^H = X_u^V, \quad J_0 X_u^V = -X_u^H$$

for $X \in \mathfrak{X}(M)$ and $u \in TM$, which is characterised by

$$d\pi(J_0 A) = -K(A), \quad K(J_0 A) = d\pi(A)$$

for $A \in \mathfrak{X}(TM)$. The pair (J_0, G_0) is an almost Hermitian structure on TM and the corresponding Kähler form coincides with the canonical symplectic form ω_0. Therefore, (TM, J_0, G_0) is an almost Kähler manifold.

Theorem 1.2 ([2])**.** *An almost Kähler manifold (TM, J_0, G_0) is integrable if and only if (M, g) is locally flat.*

Proof. Let N be the Nijenhuis tensor of J_0. Then, for any $X, Y \in \mathfrak{X}(M)$ and $u \in TM$, we have $N(X_u^H, Y_u^H) = -N(X_u^V, Y_u^V)$, $N(X_u^H, Y_u^V) = J_0 N(X_u^V, Y_u^V)$. Thus, it suffices to show that $N(X_u^V, Y_u^V) = 0$ is equivalent to $R = 0$. By Proposition 1.1, we have

$$N(X_u^V, Y_u^V) = [X_u^H, Y_u^H] - (\tilde{\nabla}_X Y)_u^H + (\tilde{\nabla}_Y X)_u^H,$$

and thus $d\pi(N(X_u^V, Y_u^V)) = 0$, $K(N(X_u^V, Y_u^V)) = -R(X, Y)u$, which completes the proof. □

2. A family of almost Hermitian structure on TM

In this section, we introduce a family of almost Hermitian structure (J, G) on TM defined by M. Tahara and Y. Watanabe ([5]) and compute the covariant derivative, exterior derivative and coderivative of the Kähler form of (J, G).

Let (M, g) be a Riemannian manifold of dimension n. We define an almost complex structure $J = J(f, h)$ on the tangent bundle TM of M by

$$\begin{aligned} JX_u^H &= fX_u^V + \frac{h-f}{t} g(X, u) u_u^V, \\ JX_u^V &= -\frac{1}{f} X_u^H + \frac{h-f}{tfh} g(X, u) u_u^H, \end{aligned}$$

for $X, Y \in T_{\pi(u)}(M)$ and $u \in TM$, where $t = \|u\|^2$ and $f, h : [0, \infty) \to \mathbb{R}$ are positive C^∞-functions such that $(f(t) - h(t))/t$ is C^∞ at $t = 0$. Moreover, we define a Riemannian metric $G = G(\alpha, \beta, f, h)$ on TM by

$$
\begin{aligned}
&G(X_u^H, Y_u^H) = \alpha g(X, Y) + \beta g(X, u) g(Y, u),\\
&G(X_u^V, Y_u^V) = \frac{\alpha}{f^2} g(X, Y) + \frac{\alpha(f^2 - h^2) + t f^2 \beta}{t f^2 h^2} g(X, u) g(Y, u),\\
&G(X_u^H, Y_u^V) = 0,
\end{aligned}
$$

for $X, Y \in T_{\pi(u)}(M)$ and $u \in TM$, where C^∞-functions $\alpha, \beta : [0, \infty) \to \mathbb{R}$ satisfy $\alpha(t) > 0$ and $\alpha(t) + t\beta(t) > 0$. It is easy to verify that (J, G) is an almost Hermitian structure on TM. In particular, $(J(1, 1), G(1, 0, 1, 1))$ coincides with the almost Kähler structure (J_0, G_0). Also, (J, G) includes the almost Hermitian structure constructed in [1] and [4], see [5] for more details.

We denote by ∇ and Ω the Riemannian connection of G and the Kähler form of (J, G), where $\Omega(\,\cdot\,, \,\cdot\,) = G(\,\cdot\,, J\,\cdot\,)$. Furthermore, we put

$$
\begin{aligned}
\psi_1 &= 2h(\log \alpha)' - \frac{f\beta}{\alpha},\\
\psi_2 &= 2h(\log f)' - \frac{h - f}{t},\\
\psi_3 &= \psi_1 - \psi_2 = 2h\left(\log \frac{\alpha}{f}\right)' - \frac{f\beta}{\alpha} + \frac{h - f}{t}.
\end{aligned}
$$

By direct (and tiresome) calculation, we obtain the following three propositions.

