12. The simple theory of types

In order to avoid the logical paradoxes, Russell invented the theory of types. The idea is to distribute all objects of thought into different types or, in other words to assume that they can be put into different layers or at different levels. We have some original objects called objects of type 0 (or 1 if one prefers). Sets of these objects or relations between them are objects of type 1. Sets of these again are objects of type 2, and so on. Further, the membership relation $x \epsilon y$ shall only have a meaning, if y is of type $n+1$ as often as x is of type n. Composite propositional functions $\phi(x)$ built up from atomic propositions $x \epsilon y$ have then only meaning if it is possible to attach numbers to the occurring variables such that always the symbol y in every occurring atomic proposition $x \epsilon y$ gets the number $n + 1$ when x gets the number n. Such expressions $\phi(x)$ are called stratified.

We may now set up the following axiom of comprehension: For any stratified $\phi(x)$ there exists a y such that the equivalence

$$x \epsilon y \longleftrightarrow \phi(x)$$

is generally valid, that is, it is valid for all x of type n if y is of type $n+1$. Since we do not introduce negative types, there will be a lowest possible type for x in $\phi(x)$, say n_0. Then the axiom asserts

$$\text{(I)} \qquad (Ey)(x)(x \epsilon y \longleftrightarrow \phi(x)),$$

where the range of the universal quantifier is the domain of all objects of type n, $n \geqq n_0$, and the range of (Ey) is the domain of objects of type $n + 1$. The identity relation $x = y$ might be introduced as an undefined notion beside the membership relation ϵ. Then we would have to set up the axiom

$$(x = y) \rightarrow (\psi(x) \longleftrightarrow \psi(y))$$

for every stratified $\psi(x)$. It is simpler, however, to use only ϵ as an undefined notion and define $=$ by letting $x = y$ stand for the validity of the equivalence,

$$\psi(x) \longleftrightarrow \psi(y)$$

for any stratified ψ. We then also need, however, the axiom of extensionality

$$\text{(II)} \qquad (z)(z \epsilon x \longleftrightarrow z \epsilon y) \rightarrow (x = y).$$

It is seen at once that the axioms of the power set and the union in the Zermelo-Fraenkel theory are valid statements here, and also the axiom of separation for stratified $C(x)$. As to the axioms of the small sets, these are also valid with the restriction that $\{a,b\}$ can be built only when a and b are of the same type. It must be noticed, however, that we get not only universal sets of different types but also null sets of different types. Indeed

$$(Ey)\,(x \epsilon y \cdot v \cdot x \bar{\epsilon} y) \text{ and } (y)(x \epsilon y \cdot \& \cdot x \bar{\epsilon} y)$$

used as $\phi(x)$ in (I) define, if y runs through all individuals of type $n + 1$, the universal set of type n respectively the null set of type n.

Because of the restriction in building the set $\{a,b\}$, we ought to look at the union and intersection of two sets. If $A(x)$ and $B(x)$ are two stratified propositional functions with only x as free variable, then also $A(x)$ & $B(x)$ and $A(x) \vee B(x)$ will be stratified. This is seen as follows: If we can attach numbers to x and the other (the bound) variables in $A(x)$ and do the same for $B(x)$, then it is possible to do this for $A(x)$ and $B(x)$ in such a way that x is assigned the same number in A and B. As a consequence of this, we can, for every type $\geqq$ a certain one, always build the union $\hat{x}(A(x) \vee B(x))$ and the intersection $\hat{x}(A(x)\ \&\ B(x))$ of the sets $\hat{x}\, A(x)$ and $\hat{x}\, B(x)$. It must be remarked that $\bar{A}(x)$ is also stratified when $A(x)$ is, so that we get a complementary set to every given set.

There is a certain difficulty with regard to relations and functions. One would have liked to be able to conceive a binary relation as a set of pairs, and it would have been nice if this set could have been of the same type as a set of single elements. However, this would require the introduction of ordered pairs, triples, and so on, as new objects of the same type as the different terms in these sequences. Thus an ordered pair (a,b), where a and b are of a certain type, should again be an object of this type. This would mean a certain complication. Instead of that one could let the sign ϵ stand for a binary relation in the case $x \epsilon y$, and a ternary if an ordered pair (x,y) is ϵz, and so on. Probably this is not advisable. The best thing to do is, I should think, to introduce the ordered pairs, triples, and so on, as sets. Also by this procedure one has to tolerate a certain complication, because the set of all x such that $A(x)$ will not be of the same type as the set of all (x,y) such that $B(x,y)$. For example if we have to do with a set N representing the number series, then the set of all primes p will be of same type as N, but the set P of all ordered pairs (x,y), where $x \epsilon N$, $y \epsilon N$, will be of a type 2 units higher. Indeed $(a,b) = \{\{a,b\}, \{a\}\}$ is of a type 2 units higher than the type of a and b. The set $\{\{p\}\}$ will however be of the same type as the set P. So far as I can see, it will be best to consider the ordered pairs, triples, etc., as sets.

