arbitrary closed subsets by the formula

$$\dim(V(\mathfrak{a})) = \dim(A/\mathfrak{a})$$

where $\mathfrak{a}$ is an arbitrary ideal of A.

If M is a finitely generated A-module we define

$$\dim(M) = \dim(\mathrm{Supp}(M)) = \dim(A/\mathrm{ann}(M)).$$

Here we use the fact, mentioned in the preliminaries, that Supp(M) is the closure in Spec(A) of Ass(M), and Ass(M) consists of the prime ideals associated to ann(M).

If $N \subset M$ is another A-module we see trivially that

$$\dim(N) \leqq \dim(M)$$

$$\dim(M/_N) \leqq \dim(M)$$

In fact $\mathrm{ann}(N) \supset \mathrm{ann}(M)$, $\mathrm{ann}(M/_N) \supset \mathrm{ann}(M)$.
A non-trivial statement, proved in Bourbaki's, chapter IV, §2, is the following:

<u>Theorem 1.2</u>. $\dim(M) = 0$ if, and only if, M has finite length, in the composition series sense.

§2. HILBERT-SAMUEL POLYNOMIAL

Let H be a graded ring, i.e.

$$H = \bigoplus_{n \geqq 0} H_n$$

where H_n are (additive) groups and $h_n \cdot h_m \in H_{n+m}$, for $h_n \in H_n$, $h_m \in H_m$. Clearly H_n is an H_0-module. We assume:

a) H_0 is an artinian ring

b) H is generated (as an H_0-algebra) by finitely many elements of H_1.

An H-module M is called graded if $M = \bigoplus_n M_n$, where M_n are H_o-modules and

$$H_n M_p \subset M_{n+p}.$$

If M is a finitely generated H-module, then M_n is a finitely generated H_o-module and (since H_o is artinian) M_n has finite length.

Definition 2.1. The Hilbert-Samuel Polynomial of M, $\chi(M, n)$, is given by

$$\chi(M, n) = \text{length}_{H_o} M_n \quad \text{for large } n.$$

Of course one needs to prove that $\chi(M, n)$ is indeed a polynomial. In fact

Theorem 2.1. (Hilbert) Let H, M be as stated above. Then there exists a polynomial $P(X) \in Q[X]$, which achieves integer values for integer values of X and such that, for all sufficiently large n,

$$\chi(M, n) = P(n)$$

Proof: Since H is finitely generated over H_o by H_1, we have a homogeneous epimorphism (of degree 0)

$$H_o[X_1,\ldots,X_r] \xrightarrow{c} H \to 0$$

and M becomes a finitely generated $H_o[X_1,\ldots,X_r]$-module. Now $\text{length}_{H_o} M_n$ is independent of whether we consider M as an H-module or an $H_o[X_1,\ldots,X_r]$-module (since c is onto). Hence we may assume $H = H_o[X_1,\ldots,X_r]$.

We proceed by induction on r. When r = 0, $H = H_o$ and, since

M is finitely generated by, say, $m_i \in M_{n_i}$, we have $M_n = 0$ if $n \geqq \max_i \{n_i\}$. Hence $\chi(M, n) = 0$ for n sufficiently large.

Let $\varphi_r : M \to M$ be given by $\varphi_r(m) = X_r \cdot m$. Then φ_r is a homogeneous morphism of degree 1 and we have

$$0 \to N \to M \xrightarrow{\varphi_r} M \to C \to 0$$

$$0 \to N_n \to M_n \to M_{n+1} \to C_{n+1} \to 0$$

Since $\text{length}_{H_0}(\cdot)$ is an additive function we have

$$\chi(M, n+1) - \chi(M, n) = \chi(C, n+1) - \chi(N, n)$$

For $n \in N$, $c \in C$ we have $X_r \cdot n = 0$, $X_r \cdot c = 0$, hence N and C are $H_0[X_1, \ldots, X_{r-1}]$ modules, and, by induction, $\chi(C, n+1) - \chi(N, n)$ is a rational polynomial in n, for sufficiently large n. A standard argument now shows that $\chi(M, n)$ is also a rational polynomial, for n sufficiently large.

For the remainder of this section we assume that A is a noetherian, semi-local ring.

<u>Definition 2.2</u>. Let $\mathfrak{q}$ <u>be an ideal of A</u>. We say that $\mathfrak{q}$ is <u>an ideal of definition of A</u>, if the ring $A/\mathfrak{q}$ is artinian.

We recall here that a ring A is called artinian if it satisfies the descending chain condition or, equivalently, if every prime ideal of A is maximal.

We assert:

<u>Proposition 2.1</u>. Let $\mathfrak{q}$ be an ideal of A. The following three conditions are equivalent.

a) $\mathfrak{q}$ is an ideal of definition of A

b) $A/\mathfrak{q}$ has finite length (in the composition series sense)

c) $\mathfrak{q} \supset \mathfrak{N}^k$, where $\mathfrak{N}$ denotes the radical of A.

Proof:

b) $\Longrightarrow$ a) is immediate, since $A/\mathfrak{q}$ satisfies both chain conditions. a) $\Longrightarrow$ b) follows from the fact that an artinian ring is also noetherian.

c) $\Longrightarrow$ a) follows from the following observation: if $\mathfrak{q} \supset \mathfrak{N}^k$ and a prime ideal $\mathfrak{p}$ contains $\mathfrak{q}$, then $\mathfrak{p}$ is one of the maximal ideals of A. To see that a) $\Longrightarrow$ c) we observe first, that since $A/\mathfrak{q}$ is artinian, $\mathrm{rad}(A/\mathfrak{q})$ = the set of nilpotents in $A/\mathfrak{q}$. Now, clearly, $\mathrm{rad}(A/\mathfrak{q}) = \varphi(\mathfrak{N})$, where $\varphi: A \to A/\mathfrak{q}$ is the canonical epimorphism.

If $\mathfrak{q}$ is an ideal of definition of A and M is a finitely generated A-module, $M/\mathfrak{q}M$ is a finitely generated $A/\mathfrak{q}$-module (in fact $M/\mathfrak{q}M \cong M \otimes_A A/\mathfrak{q}$), hence $M/\mathfrak{q}M$ has finite length.

Theorem 2.2. (Hilbert-Samuel) Let A, $\mathfrak{q}$, M be as above. Then

a) $M/\mathfrak{q}^n M$ has finite length

b) $\mathrm{length}_A(M/\mathfrak{q}^n M) = P_{\mathfrak{q}}(M, n)$ is a polynomial in n for n sufficiently large.