Proposition 2.1. *For $X, Y, Z \in T_p(M)$, $u \in TM$ $(\pi(u) = p)$, the covariant derivative $\nabla\Omega$ is given by*

$$
\begin{aligned}
(\nabla_{X_u^H}\Omega)(Y_u^H, Z_u^H) &= \frac{\alpha\psi_1}{2}\{g(X, Y)g(Z, u) - g(X, Z)g(Y, u)\} \qquad (2.1)\\
&\quad + \frac{\alpha}{2f} g(R(Y, Z)X, u),\\
(\nabla_{X_u^H}\Omega)(Y_u^V, Z_u^V) &= -\frac{\alpha\psi_1}{2fh}\{g(X, Y)g(Z, u) - g(X, Z)g(Y, u)\} \qquad (2.2)\\
&\quad + \frac{\alpha(h - f)}{2tf^3h}\{g(R(X, u)Y, u)g(Z, u)\\
&\qquad - g(R(X, u)Z, u)g(Y, u)\} - \frac{\alpha}{2f^3} g(R(Y, Z)X, u),\\
(\nabla_{X_u^V}\Omega)(Y_u^H, Z_u^V) &= \frac{\alpha}{2fh}(\psi_1 - 2\psi_2) g(X, Y) g(Z, u) \qquad (2.3)\\
&\quad - \frac{\alpha}{2f^2}(\psi_1 - 2\psi_2) g(X, Z) g(Y, u)\\
&\quad + \frac{\alpha(h - f)(\psi_1 - 2\psi_2)}{2tf^2h} g(X, u) g(Y, u) g(Z, u)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\alpha(h-f)}{2tf^3h}g(R(X,u)Y,u)g(Z,u)\\
&-\frac{\alpha}{2f^3}g(R(Y,Z)X,u),
\end{aligned}
$$

$$
(\nabla_{X_u^H}\Omega)(Y_u^H,Z_u^V)=(\nabla_{X_u^V}\Omega)(Y_u^H,Z_u^H)=(\nabla_{X_u^V}\Omega)(Y_u^V,Z_u^V)=0. \tag{2.4}
$$

In particular, if M is a space of constant curvature c, we have

$$
(\nabla_{X_u^H}\Omega)(Y_u^H,Z_u^H)=\frac{\alpha}{2}\left(\psi_1-\frac{c}{f}\right)\{g(X,Y)g(Z,u)-g(X,Z)g(Y,u)\}, \tag{2.5}
$$

$$
(\nabla_{X_u^H}\Omega)(Y_u^V,Z_u^V)=-\frac{\alpha}{2fh}\left(\psi_1-\frac{c}{f}\right)\{g(X,Y)g(Z,u)-g(X,Z)g(Y,u)\}, \tag{2.6}
$$

$$
\begin{aligned}
(\nabla_{X_u^V}\Omega)(Y_u^H,Z_u^V)=&\frac{\alpha}{2fh}\left(\psi_1-2\psi_2+\frac{c}{f}\right)g(X,Y)g(Z,u)\\
&-\frac{\alpha}{2f^2}\left(\psi_1-2\psi_2+\frac{c}{f}\right)g(X,Z)g(Y,u)\\
&+\frac{\alpha(h-f)}{2tf^2h}\left(\psi_1-2\psi_2+\frac{c}{f}\right)g(X,u)g(Y,u)g(Z,u).
\end{aligned} \tag{2.7}
$$

Proposition 2.2. *For $X,Y,Z\in T_p(M)$, $u\in TM$ ($\pi(u)=p$), the exterior derivative $d\Omega$ is given by*

$$
d\Omega(X_u^H,Y_u^V,Z_u^V)=-\frac{\alpha\psi_3}{fh}\{g(X,Y)g(Z,u)-g(X,Z)g(Y,u)\}, \tag{2.8}
$$

$$
d\Omega(X_u^H,Y_u^H,Z_u^H)=d\Omega(X_u^H,Y_u^H,Z_u^V)=d\Omega(X_u^V,Y_u^V,Z_u^V)=0. \tag{2.9}
$$

Proposition 2.3. *For $X\in T_p(M)$, $u\in TM$ ($\pi(u)=p$), the coderivative $\delta\Omega$ is given by*

$$
\delta\Omega(X_u^H)=-(n-1)\psi_3\,g(X,u), \tag{2.10}
$$

$$
\delta\Omega(X_u^V)=0. \tag{2.11}
$$

3. Conditions for each classes

First, we recall the sixteen classes of almost Hermitian manifolds established in [3]. Let $M=(M,J,g)$ be an almost Hermitian manifold and Ω the corresponding Kähler form. We denote by $\mathscr{W}$ the set of all almost Hermitian manifolds of dimension $2n$. Making use of the invariant subspaces $\mathscr{W}_1$, ..., $\mathscr{W}_4$ of the unitary representation, we can classify $\mathscr{W}$ (dimension $2n\geq 6$) into following sixteen classes.

