A Duality Operation in the Character Ring of a Finite Group of Lie Type

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INTRODUCTION

The subject of these lectures, while perhaps not a major theme in the representation theory of finite groups of Lie type, nevertheless cuts across the representation theory of these groups in interesting and sometimes unexpected ways. The main results reported on here are due to Alvis ([1], [2]), following an earlier paper [7] by the author. Some of Alvis's main results were obtained independently by Kawanaka, and a homological interpretation of the operation has been given by Deligne and Lusztig [10]

In order to describe the contents of this paper, we first require some terminology. Let H be a finite group, and let $\mathrm{ch}(\mathbb{C}H)$ denote the ring of complex valued virtual characters of H. The elements of $\mathrm{ch}(\mathbb{C}H)$ are the $\mathbb{Z}$-linear combinations of the elements of $\mathrm{Irr}\ H$, the set of irreducible characters afforded by the simple left $\mathbb{C}H$-modules. The operations of addition and multiplication of characters correspond to the operations of forming direct sums and tensor products of the corresponding $\mathbb{C}H$-modules. A <u>duality operation</u> in $ch(\mathbb{C}H)$ is a $\mathbb{Z}$-automorphism of $Ch(\mathbb{C}H)$ of order 2, which preserves the inner product

$$(f,g)_H = |H|^{-1} \sum_{x \in H} f(x)\overline{g(x)}$$

of class functions on H. Such an operation clearly permutes, up to sign, the elements of Irr H. A familiar example of a duality operation is the map $\zeta \to \overline{\zeta}$, $\zeta \in ch(\mathbb{C}H)$, where $\overline{\zeta}$ is the complex conjugate of ζ. This map corresponds to the operation of forming the contragredient module of a given $\mathbb{C}H$-module. Another example is given in § 1, for finite Coxeter groups, and consists of multiplying a character by the sign character. The duality operation described in § 2, for virtual characters of a finite group G of Lie type, defines a permutation of Irr G (up to sign), with corresponding characters not necessarily having the same degree. The degree of $\zeta \in$ Irr G and its dual $\zeta^* \in ch(\mathbb{C}G)$, with $\pm\zeta^* \in$ Irr G, always have the same p'-part, and hence differ only by a power of p, where p is the characteristic of the finite field associated with G.

In §§ 1-4, we have given a self-contained exposition, often with complete proofs, of the main results. In § 5, we survey without proof some other results, with references to the literature.

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1. DUALITY IN THE CHARACTER RING OF A FINITE COXETER GROUP.

Let (W,R) be a finite Coxeter group with distinguished generators $R = \{r_1,\dots,r_n\}$, with the presentation

$$W = \langle r_1,\dots,r_n : r_i^2 = (r_i r_j)^{n_{ij}} = 1,\ 1 \le i,\ j \le n\rangle.$$

Then the map $\varepsilon : R \to \mathbb{C}^*$ defined by $\varepsilon(r_i) = -1,\ 1 \le i \le n$, preserves the defining relations of W, and therefore can be extended to a homomorphism

$$\varepsilon : W \to \mathbb{C}^*$$

which we shall call the <u>sign representation</u> of W. For example, the symmetric group S_{n+1} is a Coxeter group with generators $r_1 = (12),\dots,r_n = (n,n+1)$ and defining relations $(r_i r_j)^2 = 1$ if $|i - j| > 1$, and $(r_i r_{i+1})^3 = 1,\ 1 \le i \le n$. In this case $\varepsilon(\sigma)$, for $\sigma \in S_n$, is the usual signature of a permutation, and is 1 if σ is even, -1 if σ is odd.

It is easily checked that the map $\mu \to \varepsilon\mu$, for $\mu \in ch(\mathbb{C}W)$, is a duality operation in $ch(\mathbb{C}W)$, and it has traditionally been used to organize the character tables of the Coxeter groups.

Our first aim is to give a geometric interpretation of ε. It is a standard result (see [3]) that W can be identified with a finite group generated by reflections on a real Euclidean space E of dimension $n = |R|$. Each reflection $r \in W$ leaves a hyperplane H pointwise fixed, and if α is a vector orthogonal to H, then

$$r_\alpha(\xi) = \xi - 2\frac{(\xi,\alpha)}{(\alpha,\alpha)}\,\alpha,\ \xi \in E,$$

where $(\ ,\ )$ is the positive definite scalar product on E. The vectors $\{\pm\alpha\}$ orthogonal to the hyperplanes fixed by the reflections in W, with their lengths suitably normalized, are permuted by the elements of W, and are called the <u>root system</u> Φ associated with W. There exists a set Π of roots $\{\alpha_1,\ldots,\alpha_n\}$, with the properties that $\{\alpha_1,\ldots,\alpha_n\}$ form a basis of E over $\mathbb{R}$, and every root α can be expressed in the form $\alpha = \sum_{i=1}^{n} c_i\alpha_i$, where the coefficients $\{c_i\}$ are either all non-negative or all non-positive. Such a set of roots is called a <u>fundamental system</u>, and there exists a fundamental system Π such that the distinguished generators $\{r_i\}_{1 \le i \le n}$ of W coincide with the reflections $\{r_{\alpha_i}\}_{\alpha_i} \in \Pi$, where r_{α_i} is the reflection defined as above, fixing the hyperplane orthogonal to α_i.

The subsets $J \subseteq R$ all define subgroups $W_J = \langle J \rangle = \langle r_j : r_j \in J \rangle$, which are themselves Coxeter groups, and are called parabolic subgroups of W.