Proof: We prove a) by induction on n. When n = 1 the assertion is precisely the observation we made previous to the statement of the theorem. Clearly, for all k, $\mathfrak{q}^k/\mathfrak{q}^{k+1}$ is a finitely generated A-module (A noetherian). Hence $(M/\mathfrak{q}M) \otimes_A \mathfrak{q}^k/\mathfrak{q}^{k+1}$ is a finitely generated A-module. The epimorphism

$$(M/\mathfrak{q}M) \otimes \mathfrak{q}^k/\mathfrak{q}^{k+1} \to \mathfrak{q}^k M/\mathfrak{q}^{k+1} M$$

given by $\overline{m} \otimes \overline{q} \rightsquigarrow \overline{qm}$ (here $\overline{m}$, $\overline{q}$ denote the equivalence classes of $m \in M$, $q \in \mathfrak{q}^k$) shows that $\mathfrak{q}^k M/\mathfrak{q}^{k+1} M$ is a finitely generated A-module. Finally the exact sequence

$$(\ast)\quad 0 \to \mathfrak{q}^n M/\mathfrak{q}^{n+1} M \to M/\mathfrak{q}^{n+1} M \to M/\mathfrak{q}^n M \to 0$$

and the induction assumption prove a).

To prove b) we define

$$H = gr_{\mathfrak{q}}(A) = \bigoplus_{i \geqq 0} (\mathfrak{q}^i/\mathfrak{q}^{i+1})$$

$$M' = gr(M) = \bigoplus_{i \geqq 0} (\mathfrak{q}^i M/\mathfrak{q}^{i+1} M)$$

where $\mathfrak{q}^0 = A$. Since $H_0 = A/\mathfrak{q}$ is artinian, H is generated over H_0 by finitely many elements of $H_1 = \mathfrak{q}/\mathfrak{q}^2$ (any A-basis of $\mathfrak{q}$ will do) and M' is a finitely generated H-module (any A-basis of M will do); we can apply Theorem 2.1 and get

$$\text{length}(\mathfrak{q}^n M/\mathfrak{q}^{n+1} M) = \text{a polynomial in } n \text{ for } n \gg 0.$$

(We write $n \gg 0$ for "...n sufficiently large".)

From the above exact sequence $(\ast)$ we get

$$\text{length}(M/\mathfrak{q}^{n+1} M) - \text{length}(M/\mathfrak{q}^n M) = \text{length}(\mathfrak{q}^n M/\mathfrak{q}^{n+1} M)$$

or

$$P_{\mathfrak{q}}(M, n+1) - P_{\mathfrak{q}}(M, n) = \text{a polynomial in } n \text{ for } n \gg 0.$$

The theorem is proved.

Note: The geometrical significance of the polynomial $P_{\mathfrak{q}}(M, n)$ was discovered by Serre, and it is the following. Let H, M' be as in the proof of the theorem. Let $X = \text{Proj}(H)$, $\mathcal{F}$ = the sheaf over Proj(H) associated to the graded module M': then for every n, $P_{\mathfrak{q}}(M', n) = \sum_i (-1)^i \text{ length } H^i(X, \mathcal{F}(n))$.

We do not go into further details, except to point out that, for $n \gg 0$ $H^1(X, \mathcal{J}(n)) = 0$, which throws a better light on the somewhat unsatisfactory statement of b), (for $n \gg 0$).

Let now A, $\mathfrak{q}$, M be as usual. A filtration $M = M_o \supset M_1 \supset \ldots \supset M_n \supset \ldots$ is called a $\mathfrak{q}$ - good filtration of M if $\mathfrak{q} M_n \subset M_{n+1}$, with equality holding for $n \geqq n_o$.

We assert

<u>Proposition 2.2</u>. Under the above hypotheses, for $n \gg 0$ $\mathrm{length}_A(M/M_n) = P((M_n), n)$ = a polynomial in n of degree and coefficient of the term of highest degree equaling those of $P_{\mathfrak{q}}(M, n)$.

<u>Proof</u>: As in the proof of theorem, we prove by induction on n that M/M_n has finite length. In fact M/M_1 is an $A/\mathfrak{q}$ - module finitely generated, and

$$0 \to M_n/M_{n+1} \to M/M_{n+1} \to M/M_n \to 0$$

and $\mathfrak{q}(M_n/M_{n+1}) = 0$, whence M_n/M_{n+1} is an $A/\mathfrak{q}$-module and has finite length.

Consider now the module M_{n_o}. It is a finitely generated A-module and $M_{n+n_o} = \mathfrak{q}^n M_{n_o}$. Hence, by theorem 2.2

$$\mathrm{length}(M_{n_o}/M_{n+n_o}) = \text{a polynomial in } n, \text{ for } n \gg 0.$$

The exact sequence

$$0 \to M_{n_o}/M_{n+n_o} \to M/M_{n+n_o} \to M/M_{n_o} \to 0$$

shows that $\mathrm{length}(M/M_n)$ is a polynomial in n for $n \gg 0$. The inclusions

$$\mathfrak{q}^{n+n_o} M \subset M_{n+n_o} \subset \mathfrak{q}^n M \subset M_n$$

give exact sequences

$$0 \to M_{n+n_o}/\mathfrak{q}^{n+n_o} M \to M/\mathfrak{q}^{n+n_o} M \to M/M_{n+n_o} \to 0$$

$$0 \to \mathfrak{q}^n M/M_{n+n_o} \to M/M_{n+n_o} \to M/\mathfrak{q}^n M \to 0$$

$$0 \to M_n/\mathfrak{q}^n M \to M/\mathfrak{q}^n M \to M/M_n \to 0$$

whence

$$P_{\mathfrak{q}}(M, n+n_o) \geqq P((M_{n+n_o}), n+n_o) \geqq P_{\mathfrak{q}}(M, n) \geqq P((M_n), n).$$

Since $P_{\mathfrak{q}}$ and P are polynomials, they must have the same degree and the same highest degree coefficient, Q.E.D.

<u>Proposition 2.3</u>. Let $\mathfrak{q}$, $\mathfrak{q}'$ be ideals of definition of A, M a finitely generated A-module. Then $P_{\mathfrak{q}}$, $P_{\mathfrak{q}'}$ are polynomials of the same degree.