(1) $\mathscr{K}$ = Kähler manifolds: $\nabla\Omega=0$.
(2) $\mathscr{W}_1=\mathscr{NK}$ = nearly Kähler manifolds: $(\nabla_X\Omega)(X,Y)=0$.
(3) $\mathscr{W}_2=\mathscr{AK}$ = almost Kähler manifolds: $d\Omega=0$.

(4) $\mathscr{W}_3 = \mathscr{H} \cap \mathscr{SK}$ = Hermitian semi-Kähler manifolds:

$$(\nabla_X \Omega)(Y, Z) - (\nabla_{JX} \Omega)(JY, Z) = \delta\Omega = 0.$$

(5) $\mathscr{W}_4$:

$$(\nabla_X \Omega)(Y, Z) = -\frac{1}{2(n-1)}\{g(X, Y)\delta\Omega(Z) - g(X, Z)\delta\Omega(Y) - g(X, JY)\delta\Omega(JZ) + g(X, JZ)\delta\Omega(JY)\}.$$

(6) $\mathscr{W}_1 \cup \mathscr{W}_2 = \mathscr{QK}$ = quasi-Kähler manifolds: $(\nabla_X \Omega)(Y, Z) + (\nabla_{JX} \Omega)(JY, Z) = 0$.

(7) $\mathscr{W}_1 \cup \mathscr{W}_3$: $(\nabla_X \Omega)(X, Y) - (\nabla_{JX} \Omega)(JX, Y) = \delta\Omega = 0$.

(8) $\mathscr{W}_1 \cup \mathscr{W}_4$:

$$(\nabla_X \Omega)(X, Y) = -\frac{1}{2(n-1)}\{\|X\|^2\delta\Omega(Y) - g(X, Y)\delta\Omega(X) - g(JX, Y)\delta\Omega(JX)\}.$$

(9) $\mathscr{W}_2 \cup \mathscr{W}_3$: $\underset{X,Y,Z}{\mathfrak{S}}\{(\nabla_X \Omega)(Y, Z) - (\nabla_{JX} \Omega)(JY, Z)\} = \delta\Omega = 0$, where $\mathfrak{S}$ denotes the cyclic sum.

(10) $\mathscr{W}_2 \cup \mathscr{W}_4$: $\underset{X,Y,Z}{\mathfrak{S}}\{(\nabla_X \Omega)(Y, Z) - g(X, JY)\delta\Omega(JZ)/(n-1)\} = 0$.

(11) $\mathscr{W}_3 \cup \mathscr{W}_4 = \mathscr{H}$ = Hermitian manifolds:

$$(\nabla_X \Omega)(Y, Z) - (\nabla_{JX} \Omega)(JY, Z) = 0.$$

(12) $\mathscr{W}_1 \cup \mathscr{W}_2 \cup \mathscr{W}_3 = \mathscr{SK}$ = semi-Kähler manifolds: $\delta\Omega = 0$.

(13) $\mathscr{W}_1 \cup \mathscr{W}_2 \cup \mathscr{W}_4$:

$$(\nabla_X \Omega)(Y, Z) + (\nabla_{JX} \Omega)(JY, Z) = -\frac{1}{n-1}\{g(X, Y)\delta\Omega(Z) - g(X, Z)\delta\Omega(Y) - g(X, JY)\delta\Omega(JZ) + g(X, JZ)\delta\Omega(JY)\}$$

(14) $\mathscr{W}_1 \cup \mathscr{W}_3 \cup \mathscr{W}_4$: $(\nabla_X \Omega)(X, Y) - (\nabla_{JX} \Omega)(JX, Y) = 0$.

(15) $\mathscr{W}_2 \cup \mathscr{W}_3 \cup \mathscr{W}_4$: $\underset{X,Y,Z}{\mathfrak{S}}\{(\nabla_X \Omega)(Y, Z) - (\nabla_{JX} \Omega)(JY, Z)\} = 0$.

(16) $\mathscr{W}$ = almost Hermitian manifolds: No condition.

In case dimension $2n = 4$, $\mathscr{W}$ can be classified into following four classes.

(1) $\mathscr{K}$ = Kähler manifolds.

(2) $\mathscr{W}_2 = \mathscr{AK}$ = almost Kähler manifolds.

(3) $\mathscr{W}_4 = \mathscr{H}$ = Hermitian manifolds.