In this set-up, the sign representation is defined by $\varepsilon(w) = \det w$, the determinant of $w \in W$ on the vector space E. We are interested in one more formula for ε, which arises from the action of W on the unit sphere S in the Euclidean space E. We shall define a natural triangulation of S defined in terms of the root system. The action of W on the simplicial complex defining the triangulation yields a representation of W on the rational homology $H_*(S)$, and we have:

(1.1) PROPOSITION. Let (W,R) be a finite Coxeter group with $n = |R| > 2$. Let Γ be the abstract simplicial complex whose simplices are the cosets $\{wW_J : w \in W, J \subsetneq R\}$ of all proper parabolic subgroups of W, with order relation (defining the faces of a simplex) given by the opposite of inclusion. Then:

(i) the geometric realization $|\Gamma|$ of Γ is homeomorphic to the unit sphere S in E.

(ii) the rational homology $H_*(\Gamma) = \oplus_{i=o}^{\infty} H_i(\Gamma)$ is given by: $H_i(\Gamma) = 0$ except in dimensions 0 and $n-1$ and $H_0(\Gamma) \cong H_{n-1}(\Gamma) \cong \mathbb{Q}$ as rational vector spaces.

(iii) the rational homology group $H_0(\Gamma)$ affords the trivial representation of W, and the homology group $H_{n-1}(\Gamma)$ affords the sign representation ε.

We shall give a sketch of the proof. For more details see Carter [4] or Bourbaki [3]. Let $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ be the fundamental system of roots such that the reflections $\{r_{\alpha_i}\}_{1 \le i \le n}$ coincide with R, and let $C = \{\xi \in E : (\xi, \alpha_i) > 0$ for all $\alpha_i \in \Pi\}$. Then C is an open simplicial cone called the <u>fundamental chamber</u> of W acting on E. The <u>walls</u> of C are the subsets $C_J \subset \overline{C}$, where $\overline{C}$ is the closure of C, and J ranges through subsets of Π, defined by

$$C_J = \left\{ \xi \in E : \begin{cases} (\xi, \alpha_i) = 0, & \alpha_i \in J, \\ (\xi, \alpha_i) > 0, & \alpha_i \in \Pi - J \end{cases} \right\}.$$

It can be proved that for each subset $J \subset \Pi$, the stabilizer of C_J in W is the parabolic subgroup W_J of W, generated by the reflections $\{r_{\alpha_j} : \alpha_j \in J\}$. For a fixed $J \subset R$, the W-translates $\{wC_J\}$ are in bijective correspondence with the cosets $\{wW_J\}$. Moreover, each vector $\xi \neq 0$ in V belongs to the interior of a unique W-translate wC_J, $J \subsetneq R$, of the chamber C or one of its faces. It follows that the intersections of all the translates $\{wC_J : w \in W, J \subsetneq R\}$ with the unit sphere S in E, defines a simplical subdivision of the sphere S. It is also clear that a simplex $wC_J \cap S$ on S is a face of $w'C_{J'} \cap S$ if and only if $wW_J \supseteq w'W_{J'}$, proving that S is the geometric realization of the simplicial complex Γ defined in the statement of the proposition, and proving the first statement. The second statement is immediate, from the properties of the rational homology of a sphere.

For the third statement, it is easily checked that $H_o(\Gamma)$ affords the trivial representation 1_W. In order to prove that $H_{n-1}(\Gamma)$ affords ε, we consider the Lefschetz character Λ of the homology representation of W on $H_*(\Gamma)$; then Λ is the virtual character of W given by

$$\Lambda = \sum_{i=o}^{n-1} (-1)^i \mathrm{Tr}(\cdot, H_i(\Gamma)).$$

For the action of a finite group (in this case W) on a finite simplicial compex Γ, it is well known that for each $w \in W$,

$$\Lambda(w) = \chi(|\Gamma|_w),$$

where $\chi(|\Gamma|_w)$ is the Euler characteristic of the fixed point set $|\Gamma|_w$ of w acting on the geometric realization $|\Gamma|$ of Γ. For a reflection r, the fixed point set of r is a sphere of one lower dimension than S, and we have

$$\Lambda(r) = 1 + (-1)^{n-2}.$$

On the other hand, by definition of Λ and the facts that $H_i(\Gamma) = 0$ except in dimensions 0 and $n-1$, and that r acts trivially on $H_o(\Gamma)$ we also have

$$\Lambda(r) = 1 + (-1)^{n-1}\mathrm{Tr}(r,H_{n-1}(\Gamma)).$$

Combining these results, we have $\mathrm{Tr}(r,H_{n-1}(\Gamma)) = -1$ for each reflection r, and hence $\mathrm{Tr}(r,H_{n-1}(\Gamma)) = \varepsilon$, completing the proof of the proposition.

As a consequence, we obtain:

(1.2) COROLLARY (L. Solomon [16]). The sign representation ε of W is given by

$$\varepsilon = \sum_{J\subseteq R} (-1)^{|J|} 1_{W_J}{}^{W},$$

where $1_{W_J}{}^{W}$ is the induced permutation representation of W on the left cosets of W_J, for $J \subseteq R$.

For the proof of the corollary, we first note that for $J \subsetneq R$, $C_J \cap S$ is an i-simplex if and only if $|\Pi - J| - 1 = i$; for example the vertices of the simplicial subdivision correspond to the faces C_J, with J a maximal subset of R, and their W-translates. We then have, for $w \in W$, since W_J is the stabilizer of C_J for each J,

$$\begin{aligned} Tr(w, C_i(\Gamma)) &= \#i \text{ simplices fixed by } \Gamma \\ &= \sum_{|\Pi - J| - 1 = i} 1_{W_J}^{W}(w), \end{aligned}$$

where $C_i(\Gamma)$ is the i^{th} chain group of Γ, with a basis consisting of the i-simplices $\{xC_J : x \in W, |\Pi - J| - 1 = i\}$. We then apply the Hopf trace formula to the Lefschetz character of the homology representation of W on $H_*(\Gamma)$, and obtain

$$\begin{aligned} \Lambda(w) &= \sum_{i=o}^{n-1} (-1)^i \, Tr(w, C_i(\Gamma)) \\ &= \sum_{J \subsetneq R} (-1)^{n-|J|-1} \, 1_{W_J}^{W}(w). \end{aligned}$$

On the other hand, from what has already been shown in part (iii) of the Proposition,

$$\Lambda(w) = 1 + (-1)^{n-1} \varepsilon(w), \quad w \in W.$$

Combining these results, we obtain Solomon's formula.