<u>Proof</u>: Since $\mathrm{rad}(\mathfrak{q}') = \mathrm{rad}(\mathfrak{q}) = \mathfrak{n}$ we have (A is noetherian) $\mathfrak{q} \supset \mathfrak{q}'^p$ and $\mathfrak{q}' \supset \mathfrak{q}^m$, for some m. Hence

$$0 \to \mathfrak{q}'^n M/\mathfrak{q}^{nm} M \to M/\mathfrak{q}^{nm} M \to M/\mathfrak{q}'^n M \to 0$$

whence $P_{\mathfrak{q}'}(M, n) \leqq P_{\mathfrak{q}}(M, mn)$ and similarly

$$P_{\mathfrak{q}}(M, n) \leqq P_{\mathfrak{q}'}(M, pn)$$

and the proposition is proved.

<u>Definition</u> 2.3. Let A, M be given as above. Then deg $P_{\mathfrak{q}}$

(which, by the proposition above is independent of $\mathfrak{q}$) is denoted by $d(M)$.

Proposition 2.4. Let A be as usual, and let

$$0 \to M' \to M \to M'' \to 0$$

be an exact sequence of finitely generated A-modules. Then, for any ideal $\mathfrak{q}$ of definition of A:

$$\deg[P_{\mathfrak{q}}(M) - P_{\mathfrak{q}}(M') - P_{\mathfrak{q}}(M'')] \leqq d(M') - 1 \leqq d(M) - 1$$

Proof: By the Artin-Rees lemma (B.C.A., III, 3, corollary 1) the submodules $M'_n = \mathfrak{q}^n M \cap M'$ of M' form a $\mathfrak{q}$- good filtration of M'. By proposition 2.2 we have (*) $P_{\mathfrak{q}}(M')$ and $P(M'_n)$ have the same degree and the same highest degree coefficient.

The exact sequence

$$0 \to \mathfrak{q}^n M \cap M' \to \mathfrak{q}^n M \to \mathfrak{q}^n M'' \to 0$$

gives an exact sequence

$$0 \to M'/\mathfrak{q}^n M \cap M' \to M/\mathfrak{q}^n M \to M''/\mathfrak{q}^n M'' \to 0$$

whence

$$P_{\mathfrak{q}}(M, n) - P_{\mathfrak{q}}(M'', n) - P(M'_n, n) = 0$$

or

$$P_{\mathfrak{q}}(M) - P_{\mathfrak{q}}(M'') - P(M'_n) = 0$$

Hence

$$P(M'_n) = P_{\mathfrak{q}}(M) - P_{\mathfrak{q}}(M'')$$

and, by (*),

$$P_{\mathfrak{q}}(M') = P_{\mathfrak{q}}(M) - P_{\mathfrak{q}}(M'') + \text{a polyn. of degree}$$
at most $d(M') - 1$.

The first inequality is proved. The second follows immediately from observing that $0 \leq P_{\mathfrak{q}}(M', n) \leq P_{\mathfrak{q}}(M, n)$, for $n \gg 0$, whence

$$\deg P_{\mathfrak{q}}(M') \leq \deg P_{\mathfrak{q}}(M).$$

Let M be a finitely generated A-module, and let $y_1, \ldots, y_k \in \mathfrak{r}$ be a set of generators of $\mathfrak{r}$. Then $M/y_1 M + \ldots + y_k M$ is an $A/\mathfrak{r}$-module and hence has finite length. With this in mind we give the following:

Definition 2.4. We denote by $s(M)$ the smallest integer k satisfying the following condition:

there exist k elements $x_1, \ldots, x_k$ in $\mathfrak{r}$ such that

$$M/x_1 M + \ldots + x_k M \text{ has finite length}$$

We are now in the position of proving the main result of dimension theory, namely

Theorem 2.3. (Krull-Chevalley-Samuel) Let A be a semi-local noetherian ring, M a finitely generated A-module. Then $\dim(M) = d(M) = s(M)$.

Proof: (Serre). We shall prove

1) $\dim(M) \leq d(M)$

2) $d(M) \leq s(M)$

3) $s(M) \leq \dim(M)$.

We start with the following

Lemma 2.1. Let $x \in \mathfrak{N}$, consider the exact sequence

$$0 \to {}_xM \to M \xrightarrow{\varphi} M \to M/xM \to 0$$

where $\varphi(m) = xm$. Then

i) $s(M) \leq s(M/xM) + 1$

ii) Let $(\mathfrak{p}_1, \ldots, \mathfrak{p}_m)$ denote those points of Supp(M) such that $\dim(A/\mathfrak{p}_i) = \dim(M)$, $i = 1, \ldots, m$. If $x \notin \bigcup_{i=1}^{m} \mathfrak{p}_i$ then $\dim(M/xM) \leq \dim(M) - 1$

iii) $\deg[P_{\mathfrak{q}}({}_xM) - P_{\mathfrak{q}}(M/xM)] \leq d(M) - 1$, where $\mathfrak{q}$ is any ideal of definition of A.

Proof:

i) Let $N = M/xM$, and let $y_1, \ldots, y_k \in \mathfrak{N}$ such that $N/y_1 N + \ldots + y_k N$ has finite length and $k = s(N)$. The isomorphism

$$N/y_1 N + \ldots + y_k N \to M/xM + y_1 M + \ldots + y_k M$$

proves i).

ii) We start with a word about the $\mathfrak{p}_i$'s. By definition we have $\dim(M) = \dim(A/_{\mathrm{ann}(M)})$. If $\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_t$, $t \geq m$, denote the prime ideals associated to ann(M) in A one easily sees that

$$\dim(M) = \max_{1 \leq i \leq t} \dim(A/\mathfrak{p}_i).$$

Hence the prime ideals mentioned in the statement of ii) are to be found among the points of Ass(M).

We have to compare dim(A/ann(M/xM)) with dim(A/ann M) Let $\mathfrak{q}_1, \ldots, \mathfrak{q}_t$ be those prime ideals in A associated to ann(M/xM) and such that dim(M/xM) = dim(A/$\mathfrak{q}_j$). Then, for some i_j, $1 \leqq i_j \leqq t$, we have $\mathfrak{q}_j \supset \mathfrak{p}_{i_j}$. Let $\mathfrak{q}'_0 \subsetneqq \mathfrak{q}'_1 \subsetneqq \ldots \subsetneqq \mathfrak{q}'_k$ be a chain of prime ideals of maximal length in A/ann(M/xM), i.e. k = dim(M/xM). The prime ideal $\mathfrak{q}'_0$ corresponds to a prime ideal $\mathfrak{q}$ of A containing ann(M/xM) and, from k = dim(M/xM) one sees that $\mathfrak{q} = \mathfrak{q}_j$ for some j. We proceed in steps.