(4) $\mathscr{W}$ = almost Hermitian manifolds.

Now, we return to our almost Hermitian manifold $TM = (TM, J, G)$ and examine the conditions for (J, G) such that TM belongs to each of these classes. For a constant c, we may consider next four conditions:

(C_0) $M = (M, g)$ is a space of constant curvature c,

$$(C_1)\quad 2h(\log\alpha)' - \frac{f\beta}{\alpha} - \frac{c}{f} = 0 \quad (\Longleftrightarrow\ \psi_1 = c/f),$$
$$(C_2)\quad 2h(\log f)' - \frac{h-f}{t} - \frac{c}{f} = 0 \quad (\Longleftrightarrow\ \psi_2 = c/f),$$
$$(C_3)\quad 2h\left(\log\frac{\alpha}{f}\right)' - \frac{f\beta}{\alpha} + \frac{h-f}{t} = 0 \quad (\Longleftrightarrow\ \psi_1 = \psi_2).$$

From (2.8)–(2.11), (C_3) is equivalent to $d\Omega = 0$ and $\delta\Omega = 0$.

Theorem 3.1. *For almost Hermitian manifold* $TM = (TM, J(f,h), G(\alpha,\beta,f,h))$, *we have the following:*

(1) $TM \in \mathcal{K}$ *if and only if* (C_0), (C_1), (C_2) *and* (C_3).
(2) $TM \in \mathscr{W}_1 = \mathscr{NK}$ *if and only if* (C_0), (C_1), (C_2) *and* (C_3).
(3) $TM \in \mathscr{W}_2 = \mathscr{AK}$ *if and only if* (C_3).
(4) $TM \in \mathscr{W}_3 = \mathscr{H} \cap \mathscr{SK}$ *if and only if* (C_0), (C_1), (C_2) *and* (C_3).
(5) $TM \in \mathscr{W}_4$ *if and only if* (C_0) *and* (C_2).
(6) $TM \in \mathscr{W}_1 \cup \mathscr{W}_2 = \mathscr{QK}$ *if and only if* (C_3).
(7) $TM \in \mathscr{W}_1 \cup \mathscr{W}_3$ *if and only if* (C_0), (C_1), (C_2) *and* (C_3).
(8) $TM \in \mathscr{W}_1 \cup \mathscr{W}_4$ *if and only if* (C_0) *and* (C_2).
(9) $TM \in \mathscr{W}_2 \cup \mathscr{W}_3$ *if and only if* (C_3).
(10) $TM \in \mathscr{W}_2 \cup \mathscr{W}_4$ *for any* f, h, α, β.
(11) $TM \in \mathscr{W}_3 \cup \mathscr{W}_4 = \mathscr{H}$ *if and only if* (C_0) *and* (C_2).
(12) $TM \in \mathscr{W}_1 \cup \mathscr{W}_2 \cup \mathscr{W}_3 = \mathscr{SK}$ *if and only if* (C_3).
(13) $TM \in \mathscr{W}_1 \cup \mathscr{W}_2 \cup \mathscr{W}_4$ *for any* f, h, α, β.
(14) $TM \in \mathscr{W}_1 \cup \mathscr{W}_3 \cup \mathscr{W}_4$ *if and only if* (C_0) *and* (C_2).
(15) $TM \in \mathscr{W}_2 \cup \mathscr{W}_3 \cup \mathscr{W}_4$ *for any* f, h, α, β.

Proof. (1): First, we consider the case $\dim TM \geq 6$ ($\dim M \geq 3$). Assume that M is not a space of constant curvature. Then, from (2.1), the condition $\nabla\Omega = 0$ requires $\alpha = 0$. Thus, M must have a constant curvature c. The, form (2.5)–(2.7), we observe that $\nabla\Omega = 0$ if and only if $\psi_1 - c/f = 0$ and $\psi_1 - 2\psi_2 + c/f = 0$, namely (C_1), (C_2) and (C_3). If $\dim TM = 4$ ($\dim M = 2$), the curvature tensor R is of the form

$$g(R(X,Y)Z,W) = k(p)\{g(X,W)g(Y,Z) - g(X,Z)g(Y,W)\},$$

for $X, Y, Z, W \in T_p(M)$. Therefore, the equalities (2.5)–(2.7) are valid if we replace c with $k(p)$. Then, form (2.5), we may show that $k(p)$ must be constant. So, the argument comes down to the case of constant curvature. Hence, (1) follows.

Using (2.1)–(2.11) and making a similar argument as above if necessary, we can prove (2)–(15). □

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