A final corollary gives an expression of the duality operation $\mu \to \varepsilon\mu$, for virtual characters $\mu \in ch(\mathbb{C}W)$, as an alternating sum of induced characters, which will point the way toward the duality operation we shall define for characters of finite groups of Lie type.

(1.3) COROLLARY. For each virtual character $\mu \in ch(\mathbb{C}W)$, we have

$$\varepsilon\mu = \sum_{J \subseteq R} (-1)^{|J|} (\mu|_{W_J})^W,$$

where $\mu|_{W_J}$ is the restriction of $\mu \in ch(\mathbb{C}W)$ to W_J.

By Solomon's formula we have

$$\varepsilon = \Sigma(-1)^{|J|} 1_{W_J}{}^W.$$

Multiplying by μ, we have

$$\mu\varepsilon = \Sigma(-1)^{|J|} \mu(1_{W_J}{}^W).$$

A well-known identity for induced characters then gives

$$\mu(1_{W_J}{}^W) = (\mu|_{W_J})^W,$$

completing the proof.

2. TRUNCATION AND DUALITY IN THE CHARACTER RING OF A FINITE GROUP OF LIE TYPE.

In this section, G denotes a finite group of Lie type. Such a group may be described in various ways. For example, we may assume G is a Chevalley group over a finite field $\mathbb{F}_q$ (or a twisted type of Chevalley group) as in Carter [4] or Steinberg [18]. Alternatively, let $\underline{\underline{G}}$ be a connected reductive affine algebraic group over an algebraically closed field K of characteristic

$p > 0$, with a rational structure defined by a surjective endomorphism $\sigma : \underline{\underline{G}} \to \underline{\underline{G}}$ such that the subgroup $\underline{\underline{G}}_\sigma$, of elements of $\underline{\underline{G}}$ fixed by σ, is finite. Then $G = \underline{\underline{G}}_\sigma$ is a finite group of Lie type, and the two descriptions are essentially equivalent (see Steinberg [19]).

As examples to keep in mind, let $\underline{\underline{G}} = GL_n(K)$, and let $q = p^a$, $a > 0$, where p is the characteristic of K. Then $\underline{\underline{G}}$ has two rational structures, defined by the maps

$$\sigma_1 : (x_{ij}) \to (x_{ij}{}^q) \quad \text{and} \quad \sigma_2 : (x_{ij}) \to {}^t(x_{ij}{}^q)^{-1}.$$

In the first case, the resulting finite group of Lie type is the untwisted group $GL_n(\mathbb{F}_q)$, while in the second case, $\underline{\underline{G}}_{\sigma_2}$ is the unitary group $U_n(\mathbb{F}_{q^2})$, a twisted type of Chevalley group.

In this section and the next, the properties of G we shall require can be summarized in the statement that G has a split (B,N)-pair of characteristic $p > 0$, for a fixed prime p, with Weyl group W, and satisfies the Chevalley commutator relations (see Richen [14] or Curtis [6]). Thus G has a (B,N) pair with Borel subgroup B, and a subgroup N, satisfying the usual axioms. In particular, we have:

(i) $G = \langle B,N \rangle$;

(ii) If $H = B \cap N$, then $H \trianglelefteq N$, and the group

$W = N/H$ (the Weyl group of G) is a finite group generated by a set of involutions $R = \{r_1,\ldots,r_n\}$, satisfying the additional conditions:

(iii) $r_i Bw \subseteq BwB \cup Br_i wB$ (where we use the notation Bw, for $w \in W = N/B \cap N$, to denote Bn, where $n \in N$ corresponds to w under the natural map $N \to N/H$), and

(iv) $r_i B r_i \neq B$ for all $r_i \in R$.

From the axioms, it follows that (W,R) is a Coxeter system, with distinguished generators R, and we have the Bruhat decomposition of G:

$$G = \bigcup_{w \in W} BwB,$$

with $w \to BWB$ a bijection from W to the double cosets $B\backslash G/B$.

The split (B,N)-pair in G is defined by a splitting of B as a semidirect product:

$$B = UH, \quad U \trianglelefteq B, \quad U \cap H = \{1\},$$

with $U = O_p(B)$, the unique maximal normal p-subgroup of B, and H an abelian p'-group.

For example, in the case of $GL_n(\mathbb{F}_q)$, a split (B,N)-pair is given as follows:

$$B = \left\{\begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix}\right\} \qquad \text{(upper-triangular matrices)},$$

N = {monomial matrices, with exactly one nonzero entry in each row and column},

$H = B \cap N$ = {diagonal matrices},

$$U = \left\{ \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \right\} = \text{\{unit upper-triangular matrices\}}.$$

Then the Weyl group $W = N/H$, in this case, is isomorphic to the symmetric group S_n, and defines a split (B,N)-pair of rank $n-1$.

The <u>standard parabolic subgroups</u> of G (in the general case) are the subgroups containing B, and are defined in terms of the Bruhat decomposition by

$$P_J = BW_JB = \bigcup_{w \in W_J} BwB,$$

where W_J is a parabolic subgroup of W, and J ranges over the subsets of R. It follows from the (B,N)-properties and the commutator formulas (see Curtis [6]) that each standard parabolic subgroup has a decomposition as a semidirect product (called a <u>Levi decomposition</u>)

$$P_J = L_JV_J, \quad \text{with} \quad V_J = O_p(P_J) \trianglelefteq P_J,$$

where L_J, called the <u>Levi factor</u>, is a group with a split (B,N)-pair with Weyl group (W_J,J). The group V_J is often called the <u>unipotent radical</u> of P_J, and is the group of $\mathbb{F}_q$-rational points on the unipotent radical $R_u(\underline{P}_J)$ of a parabolic subgroup $\underline{P}_J$ defined over $\mathbb{F}_q$, in case G arises from an algebraic group $\underline{G}$ as described earlier.