<u>Case 1</u>. $\mathfrak{q}_j \supset \mathfrak{p}_{i_j}$, $i_j > m$. Then

$$\dim(M/xM) = \dim A/\mathfrak{q}_j \leqq \dim A/\mathfrak{p}_{i_j} < \dim(M)$$

and ii) is proved in this case.

<u>Case 2</u>. $\mathfrak{q}_j \supset \mathfrak{p}_{i_j}$, $i_j \leqq m$. Then (since $x \in \mathfrak{q}_j$), $\mathfrak{q}_j \supsetneqq \mathfrak{p}_{i_j}$ and the chain $\mathfrak{p}_{i_j} \subsetneqq \mathfrak{q}_j \subsetneqq \ldots$ shows that dim(M) $\geqq$ k + 1 and ii) is proved in this case also.

iii). We have two exact sequences

$$0 \to {}_xM \to M \to xM \to 0$$

$$0 \to xM \to M \to M/xM \to 0$$

Now

$$\deg[P_{\mathfrak{q}}({}_xM) - P_{\mathfrak{q}}(M/xM)] =$$

$$\deg[(P_{\mathfrak{q}}({}_xM) + P_{\mathfrak{q}}(xM) - P_{\mathfrak{q}}(M)) + (P_{\mathfrak{q}}(M) - P_{\mathfrak{q}}(xM) - P_{\mathfrak{q}}(M/xM)]$$

and, by proposition 2.2 the right hand side is the degree of the sum of two polynomials, one of degree $\leqq d({}_xM) - 1 \leqq d(M) - 1$, the other of degree $\leqq d(xM) - 1 \leqq d(M) - 1$. The lemma is proved. Now we return to the proof of the theorem.

1) $\dim(M) \leqq d(M)$. We proceed by induction on $d(M)$. $d(M) = 0$. Then $P_{\mathfrak{q}}(M) =$ constant, whence

$$\text{length } (M/\mathfrak{q}^n M) = \text{length } (M/\mathfrak{q}^{n+1} M) \text{ for } n \gg 0.$$

The exact sequence

$$0 \to \mathfrak{q}^n M/\mathfrak{q}^{n+1} M/ \to M/\mathfrak{q}^{n+1} M \to M/\mathfrak{q}^n M \to 0$$

shows length $(\mathfrak{q}^n M/\mathfrak{q}^{n+1} M) = 0$ whence $\mathfrak{q}^n M = \mathfrak{q}^{n+1} M$. Now, we take $\mathfrak{q} = \mathfrak{r}$, and then we have $\bigcap_{n \geqq 0} \mathfrak{r}^n = (0)$, whence $\mathfrak{r}^n M = 0$ for $n \gg 0$. Hence M is an $A/\mathfrak{r}^n$-module, and since $A/\mathfrak{r}^n$ is artinian, its dimension is 0, whence $\dim(M) = 0$. Hence 1 holds when $d(M) = 0$.

Choose a prime $\mathfrak{p}_0 \in \mathrm{Ass}(M)$ such that $\dim(M) = \dim(A/\mathfrak{p}_0)$. Since $\mathfrak{p}_0$ is <u>the annihilator</u> of an element $m \in M$, the submodule $N = Am \subset M$ is isomorphic to $A/\mathfrak{p}_0$. By proposition 2.4 we have

$$d(N) \leqq d(M)$$

and

$$\dim(N) = \dim(M)$$

Hence it suffices to prove 1) for N. Let

$\mathfrak{p}_0 \underset{\neq}{\subset} \mathfrak{p}_1 \underset{\neq}{\subset} \mathfrak{p}_2 \cdots \underset{\neq}{\subset} \mathfrak{p}_n$ be a chain of maximal length in A, corresponding to a chain of maximal length in $A/\mathfrak{p}_0$ (note that $n = +\infty$ is a priori possible). If $\mathfrak{p}_1 \cap \mathfrak{r} \subset \mathfrak{p}_0$, then $\mathfrak{p}_0 \supset \mathfrak{r}$, whence is maximal (because A is semi-local), a contradiction.

Choose $x \in \mathfrak{p}_1 \cap \mathfrak{r}$, $x \notin \mathfrak{p}_0$.

We have

$$N/xN = (A/xA) \otimes_A N$$

and, from proposition 18 of B.C.A., II, §4 we get

$$\mathrm{Supp}(N/xN) = \mathrm{Supp}(N) \cap V(x).$$

Hence $\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_n \in \mathrm{Supp}(N/xN)$, whence $\dim(N/xN) \geqq n - 1$ (in particular, if $\dim(N/xN)$ is finite, so is n). Now trivially the homomorphism $A/\mathfrak{p}_0 \to A/\mathfrak{p}_0$ given by $\bar{a} \rightsquigarrow x\bar{a}$ is injective, hence ${}_xN = 0$. By lemma 2.1 we get $d(N/xN) \leqq d(N) - 1 \leqq d(M) - 1$, and by induction $\dim(N/xN) \leqq d(N/xN)$ (and we have proved that n is finite). Now

$$\dim(M) = n \leqq \dim(N/xN) + 1 \leqq d(N/xN) + 1 \leqq d(M)$$

and 1) is proved.

We observe here that we have actually shown $\dim(M) < + \infty$.