If we should try to develop mathematics, basing it on the simple theory of types, it would be desirable to have an axiom of infinity for the things of type 0. Indeed, if there is only a finite number of individuals of type 0, there can be only a finite number of each of the higher types. The development of arithmetic will then already be difficult and analysis would scarcely be possible. Now the axiom of infinity might be set up in different ways. We might assume a one-to-one correspondence f given between the set V of all things of type 0 and a proper subset V' of V. This mapping f would then be a fundamental notion in the theory beside the relation ϵ . We may manage so that we don't introduce such an extra notion. We may assume the axiom

(III) $\qquad (x)(x \text{ is inductive finite} \rightarrow (Ey)(y\, \bar{\epsilon}\, x)).$

where y runs through all objects of type 0, x all objects of type 1. Then there will exist sets x of type 1 with 0,1,2,... elements. Introducing the notion cardinal number for the sets of type 1, every one of these cardinals is a set of type 2, and the finite cardinals constitute a set of type 3 which can be taken as the natural number series. Starting with this, the introduction of negative integers, fractions, real numbers, etc., can be performed in just the usual way. One has to take care of the type distinctions, but it is quite easy to develop ordinally mathematics in this way.

Some small changes will often be necessary to carry over the theorems and their proofs from the Zermelo-Fraenkel theory to the simple theory of types. Bernstein's equivalence theorem with its proof remains unchanged. Cantor's theorem that UM is always of higher cardinality than M must be expressed thus: Let EM be the set of all unit sets $\{m\}$ contained in M. Then $\overline{\overline{EM}} < \overline{\overline{UM}}$. The previous definition of well-ordering (see § 4) must be slightly changed to this wording: A set M is well-ordered, if there is a function R from EM to UM such that, for $0 \subset N \subseteq M$, there is a unique $n \epsilon N$ such that $N \subseteq R(\{n\})$. The wording of Theorem 10 must now be: Let a function ϕ be given such that $\phi(A)$, for every A such that $0 \subset A \subseteq M$, denotes a unit subset of A. Then there is a subset $\mathfrak{M}$ of UM such that to every $N \subseteq M$ there is one and only one element N_0 of $\mathfrak{M}$ such that $N \subseteq N_0$ and $\phi(N_0) \subseteq N$. Such slight changes will be necessary in many of the previous theorems and proofs. If we look at Theorem 6 for example, there can be no meaning in an equivalence between $M + N$ and $M \cdot N$ or even $M \times N$, because the elements of $M \cdot N$ are of type $t + 1$ and those of $M \times N$ are of type $t + 2$ when those of M and N are of type t. If, however, we replace M by its sets of unit subsets EM and N by EN, then $EM + EN$ and $M \cdot N$ will be of same type, and an equivalence between these two sets will be meaningful. Similarly we can compare $EEM + EEN$ and $M \times M$. I don't think it is necessary to carry out in detail these small changes in the considerations. By the way, it may be remarked that functions may well be introduced such that arguments and values are not of same type, but if functions should be conceived as special cases of relations, and relations as sets of sequences conceived as sets, such a procedure must be avoided.

13. The theory of Quine

There have been many attempts to avoid the introduction of types, which are inconvenient. One of these is the theory of Quine. An exposition of this can be found in the book "Logic for Mathematicians" recently published by B. Rosser. Quine's theory is something intermediate between the axiomatic theory of Zermelo-Fraenkel and Russell's type theory. It has in common with the former the feature that there are no type distinctions. On the other hand it has in common with the latter the feature that only stratified propositional functions are admitted for the definition of new sets. Indeed we have in Quine's theory the following axiom of comprehension:

$$(Ey)(x)(x \epsilon y \longleftrightarrow \phi(x))$$

with the whole domain of objects as range of variation of x and y. Of course y must not occur in $\phi(x)$.

It is easy to see that here we again get only one null set A and only one universal set V. We may for example use these definitions:

$$x \epsilon A \longleftrightarrow (y)(x \epsilon y \ \& \ x \bar{\epsilon} y), \qquad x \epsilon V \longleftrightarrow (Ey)(x \epsilon y \cdot v \cdot x \bar{\epsilon} y) .$$

Obviously the set **V** is ϵV. Nevertheless Russell's antinomy cannot be deduced, because the propositional function $x \epsilon x$ is not stratified, so that no