In case $P_J = B$, the Borel subgroup, with $J = \phi$, the Levi decomposition is given by

$$B = UH, \quad \text{with} \quad V_\phi = U, \; L_\phi = H,$$

and is part of the definition of split (B,N)-pair.

The <u>parabolic subgroups</u> of G are the standard parabolic subgroups $\{P_J\}_{J \subseteq R}$, and their conjugates by elements of G. In the case of $GL_n(\mathbb{F}_q)$, they are the stabilizers of all flags in the underlying vector space on which $GL_n(\mathbb{F}_q)$ acts, where a flag is a chain of subspaces

$$W_1 \subset W_2 \subset \dots \subset W_s, \; s \geq 1.$$

Thus the Borel subgroups are conjugates of the stabilizers of complete flags (with $\dim W_i = i$, $1 \leq i \leq s$ and $s = n - 1$).

The definition of the duality operation involves restriction and induction from parabolic subgroups P_J, $J \subseteq R$. The usual definitions, however, have to be modified to take into account the unipotent radicals V_J, so that we really are setting up relationships between characters of G and characters of the Levi subgroups

L_J of P_J, for $J \subseteq R$. The process can be described in a general way as follows. Let G be a finite, H a subgroup of G, and V a normal subgroup of H, so that $V \trianglelefteq H \leq G$. Let X be a left $\mathbb{C}G$-module with character ξ. Then the restriction $\xi|_H = \mathrm{res}_H^G \xi$ is a sum of two characters of H,

$$\xi|_H = \xi_H' + \xi_H'',$$

where ξ_H' is the contribution to $\xi|_H$ from irreducible characters of H having V in their kernels, and ξ_H'' is the contribution from those that do not. Note that ξ_H' can be viewed as a character of H/V. On the other hand, let $\mathrm{inv}_V(X)$ be the subspace of X affording the trivial representation of V; then $\mathrm{inv}_V(X)$ is a $\mathbb{C}H$-module since $V \trianglelefteq H$, and, in fact, is a $\mathbb{C}(H/V)$-module since V acts trivially on $\mathrm{inv}_V(X)$. Let $\xi_{(H/V)}$ be the character of H/V afforded by $\mathrm{inv}_V(X)$. Finally, for each character λ of H/V, let $\tilde{\lambda}$ denote the character of H with V in its kernel defined by the natural map $H \to H/V$. These ideas are connected by the following result, whose proof is left as an exercise.

(2.1) PROPOSITION. Keeping the above notations, the following characters of H/V coincide:

$$\xi_H' = \xi_{(H/V)} = (\xi|_H)_{(H/V)} = \sum_{\lambda \in \mathrm{Irr}(H/V)} (\xi, \tilde{\lambda}^G)\lambda.$$

An easy lemma from the Feit-Thompson paper ([11], p. 783) implies that for an element $x \in H$ such that $C_G(x) \cap V = \{1\}$, we have

$$\xi(x) = \xi_{(H/V)}(x),$$

which is a kind of reduction theorem for character values on certain elements in terms of characters of the smaller group H/V.

From this point on, we shall often be following the approach of Alvis [1] (see also Curtis [7]).

(2.2) DEFINITION. Let G be a finite group of Lie type, with Coxeter system (W,R), and let $J \subseteq R$. The operation of <u>truncation</u> $T_J : ch(\mathbb{C}G) \to ch(\mathbb{C}L_J)$ is a homomorphism of $\mathbb{Z}$-modules, which assigns to each virtual character $\mu \in ch(\mathbb{C}G)$ the virtual character $T_J\mu \in ch(\mathbb{C}L_J)$, where

$$T_J\mu(\ell) = |V_J|^{-1} \sum_{v \in V_J} \mu(v\ell), \quad \ell \in L_J.$$

The fact that $T_J\mu$ is in fact a virtual character of L_J, and connections with Proposition 2.1, are given by:

(2.3) PROPOSITION. Let $\mu \in ch(\mathbb{C}G)$ be a character afforded by some $\mathbb{C}G$-module M. Then

$$T_J\mu = \mu_{(P_J/V_J)} = \sum_{\lambda \in \mathrm{Irr}\, L_J} (\tilde{\lambda}^G, \mu)\lambda = (\mu|_{P_J})_{(P_J/V_J)}.$$

For the proof, let $e_J = |V_J|^{-1}\Sigma_{v\in V_J} v$. Then $e_J M = \mathrm{inv}_{V_J} M$, and for $\ell \in L_J$, $\ell e_J = e_J \ell$ since L_J normalizes V_J. Then

$$\mu(e_J \ell) = \mathrm{Tr}(\ell, e_J M),$$

and hence $T_J \mu = \mu_{(P_J/V_J)}$.

Corresponding to the operation of truncation, we have a second operation, which is a homomorphism of $\mathbb{Z}$-modules

$$I_J : \mathrm{ch}(\mathbb{C}L_J) \to \mathrm{ch}(\mathbb{C}G),$$

given by $I_J \lambda = \tilde{\lambda}^G$, for each character λ of L_J, where $\tilde{\lambda}$ is the lift of λ from $L_J \cong P_J/V_J$ to P_J defined previously. The operations I_J and T_J are adjoint with respect to the scalar product of characters (see (2.5)).