2) $d(M) \leqq s(M)$. Let $\{x_i\}_{1 \leqq i \leqq n}$ be elements of $\mathfrak{r}$ such that, letting $\mathfrak{a} = x_1 A + \ldots + x_n A$, we have length $(M/\mathfrak{a} M) < + \infty$ and $n = s(M)$. Let $\mathfrak{q} = \mathfrak{a} + \mathfrak{r} \cap \mathrm{ann}(M)$. We have $\mathrm{ann}(M/\mathfrak{q} M) \supset \mathfrak{a}$, hence the prime ideals in $\mathrm{Ass}(M/\mathfrak{q}M)$ are maximal, and therefore $\mathfrak{q} \supset \mathfrak{r}^k$ for some k, i.e. $\mathfrak{q}$ is an ideal of definition of A. Now clearly $\mathfrak{q}^m M = \mathfrak{a}^m M$, whence $\mathfrak{q}^m M/\mathfrak{q}^{m+1} M = \mathfrak{a}^m M/\mathfrak{a}^{m+1} M$. Let $z_1, \ldots, z_r$ be a minimal set of generators of M over A. Then the elements $\{x_1^{\nu_1} \ldots x_n^{\nu_n} \cdot z_i\}$ $1 \leq i \leq r$, $\nu_1 + \ldots + \nu_n = m$ are a set of generators of $\mathfrak{a}^m M/\mathfrak{a}^{m+1} M$ over $A/\mathfrak{q}$. Let length $(A/\mathfrak{q}) =$

$a (a < +\infty$ since $A/\mathfrak{q}$ is artinian). Now

$$\text{length } (\mathfrak{q}^m M/\mathfrak{q}^{m+1} M) = \text{length } (\mathfrak{a}^m M/\mathfrak{a}^{m+1} M) \leqq$$

$$\text{a.r.} \binom{n+m-1}{n-1} = \text{a polyn. in m of degree } n-1.$$

The exact sequence

$$0 \to \mathfrak{q}^m M/\mathfrak{q}^{m+1} M \to M/\mathfrak{q}^{m+1} M \to M/\mathfrak{q}^m M \to 0$$

shows 2).

3) $s(M) \leqq \dim(M)$. We proceed by induction on $\dim(M)$ (which is finite by 1).

$\dim(M) = 0$. Then length $(M) < +\infty$ (since $A/\text{ann } M$ is artinian) and no elements of $\mathfrak{r}$ are needed to have length $(M/x_1M+\ldots+x_kM) < +\infty$. Hence $s(M) = 0$ and 3) holds. Let $n = \dim(M) \geqq 1$. Let $\{\mathfrak{p}_i\}_{1\leqq i\leqq m}$ be those elements of $\text{Ass}(M)$ such that $\dim(M) = \dim(A/\mathfrak{p}_i)$. Since $n \geqq 1$ the $\mathfrak{p}_i$ are not maximal. We assert:

$\mathfrak{r} \not\subset \bigcup_{i=1}^{m} \mathfrak{p}_i$. In fact, if $\mathfrak{r} \subset \bigcup_{i=1}^{m} \mathfrak{p}_i$, then, by proposition 2 of B.C.A., II, §1, we have $\mathfrak{r} \subset \mathfrak{p}_i$ for some i, a contradiction, since $\mathfrak{p}_i$ is not maximal. Hence we can choose $x \in \mathfrak{r}$, $x \notin \bigcup_{i=1}^{m} \mathfrak{p}_i$. By lemma 2.1 we have

$$s(M) \leqq s(M/xM) + 1$$

and $\dim(M/xM) \leqq \dim(M) - 1$. Hence, by induction

$$s(M/xM) \leqq \dim(M/xM)$$

and finally

$$s(M) \leqq s(M/xM) + 1 \leqq \dim(M/xM) + 1 \leqq \dim(M),$$

Q.E.D.

Appendix

We give a brief description of the geometrical meaning of the three numbers dim(M), s(M), d(M).

We admit right off that d(M) is a far-reaching concept leading in particular to certain results of intersection theory, and we shall limit ourselves to a geometrical interpretation of dim(M) and s(M).

dim(M) is the simplest of the two. It simply gives the maximal length of irredundant descending chains of irreducible subsets of Supp(M). (Such chains must necessarily terminate with a closed point.)

s(M) has a somewhat more sophisticated interpretation. Remembering that $\operatorname{Supp}(M/xM) = \operatorname{Supp}(M) \cap V(x)$ and that length $(M) < +\infty \Longleftrightarrow \dim(M) = 0 \Longleftrightarrow \dim(\operatorname{Supp}(M)) = 0 \Longleftrightarrow$ (by above remark) $\Longleftrightarrow$ Supp(M) consists of a finite number of closed points. We see that s(M) is the smallest number of "hypersurfaces" (the V(x)'s) such that their intersection with Supp(M) is zero dimensional.

There is a fourth integer that one should introduce in this connection, but which is related to the previous three, in general, by an inequality rather than equality.

Let A be a local ring, $\mathfrak{m}$ its maximal ideal. The A-module $\mathfrak{m}/\mathfrak{m}^2$ is (clearly!) annihilated by $\mathfrak{m}$, hence $\mathfrak{m}/\mathfrak{m}^2$ is an $A/\mathfrak{m}$ module, i.e. a vector space over $k = A/\mathfrak{m}$. $\dim_k(\mathfrak{m}/\mathfrak{m}^2)$ is the fourth integer we wish to consider. We assert:

Proposition 2.5.

$$s(A) \leq \dim_k(\mathfrak{m}/\mathfrak{m}^2).$$

Proof: Let $x_1,\dots,x_n$ be elements of $\mathfrak{m}$ such that their equivalence classes (mod $\mathfrak{m}^2$) form a basis of $\mathfrak{m}/\mathfrak{m}^2$ over $A/\mathfrak{m}$. We assert that $x_1,\dots,x_n$ form a system of generators of $\mathfrak{m}$. Let

$$M = x_1\, A \oplus x_2\, A \oplus \dots \oplus x_n\, A$$
$$N = \mathfrak{m}$$

and let $u: M \to N$ be defined by $u(a_1x_1 \oplus \dots \oplus a_nx_n) = \sum a_ix_i$. Let $\mathfrak{n} = \mathfrak{m}^2 \subset \operatorname{rad}(A) = \mathfrak{m}$. Now

$$N \otimes A/\mathfrak{m}^2 \simeq \mathfrak{m}/\mathfrak{m}^2$$

and

$$u \otimes \mathrm{id}_{A/\mathfrak{m}^2} : M \otimes A/\mathfrak{m}^2 \to \mathfrak{m}/\mathfrak{m}^2$$

is surjective, since we have the commutative diagram

$$\begin{tikzcd}
M \otimes A/\mathfrak{m} \simeq \displaystyle\bigoplus_{i=1}^{n} k \arrow[dr, "\Psi"] & \\
M \otimes A/\mathfrak{m}^2 \arrow[u, "\varphi"] \arrow[r, "u \otimes \mathrm{id}_{A/\mathfrak{m}^2}"'] & \mathfrak{m}/\mathfrak{m}^2
\end{tikzcd}$$

and φ, Ψ are surjective. By Nakayama's lemma we have that u is surjective, which proves that $x_1,\dots,x_n$ form a system of generators of $\mathfrak{m}$. Hence $A/x_1A + \dots + x_nA = k$ and $\operatorname{length}_A(k) < +\infty$.

Hence $s(A) \leqq n = \operatorname{rank}_k(\mathfrak{m}/\mathfrak{m}^2)$, Q.E.D.

We show with an example that $s(A) < \operatorname{rank}_k(\mathfrak{m}/\mathfrak{m}^2)$ does happen. We observe first of all that (trivially) <u>any</u> set of generators of $\mathfrak{m}$ gives rise to a set of generators of $\mathfrak{m}/\mathfrak{m}^2$ over k. Hence $\dim_k(\mathfrak{m}/\mathfrak{m}^2)$ = smallest number of generators of $\mathfrak{m}$. Let now

$$R = \mathbb{C}[X, Y]/_{(Y^2 - X^3)} = \mathbb{C}[x, y]$$

$$\mathfrak{p} = xR + yR$$

$$A = R_{\mathfrak{p}}$$

$$\mathfrak{m} = \mathfrak{p} A_{\mathfrak{p}} \quad .$$

We make (without proof) the following assertions: (b), c) have easy proofs)

a) $\dim R = 1$

b) R is an integral domain

c) $\mathfrak{p}$ is prime

Hence it follows that $\mathfrak{p}$ is maximal and that $s(A) = \dim(A) = 1$. But $\mathfrak{m}$ is <u>not</u> principal, in fact $\dim_{A/\mathfrak{m}} (\mathfrak{m}/_{\mathfrak{m}^2}) = 2$. To see this, consider the diagram

$$\begin{tikzcd} & \mathfrak{m} \arrow[r, dash] \arrow[d, dash] & A \arrow[d, dash] \\ \mathfrak{m} \cap R = & \mathfrak{p} \arrow[r, dash] & R \end{tikzcd}$$

We see that $\dim_{A/\mathfrak{m}}(\mathfrak{m}/_{\mathfrak{m}^2})$ = smallest no. of generators of $\mathfrak{m} \leqq 2$ (x, y generate $\mathfrak{m}$). However, were $\mathfrak{m}$ principal, so would $\mathfrak{p}$ be. Now were it so, the inverse image of $\mathfrak{p}$ under $\mathbb{C}[X, Y] \to R$ would be principal $\mathrm{mod}(Y^2 - X^3)$, which is easily seen to be impossible. Hence $\dim_{\mathbb{C}}(\mathfrak{m}/_{\mathfrak{m}^2}) = 2$. (Note that $A/\mathfrak{m} = \mathbb{C}$). From $\dim R = 1$ one obtains $\dim(A) = 1$, whence $s(A) = 1 < \dim_{\mathbb{C}}(\mathfrak{m}/_{\mathfrak{m}^2})$.

When the local ring A is such that $s(A) = \dim_{A/\mathfrak{m}} (\mathfrak{m}/_{\mathfrak{m}^2})$ we say that A is a regular local ring.

The geometrical interpretation of the number $\dim_{A/\mathfrak{m}}(\mathfrak{m}/\mathfrak{m}^2)$ is the following: it is the number of linearly independent linear forms (modulo forms of higher degree). This corresponds to the classical concept of the dimension of the tangent space.

If A is not a local ring, one can still talk about dim(A), and one trivially gets the formula

$$\dim(A) = \operatorname{Sup}_{\mathfrak{m}}(\dim(A_{\mathfrak{m}}))$$

where $\mathfrak{m}$ ranges over the maximal ideals of A.

We give a brief description of the situation when dim(A) = 0, 1.

dim(A) = 0. Then A is artinian, hence semi-local. Let $\mathfrak{N}$ = nil radical(A). We get $A/\mathfrak{N} \simeq \oplus A/\mathfrak{m}_i$, i.e. $A/\mathfrak{N}$ is a direct sum of fields. Spec(A) consists of a finite number of closed points, and the local rings are primary rings (i.e. some power of the maximal ideal is 0). In fact, since A is artinian, so is $A_{\mathfrak{m}_i}$, whence $(\mathfrak{m}_i A_{\mathfrak{m}_i})^n = (\mathfrak{m}_i A_{\mathfrak{m}_i})^{n+1}$, $n \gg 0$, and $\bigcap_n (\mathfrak{m}_i A_{\mathfrak{m}_i})^n = (0)$. Furthermore we have

$$A = \Gamma(\operatorname{Spec} A, \tilde{A}) = \oplus A_{\mathfrak{m}_i}$$

which is easily seen from the fact that Spec(A) consists of a finite number of closed points.

dim(A) = 1. In this case the prime ideals of A are either minimal or maximal, and there are only finitely many minimal primes, with at least one, say $\mathfrak{p}$, such that $\dim(A/\mathfrak{p}) = 1$. If A is local, all minimal primes have this property. There are infinitely many maximal primes, if A is not semi-local.

A typical example of this case are the Dedekind rings, i.e. noetherian, integrally closed domains A such that every prime ideal $\mathfrak{p} \subset A$, $\mathfrak{p} \neq (0)$ is maximal. It follows that all local rings $A_{\mathfrak{p}}$ are valuation rings.

We note however that, while in the case $A = \mathbb{C}[X]$ all local rings $A_{\mathfrak{p}}$ are isomorphic, when $A = Z$ we obtain distinct local rings, for distinct $\mathfrak{p}$.

One can get more one-dimensional examples in the following way: Let A be a Dedekind ring, K its field of quotients, L a finite extension of K. Then any ring B, with $A \subset B \subset L$, is one dimensional (and need not be Dedekind). (Krull-Akizuki theorem, B.C.A., VII, §2.) Other examples are the orders of A in L, i.e. rings contained in A, with field of quotients L (hence not integrally closed when they are different from A).

If A is one dimensional local ring which is a Dedekind domain (i.e. integrally closed), then A is a valuation ring (See Lang, "Introduction to Algebraic Geometry", theorem 1, p. 151, or B.C.A., VI).

The geometrical interpretation of the notion of Dedekind rings is seen by observing that, if A is a Dedekind domain, Spec(A) consists of one minimal prime and maximal primes whose local rings are integrally closed whence regular. Classically this corresponds to the notion of an irreducible, non-singular curve.