REMARK. These ideas can be used to describe Harish-Chandra's organization of the character theory of G. An irreducible character $\zeta \in \mathrm{Irr}\ G$ is <u>cuspidal</u> (or <u>discrete series</u>) if $T_J \zeta = 0$ for all $J \subsetneq R$. It is an easy exercise to show that ζ is cuspidal if and only if $(\zeta, I_J \lambda) = 0$ for all $J \subsetneq R$ and all $\lambda \in \mathrm{Irr}\ L_J$; in other words ζ is missed by the process of lifting and induction from all characters of Levi factors of proper parabolic subgroups of G. A basic result (which we do not prove here) asserts that every character $\zeta \in \mathrm{Irr}(G)$ is either cuspidal, or is a constituent of $I_J\lambda$, for some irreducible and cuspidal character λ of L_J, for $J \subsetneq R$, where λ and the Levi factor L_J, are uniquely determined (by ζ) up to conjugacy. Thus the main problems in the character theory of G are to find all cuspidal characters, and to decompose the induced characters $I_J\lambda$ from cuspidal irreducible characters of L_J.

Returning to our main theme, we have:

(2.4) DEFINITION. The <u>duality operation</u> is a $\mathbb{Z}$-endomorphism $\zeta \to \zeta^*$ of $\mathrm{ch}(\mathbb{C}G)$ defined by

$$\zeta^* = \sum_{J \subseteq R} (-1)^{|J|} I_J T_J \zeta, \quad \zeta \in \mathrm{ch}(\mathbb{C}G).$$

REMARKS AND EXAMPLES. i) Note the parallel between this definition and the duality operation in $\mathrm{ch}(\mathbb{C}W)$ (see Corollary 1.3). The fact that $\zeta \to \zeta^*$ is a duality operation in $\mathrm{ch}(\mathbb{C}G)$ will be proved in § 3.

ii) The dual of the trivial character is the Steinberg character of G, $1_G{}^* = \mathrm{St}_G$, where $\mathrm{St}_G \in \mathrm{Irr}(G)$ is a character of degree $|G|_p$, the p-part of the order of G.

iii) If $\zeta \in \mathrm{Irr}\ G$ is cuspidal, then $\zeta^* = \pm\zeta$, since all the truncations $T_J\zeta$, for $J \subsetneq R$, are zero.

We next derive some properties of truncation and induction, which will be needed to prove the main results about the duality operation.

For subsets $K \subseteq J \subseteq R$, let

$$P_{J,K} = L_J \cap P_K;$$

these are the standard parabolic subgroups of L_J, containing the Borel subgroup $B_J = B \cap L_J$. Their Levi decompositions are:

$$P_{J,K} = L_K \cdot V_{J,K},\ V_{J,K} = L_J \cap V_K,$$

and we have

$$V_K = V_{J,K}V_J \quad \text{(semidirect, with } V_J \trianglelefteq V_K).$$

Then we have additive maps, for $K \subseteq J \subseteq R$,

$$T_K^J : ch(\mathbb{C}L_J) \to ch(\mathbb{C}L_K),\ I_K^J : ch(\mathbb{C}L_K) \to ch(\mathbb{C}L_J),$$

defined as above.

The following result is easily proved, and is left as an exercise:

(2.5) PROPOSITION. Let $K \subseteq J \subseteq R$. Then we have:

(i) (Transitivity) $I_K = I_J I_K^J$, and $T_K = T_K^J T_J$;

(ii) (Frobenius reciprocity) For $\xi \in ch(\mathbb{C}L_J)$, $\eta \in ch(\mathbb{C}L_K)$, $(\xi, I_K^J\eta)_{L_J} = (T_K^J\xi, \eta)L_K$.

For the next result, let $J, J' \subseteq R$, and let $D_{JJ'}$

be the distinguished double coset representatives for $W_J\backslash W/W_{J'}$. For a subgroup $H \le G$ and $\lambda \in ch(\mathbb{C}H)$, and $x \in G$, we define the conjugate virtual character ${}^x\lambda \in ch(\mathbb{C}({}^xH))$ by ${}^x\lambda({}^xh) = \lambda(h)$, $h \in H$, where ${}^xh = xhx^{-1}$. We then have:

(2.6) PROPOSITION (Intertwining Number Theorem). Let $J,J' \subseteq R$, and let $\lambda \in ch(\mathbb{C}L_J)$, $\lambda' \in ch(\mathbb{C}L_{J'})$. Then

$$(I_J\lambda, I_{J'}\lambda')_G = \sum_{d\in D_{JJ'}} (T_K^J\lambda, {}^dT_{K'}^{J'}\lambda')_{L_K},$$

where $K = J \cap {}^dJ' = {}^dK'$, so $L_K = {}^dL_{K'}$.

For the proof, we first apply the usual intertwining number theorem, and obtain

$$(I_J\lambda, I_{J'}\lambda') = \sum_{d\in D_{JJ'}} (\tilde{\lambda}|_{P_J \cap {}^dP_{J'}}, {}^d\tilde{\lambda}'|_{P_J \cap {}^dP_{J'}})_{P_J \cap {}^dP_{J'}}.$$

We have a factorization of $P_J \cap {}^dP_{J'}$ (see Curtis [6], § 2),

$$P_J \cap {}^dP_{J'} = L_K(L_J \cap {}^dV_{J'})(V_J \cap {}^dL_{J'})(V_J \cap {}^dV_{J'}),$$

with uniqueness of expression. Then

$$(\tilde{\lambda}|_{P_J \cap {}^dP_{J'}}, {}^d\tilde{\lambda}'|_{P_J \cap {}^dP_{J'}}) =$$

$$|P_J \cap {}^dP_{J'}|^{-1} \sum_{\ell,v,y,z} \tilde{\lambda}(\ell v {}^dyz)\overline{{}^d\tilde{\lambda}'(\ell v {}^dyz)},$$

where $v \in V_{J,K}$, $y \in V_{J',K'}$, so ${}^dyz \in V_J$, $vz \in {}^dV_{J'}$, and the expression becomes

$$|L_K|^{-1}|V_{JK}|^{-1}|V_{J'K'}|^{-1} \Sigma \ \tilde{\lambda}(\ell v)^{d}\tilde{\lambda}'(\ell^{d}y),$$

completing the proof.

3. MAIN THEOREMS ON DUALITY.

In this section, G denotes a finite group of Lie type as in § 2.

(3.1) THEOREM. Let $\zeta \in ch(\mathbb{C}G)$ and $J \subseteq R$. Then

$$T_J(\zeta^*) = (T_J\zeta)^*,$$

where $(T_J\zeta)^*$ is the dual of $T_J\zeta$ in $ch(\mathbb{C}L_J)$. (In other words the operation of truncation intertwines the duality operation.)

We give a sketch of the proof. For more details see Curtis [7]. We have to prove that

$$\sum_{J'\subseteq R} (-1)^{|J'|} T_J I_{J'} T_{J'} \zeta = \sum_{K\subseteq J} (-1)^{|K|} I_K^J T_K^J T_J \zeta .$$

A typical term on the left hand side is

$$\begin{aligned} T_J(I_{J'}T_{J'}\zeta) &= T_J(I_{J'}T_{J'}\zeta|_{P_J}) \\ &= T_J\Big(\sum_{d\in D_{JJ'}} {}^{d}(T_{J'}\zeta)_{{}^{d}P_{J'} \cap P_J}^{\ P_J}\Big) \end{aligned}$$

by Mackey's sugbroup Theorem. We then have

(3.2) PROPOSITION. $T_J({}^{d}(T_{J'}\zeta)_{{}^{d}P_{J'} \cap P_J}^{\ P_J}) = I_K^J T_K^J T_J \zeta,$

for all $J,J' \subseteq R$, $d \in D_{JJ'}$, where $K = {}^{d}J' \cap J$.

This result is Prop. 2.1 of Curtis [7], and will not be proved here.

Applying (3.2) to the left hand side, we obtain

$$\begin{aligned} T_J(\zeta^*) &= \sum_{J'\subseteq R} (-1)^{|J'|}(T_J(I_{J'},T_{J'},\zeta)|_{P_J}) \\ &= \sum_{J'\subseteq R} (-1)^{|J'|} \sum_{d\in D_{JJ'}} T_J|{}^{d}T_{J'},\zeta|_{{}^{d}P_{J'}\cap P_J}^{P_J}) \\ &= \sum_{J'\subseteq R} (-1)^{|J'|} \sum_{\substack{d\in D_{JJ'} \\ {}^{d}J'\cap J=K}} I_K^J T_K^J T_J\zeta \\ &= \sum_{K\subseteq J} \Big(\sum_{J'\subseteq R} (-1)^{|J'|} a_{J'JK}\Big) I_K^J T_K \zeta \quad \text{(by (2.5))}, \end{aligned}$$

where

$$a_{J'JK} = \mathrm{card}\{d \in D_{JJ'} : {}^{d}J' \cap J = K\}.$$

The proof is completed using the following result.

(3.3) PROPOSITION. Let $a_{J'JK}$ be defined as above, for $J',J,K \subseteq R$. Then

$$\sum_{J'\subseteq R} (-1)^{|J'|} a_{J'JK} = (-1)^{|K|}.$$

For a proof, see Curtis ([7], Lemma 2.5).

As an immediate consequence of Theorem 3.1, we have:

(3.4) COROLLARY. For all $J \subseteq R$, $T_J St_G = St_{L_J}$, where St_G and St_{L_J} are the Steinberg characters of G and L_J, respectively.

(3.5) THEOREM. (Alvis [2]) The duality map $\zeta \to \zeta^*$ is a self-adjoint isometry of order 2, in $ch(\mathbb{C}G)$. Thus

$$(\zeta^*,\eta) = (\zeta,\eta^*) \quad \text{and} \quad \zeta^{**} = \zeta,$$

for all $\zeta,\eta \in ch(\mathbb{C}G)$.

To begin the proof, we recall (prop. 2.3) that for all $J \subseteq R$, and $\zeta \in ch(\mathbb{C}G)$,

$$T_J\zeta = \sum_{\varphi\in \mathrm{Irr}\, L_J} (\zeta,\tilde{\varphi}^G)\varphi = \sum_{\varphi\in \mathrm{Irr}\, L_J} (\zeta,I_J\varphi)\varphi,$$

and by Theorem 3.1, we also have

$$T_J(\zeta^*) = (T_J\zeta)^*.$$

Let $\zeta,\eta \in ch(\mathbb{C}G)$. Then

$$\begin{aligned}(\zeta^*,\eta) &= \sum_{J\subseteq R}(-1)^{|J|}(I_JT_J\zeta,\eta)\\ &= \sum_{J\subseteq R}(-1)^{|J|}(I_J\sum_{\varphi\in \mathrm{Irr}\, L_J}(\zeta,I_J\varphi)\varphi,\eta)\\ &= \sum_{J\subseteq R}(-1)^{|J|}\sum_{\varphi\in \mathrm{Irr}\, L_J}(\zeta,I_J\varphi)(I_J\varphi,\eta)\\ &= (\zeta,\eta^*)\end{aligned}$$

by symmetry.