Let $A = \mathbb{C}[X, Y]$, and let $f(X, Y) \in \mathbb{C}[X, Y]$. Then a classical statement in Algebraic Geometry is that the irreducible components (in the Zariski topology) of the variety of zeros of $f(X, Y)$ have codimension $\leqq 1$. We generalize the above situation with the following:

Theorem 2.4. Let A be a noetherian ring, $x_1, \ldots, x_n \in A$, $\mathfrak{a} = x_1 A + \ldots + x_n A$. Let $\mathfrak{p}$ be a minimal prime in $\mathrm{Ass}(A/\mathfrak{a})$. Then $\mathrm{codim}(V(\mathfrak{p})) = \dim(A_{\mathfrak{p}}) \leqq n$

(When $n = 1$ this is the well-known "Hauptidealsatz").

Proof. We have the inclusions $A_{\mathfrak{p}} \supset \mathfrak{p} A_{\mathfrak{p}} \supset \mathfrak{a} A_{\mathfrak{p}}$. Since $\mathfrak{p}$ is minimal in $\mathrm{Ass}(A/\mathfrak{a})$, there are no primes of A properly included between $\mathfrak{p}$ and $\mathfrak{a}$, hence $A_{\mathfrak{p}}/\mathfrak{a} A_{\mathfrak{p}}$ has a unique prime ideal (namely $\mathfrak{p}(A_{\mathfrak{p}}/\mathfrak{a} A_{\mathfrak{p}})$.), and is therefore Artinian, whence of finite length. Now $A_{\mathfrak{p}}/\mathfrak{a} A_{\mathfrak{p}} \simeq A_{\mathfrak{p}}/x_1 A_{\mathfrak{p}} + \ldots + x_n A_{\mathfrak{p}}$, whence $\mathrm{codim}(V(\mathfrak{p})) = \dim A_{\mathfrak{p}} = s(A_{\mathfrak{p}}) \leqq n$, Q.E.D.

Theorem 2.4 is an example of how we can apply our local dimension theory to a global situation.

Some final results concerning the notion of dimension:

Theorem 2.5. (Artin-Tate). Let A be a noetherian integral domain. Then the following conditions are equivalent:

a) A is semi-local of dimension $\leqq 1$

b) (0) is an isolated point in Spec(A)

c) there exists an $f \in A$ such that A_f is a field.

Proof: We give a cyclic proof.

a) $\Longrightarrow$ b). Since A is integral, $(0) \in \mathrm{Spec}(A)$. Since A is semi-local, there are a finite number of closed points, $\{\mathfrak{m}_1\}, \ldots, \{\mathfrak{m}_n\}$ in Spec(A). Since $\dim(A) \leqq 1$, Spec(A) consists precisely of $\{(0)\}$, $\{\mathfrak{m}_1\}, \ldots, \{\mathfrak{m}_n\}$ and b) follows.

b) $\Longrightarrow$ c) Since (0) is isolated in Spec(A), and the open subsets $\{D(f)\}_{f \in A}$ form a basis for the Zariski topology of Spec(A), there exists $f \in A$ such that $D(f) = (0)$. But $D(f) = \operatorname{Spec} A_f$, whence A_f has <u>only one</u> prime ideal, namely (0), and c) follows.

c) $\Longrightarrow$ a) Let $\mathfrak{p} \neq (0)$ be any point of Spec(A). The injection $A \to A_f$ shows, since A_f is a field, that $1 \in \mathfrak{p} A_f$. Hence $f \in \mathfrak{p}$. We assert:

(*) every minimal prime ideal of A/fA is maximal.

In fact, since A/fA is noetherian, let $\mathfrak{p}_1, \ldots, \mathfrak{p}_k$ be the minimal prime ideals of A/fA. Assume that one of them, say $\mathfrak{p}_1$, is not maximal. Let $\mathfrak{m} \underset{\neq}{\supset} \mathfrak{p}_1$ be maximal. Since $\mathfrak{p}_j$ is minimal, we have $\mathfrak{m} \neq \mathfrak{p}_j$, $j = 2, \ldots, k$. If $\mathfrak{m} \subset \bigcup_{j=1}^{k} \mathfrak{p}_j$, then $\mathfrak{m} = \mathfrak{p}_j$ for some j, which we have just shown not to be the case. So $\mathfrak{m} \not\subset \bigcup_{j=1}^{k} \mathfrak{p}_j$ i.e. there exists $g' \in \mathfrak{m}$ such that $g' \notin \mathfrak{p}_j$, $j = 1, \ldots, k$. Let $g \in A$ such that $g' = g + fA$. Let $\mathfrak{q}$ be a minimal ideal of Ass(A/gA). By theorem 2.4 $\operatorname{Codim}(V(\mathfrak{q})) \leqq 1$, and clearly $\operatorname{Codim}(V(\mathfrak{q})) = 1$, since $\mathfrak{q} \neq (0)$ and A is an integral domain. Therefore $\mathfrak{q}$ is a minimal prime of A, hence $f \in \mathfrak{q}$ and $\mathfrak{q} \cdot A/fA$ is a minimal prime of A/fA, i.e. $\mathfrak{q} \cdot A/fA = \mathfrak{p}_j$ for some j. Clearly $g \in \mathfrak{q}$, hence $g' \in \mathfrak{p}_j$, is a contradiction. Therefore assertion (*) above is proved, and every non zero prime ideal of A is hence maximal. Furthermore the <u>only</u> prime ideals of A are (0) and the inverse images of $\mathfrak{p}_1, \ldots, \mathfrak{p}_k$. Hence A is semi-local and dim(A) = 1.

Proposition 2.6. Let A be a noetherian semi-local ring, M a finitely generated A-module, $x \in \mathfrak{r} = \mathrm{rad}(A)$. Then

$$\dim(M/xM) \geqq \dim(M) - 1$$

and equality holds if, and only if, x belongs to none of those minimal primes $\mathfrak{p} \in \mathrm{Ass}(M)$ such that $\dim(M) = \dim(A/\mathfrak{p})$.