We now have, for $\zeta \in ch(\mathbb{C}G)$

$$\zeta^{**} = \sum_{J\subseteq R}(-1)^{|J|}I_J(T_J(\zeta^*))$$

$$= \sum_{J \subseteq R} (-1)^{|J|} I_J (T_J \zeta)^* \quad \text{(by (3.1))}$$

$$= \sum_{J \subseteq R} (-1)^{|J|} \sum_{K \subseteq J} (-1)^{|K|} I_J I_K^J T_K^J T_J \zeta$$

$$= \sum_{J \subseteq R} (-1)^{|J|} \sum_{K \subseteq J} (-1)^{|K|} I_K T_K \zeta \quad \text{(by (2.5))}$$

$$= \sum_{K \subseteq R} (-1)^{|K|} \Big(\sum_{J \subseteq K} (-1)^{|J|} \Big) I_K T_K \zeta$$

$$= (-1)^{|R|} (-1)^{|R|} \zeta = \zeta,$$

since $\sum_{J \supseteq K} (-1)^{|J|} = 0$ if $K \neq R$.

Finally, to prove that $\zeta \to \zeta^*$ is an isometry, we have, for $\zeta, \eta \in \mathrm{ch}(\mathbb{C}G)$

$$(\zeta^*, \eta^*) = (\zeta, \eta^{**}) = (\zeta, \eta),$$

completing the proof.

(3.6) COROLLARY. (Alvis) The duality operation permutes, up to sign, the irreducible characters of G.

Let $\zeta \in \mathrm{Irr}\, G$. Then $(\zeta,\zeta) = 1$, so ζ^* is a virtual character of G such that $(\zeta^*,\zeta^*) = 1$, by Theorem 3.5. Thus $\pm\zeta^* \in \mathrm{Irr}\, G$, as required

The next result shows how the duality operation interacts with Harish-Chandra's "philosophy of cusp forms", discussed in § 2.

(3.7) COROLLARY. Let φ be a cuspidal irreducible character of L_J, for some $J \subseteq R$ (so $T_K^J\varphi = 0$ for all $K \subsetneq J$). Then

$$(I_J\varphi)^* = (-1)^{|J|} I_J\varphi .$$

Moreover, for $\zeta \in \mathrm{Irr}(G)$,

$$(\zeta, I_J\varphi) \neq 0 \implies (\zeta^*, I_J\varphi) \neq 0,$$

so $*$ permutes the irreducible components of $I_J\varphi$, for φ as above.

We have, for $\zeta \in \mathrm{Irr}\, G$,

$$\begin{aligned}((I_J\varphi)^*, \zeta) &= (I_J\varphi, \zeta^*)_G = (\varphi, T_J(\zeta^*))_{L_J} \quad \text{(by (2.5))} \\ &= (\varphi, (T_J\zeta)^*)_{L_J} \quad \text{(by (3.1))} \\ &= (\varphi^*, T_J\zeta)_{L_J} \quad \text{(by (3.5))} \\ &= (-1)^{|J|}(\varphi, T_J\zeta)_{L_J} = (-1)^{|J|}(I_J\varphi, \zeta) \quad \text{(by (2.5))},\end{aligned}$$

using also the fact that if φ is cuspidal in $\mathrm{Irr}(L_J)$, then

$$\varphi^* = (-1)^{|J|}\varphi .$$

This completes the proof of the corollary.

4. APPLICATIONS TO SPRINGER'S THEOREM AND A CONJECTURE OF MACDONALD.

All the results in this section are due to Alvis [1]. Let G be a finite group of Lie type, and p the prime associated with G (i.e. the characteristic of the field, for a finite Chevalley group). Let V be the set of all p-elements in G. (Note that in case $G = \underline{\underline{G}}_\sigma$ for a connected reductive affine algebraic group $\underline{\underline{G}}$, as in § 2, then V is the set of all unipotent elements in G.) We shall call V the <u>unipotent set</u> in G.

It is necessary to extend the operations T_J, I_J, $\zeta \to \zeta^*$, etc. to the vector space $cf_{\mathbb{C}}G$ of complex valued class functions on G; then the usual results continue to hold, by linearity, since $\operatorname{Irr} G$ is a basis for $cf_{\mathbb{C}}G$.

The first main result is a variation by Alvis of a theorem of Springer [17].

(4.1) THEOREM. Let ρ be the regular character of G, and χ_V the characteristic function on V (i.e. the function which is 1 on V and 0 on the complement of V). Then

$$\rho^* = |G|_{p'}\chi_V,$$

where $|G|_{p'}$ is the p'-part of the order of G.

We first require:

(4.2) LEMMA. For all $J \subseteq R$, $\widetilde{T_J\chi_V} = \chi_V|_{P_J}$

We first note that for a class function ξ on L_J, $\tilde{\xi}$ is the class function on P_J defined by $\tilde{\xi}(\ell u) = \xi(\ell)$, $\ell \in L_J$, $u \in V_J$. We have to show that for $\ell \in L_J$, $u \in V_J$,

$$\widetilde{T_J\chi_V}(\ell u) = T_J\chi_V(\ell) = \chi_V(\ell u),$$

which is the same as

$$|V_J|^{-1} \sum_{u \in V_J} \chi_V(gu) = \chi_V(g),$$

for $g \in P_J$. On the right hand side, we get 1 if g is a p-element, and zero otherwise. On the left side, it suffices to show that for $u \in V_J$, $g \in P_J$, gu is a p-element if and only if g is a p-element. This is easily checked, and the Lemma follows.

For the proof of the Theorem, we also require the facts that $St_G(1) = |G|_p$, and hence that St_G vanishes on p-irregular elements (see Curtis [51]). Then we have

$$\begin{aligned} |G|_{p'}^{-1}\rho &= \chi_V \cdot St_G \quad \text{(by direct verification)} \\ &= \Sigma(-1)^{|J|}\chi_V \cdot 1_{P_J}{}^G \quad (\text{since } St_G = 1_G{}^*) \\ &= \Sigma(-1)^{|J|}(\chi_V|_{P_J})^G \end{aligned}$$

$$= \Sigma(-1)^{|J|} I_J T_J \chi_V = \chi_V^*,$$

completing the proof.

As a more or less direct consequence, we obtain the following result, the last part of which is related to conjecture of MacDonald which remained unproved after the others were settled by Deligne and Lusztig (see Alvis [2] for further discussion).

(4.3) THEOREM. Let $\zeta \in \mathrm{Irr}\, G$. Then the following statements hold

(i) $\sum_{u \in V} \zeta(u) = |G|_p \cdot \zeta^*(1)$.

(ii) $|\zeta^*(1)|_{p'} = \zeta(1)_{p'}$.

(iii) $\zeta(1)^{-1} \sum_{u \in V} \zeta(u)$ is, up to sign, a power of p.

For the proof of (i), we have

$$\sum_{u \in V} \zeta(u) = |G|(\zeta, \chi_V)_G = |G|(\zeta^*, \chi_V^*) \quad \text{(by (3.5))}$$

$$= |G|(\zeta^*, |G|_{p'}^{-1}\rho) \quad \text{(by (4.1))}$$

$$= |G|_p(\zeta^*, \rho) = |G_p|\zeta^*(1) \quad (\text{since } \pm\zeta^* \in \mathrm{Irr}\, G).$$

For part (ii), we first obtain

$$\zeta(1)^{-1} \sum_{u \in V} \zeta(u) = \sum_{\mathfrak{C}} \frac{|\mathfrak{C}|\, \zeta(u_{\mathfrak{C}})}{\zeta(1)} \in \text{alg·int } \mathbb{C} \cap \mathbb{Q} = \mathbb{Z},$$

where $\{\mathfrak{C}\}$ ranges over the conjugacy classes of p-elements in G, and $u_{\mathfrak{C}} \in \mathfrak{C}$, for each $\mathfrak{C}$. We have also used the standard result from character theory that

$$\frac{|\mathfrak{C}|\zeta(u_{\mathfrak{C}})}{\zeta(1)} \in \text{alg}\cdot\text{int}.\mathbb{C} \quad \text{(the algebraic integers in } \mathbb{C}\text{)},$$

for any irreducible character ζ and conjugacy class $\mathfrak{C}$. Thus

$$\frac{|G|_p\zeta^*(1)}{\zeta(1)} \in \mathbb{Z}$$

by part (i). Since $\pm\zeta^* \in \operatorname{Irr} G$, by (3.5), we can replace ζ by ζ^* in the above argument, and obtain also

$$\frac{|G|_p\zeta(1)}{\zeta^*(1)} \in \mathbb{Z} .$$

Combining these results we obtain (ii). Finally, (iii) follows at once from parts (i) and (ii).

As a further application, we obtain another proof of a result of Steinberg [19].

(4.4) COROLLARY. $|V| = |G|_p^2$.

The proof is immediate, upon applying (i) to the trivial character, since $1^* = St_G$, and $St_G(1) = |G|_p$.

5. SOME ADDITIONAL RESULTS.

We shall describe briefly, without proofs, some other results about the duality operation.

5.a. THE PERMUTATION OF THE IRREDUCIBLE COMPONENTS OF $I_J\varphi$, FOR φ CUSPIDAL IN Irr L_J.

By Corollary 3.7, the duality operation permutes (up to sign) the irreducible components of $I_J\varphi$, for $J \subseteq R$ and φ a cuspidal irreducible character of L_J. The precise nature of this permutation has now been determined. To describe the main idea, we begin with the simplest case, with $J = \phi$, $\varphi = 1$, so $I_\phi 1 = 1_B{}^G$, the permutation character of the action of G on the cosets G/B. Using the theory of Hecke algebras, it is known, in this case, that there is a bijection $\psi \to \zeta_\psi$ from Irr W to the characters $\{\zeta \in \text{Irr } G: (\zeta, 1_B{}^G) \neq 0\}$, with the property that

$$(\zeta_\psi, 1_{P_J}{}^G) = (\psi, 1_{W_J}{}^W),$$

for all $J \subseteq R$. Let ε be the sign character of W. We then have:

(5.1) THEOREM (Curtis [7]). For all $\psi \in$ Irr W. we have

$$\zeta_\psi{}^* = \zeta_{\varepsilon\psi}.$$

Thus the duality operation, for the characters in $1_B{}^G$, corresponds exactly to the duality operation of $ch(\mathbb{C}W)$ described in § 1. This result has been extended to components of $\lambda_B{}^G$ by McGovern [13], and to the general case, for components of $I_J\varphi$ as in (3.7), by Howlett and Lehrer [12].

5.b. HOMOLOGICAL INTERPRETATION OF DUALITY

In § 1, we interpreted the duality operation in $ch(\mathbb{C}W)$ in terms of the homology representation of W on $H_*(\Gamma)$. A similar interpretation of the duality operation is possible for a finite group G of Lie type. Let Δ be the combinatorial building of G; then Δ is the finite simplicial complex whose simplices are the proper parabolic subgroups of G, with G-action given by conjugation, and with the order relation given by the opposite of inclusion. Thus the vertices of Δ are the maximal parabolic subgroups of G. We first have (assuming the rank n of the BN-pair in G is ≥ 2),

(5.2) THEOREM (Solomon-Tits [16]). The rational homology $H_*(\Delta)$ is zero except in dimensions 0 and $n-1$, and in these dimensions affords 1_G and St_G, respectively.

Since $St_G = 1_G{}^*$, this result suggests the possibility of a homological interpretation of the duality operation, in the general case, using a suitable coefficient system over the building Δ. Such a result has been obtained by Deligne and Lusztig [10]. (See also Curtis-Lehrer [8] for a proof of (5.2) in terms of a comparison of $\mathrm{End}_{\mathbb{Q}G}(H_*(\Delta))$ with $\mathrm{End}_{\mathbb{Q}W}(H_*(\Gamma))$, and [9] for extensions of this idea to the homology of the building over certain coefficient systems.)

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