Proof: By theorem 2.3 and lemma 2.1 we have

$$\dim(M/xM) = s(M/xM) \geqq s(M) - 1 = \dim(M) - 1.$$

Now assume that x belongs to none of those minimal primes $\mathfrak{p} \in \mathrm{Ass}(M)$ such that $\dim(M) = \dim(A/\mathfrak{p})$. Again by theorem 2.4 and lemma 2.1 we have

$$\dim(M/xM) \leqq \dim(M) - 1$$

whence equality holds. Conversely, assume that equality holds. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_k \in \mathrm{Ass}(M)$ such that $\dim(M) = \dim(A/\mathfrak{p}_j)$, $j = 1, \ldots, k$. Then clearly $\mathfrak{p}_j \notin \mathrm{Supp}(M/xM)$ (since, for any M, $\dim(M) = \dim(\mathrm{Supp}(M)) = \sup\limits_{\mathfrak{p} \in \mathrm{Supp}(M)} (\dim A/\mathfrak{p}) = \sup\limits_{\mathfrak{p} \in \mathrm{Ass}(M)} (\dim A/\mathfrak{p})$).

More quickly, since $\mathfrak{p}_j \in \mathrm{Supp}(M)$ and $\mathrm{Supp}(M/xM) = \mathrm{Supp}(M) \cap V(x)$, $x \notin \mathfrak{p}_j$. Q.E.D.

We define a notion extensively used in Algebraic Geometry.

Definition 2.5. Let A be a noetherian semi-local ring. A set of elements $x_1, \ldots, x_n \in \mathfrak{r}$ is called a system of parameters of the finitely generated A-module M if $n = \dim(M)$

and $M/x_1 M + \dots + x_n M$ has finite length.

Note that, by the remark preceding definition 2.5 and theorem 2.4 every A-module admits a system of parameters.

We prove

Proposition 2.7. Let A, M be as in the above definition. Let $x_1, \dots, x_k \in \mathfrak{m}$. Then

$$\dim(M/x_1 M + \dots + x_k M) \geqq n - k$$

and equality holds if, and only if, the system $x_1, \dots, x_k$ can be imbedded in a system of parameters of M.

Proof: We proceed by induction on k.

When $k = 1$ the inequality holds by Proposition 2.6. Furthermore equality holds if and only if x belongs to none of the primes $\mathfrak{p}$ in Ass(M) with $\dim(M) = \dim(A/\mathfrak{p})$. Let $x_1, \dots, x_{n-1} \in \mathfrak{m}$ such that $s(M/xM) = n - 1$, $(M/xM)/x_1(M/xM) + \dots + x_{n-1}(M/xM)$ has finite length. (See definition 2.5) Then $x, x_1, \dots, x_{n-1}$ is a system of parameters of M. Conversely, if x can be imbedded in a system of parameters, say $x, x_1, \dots, x_{n-1}$ then $s(M/xM) \leqq n - 1$ and, by Proposition 2.6, $\dim(M/xM) = n - 1$.

Q.E.D.

The equality

$$M/x_1 M + \dots + x_k M = (M/x_1 M + \dots + x_{k-1} M)/x_k(M/x_1 M + \dots + x_{k-1} M)$$

shows, by the induction assumption, the desired inequality. Assume now $\dim(M/x_1 M + \dots + x_k M) = n - k$. Then, letting $N = M/x_1 M$

$$\dim(N/x_2 N + \dots + x_k N) = (n - 1) - (k - 1)$$

and

$$(n-1) - (k-1) \geqq \dim(N) - (k-1) \geqq \dim(M) - 1 - (k-1) = n-k$$

whence $\dim(N) - k + 1 = n - k$ or $\dim(N) = n - 1$. By the induction assumption, $\{x_2, \ldots, x_k\}$ can be imbedded in a system of parameters of N, say $\{x_2, \ldots, x_k, x_{k+1}, \ldots, x_n\}$ (here we must use $\dim(N) = n - 1$). Then clearly $\{x_1, x_2, \ldots, x_n\}$ is a system of parameters of M.

Conversely, if $\{x_1, x_2, \ldots, x_k, x_{k+1}, \ldots, x_n\}$ is a system of parameters of M, let $N = M/x_1 M$. Then $N/x_2 N + \ldots + x_n N$ has finite length, whence $s(N) \leqq n - 1$. By Proposition 2.6 we have

$$n - 1 = \dim(M) - 1 \leqq \dim(N) = s(N) \leqq n - 1$$

whence $\dim(N) = n - 1$. Hence $\{x_2, \ldots, x_k, \ldots, x_n\}$ is a system of parameters of N, and, by the induction assumption

$$\dim(N/x_2 N + \ldots + x_k N) = (n - 1) - (k - 1) = n - k$$

The proposition is proved.

We finish this section with a few remarks about the nature of the function $\Psi: \mathrm{Spec}(A) \to \mathbb{N}$ given by

$$\Psi(\mathfrak{p}) = \dim(A_{\mathfrak{p}})$$

where A is any noetherian ring. It is obviously not continuous, otherwise it would have to be constant when Spec(A) is connected (e.g. when A is an integral domain), and trivial examples show this is not the case (say $A = k[X, Y]$).

We do nevertheless have some information, namely, by proposition 1.1,

$$\dim(A_{\mathfrak{p}}) \geqq \dim(A)$$

and

$$\dim(A/\mathfrak{p}) \leqq \dim(A).$$

The latter is geometrically interpreted as follows: If $x \in \overline{y}$, then $\dim(V(j_x)) \leqq \dim(V(j_y))$.

Dimension is a very coarse invariant, i.e. were we to consider the equivalence classes of affine varieties of a given dimension, we would obtain huge classes of highly non isomorphic varieties.

§3. DEPTH

The next numerical invariant we shall study in the notion of depth. We assume throughout this section that A is a noetherian local ring with maximal ideal $\mathfrak{m}$, and that M is a finitely generated A-module.

Definition 3.1. a) an element $x \in A$ is called M-regular if the homomorphism $\varphi: M \to M$ given by $\varphi(m) = xm$ is injective.

b) a sequence $\{x_1,\ldots,x_n\}$ of elements of A is called M-regular if x_i is $M/x_1 M + \ldots + x_{i-1} M$ regular, $1 \leqq i \leqq n$.

Remark. Clearly every $x \notin \mathfrak{m}$ being invertible is M-regular for every module M. Hence we shall confine our attention to those M-regular elements which belong to $\mathfrak{m}$. With regard to b) we state, without proof, the fact that the sequence $\{x_1,\ldots,x_n\}$ is M-regular if, and only if all sequences $\{x_{\sigma(1)},\ldots,x_{\sigma(n)}\}$ $\sigma \in S_n$ are M-regular, where S_n denotes the group of permutations on n symbols. (Grothendieck, E.G.A., Ch. O, §15.1, I.H.E.S. no 20) The above statement is false if A is not noetherian.