Chapter 7

THE FIRST APPROXIMATION THEOREM

1. The properties A_d, B, and C.

While the last two chapters depended on purely algebraic ideas, we now introduce real and g-adic algebraic numbers and study their rational approximations with respect to the corresponding absolute value or g-adic value, respectively. Here, as usual,

$$g = p_1^{e_1} \dots p_r^{e_r} \geq 2,$$

where $p_1, \dots, p_r$ are distinct primes, and $e_1, \dots, e_r$ are positive integers; the g-adic value $|A|_g$ of $A \leftrightarrow (\alpha_1, \dots, \alpha_r)$ is defined by

$$|A|_g = \max\left(|\alpha_1|_{p_1}^{\frac{\log g}{e_1 \log p_1}}, \dots, |\alpha_r|_{p_r}^{\frac{\log g}{l_r \log p_r}}\right).$$

The later occurring g'-adic and g''-adic values $|a|_{g'}$ and $|a|_{g''}$ are defined analogously.

The letter ξ always denotes a fixed real algebraic number, and the letter Ξ a fixed g-adic algebraic number. Only ξ satisfying

$$\xi \neq 0$$

and only $\Xi \leftrightarrow (\xi_1, \dots, \xi_r)$ satisfying

$$\xi_1 \neq 0, \dots, \xi_r \neq 0$$

will be considered. We denote by

$$F(x) = F_0 x^f + F_1 x^{f-1} + \dots + F_f, \qquad \text{where} \qquad f \geq 1,\ F_0 \neq 0,\ F_f \neq 0,$$

a polynomial of lowest degree with integral coefficients having either ξ, or Ξ, or both ξ and Ξ, as zeros; hence, by Chapter 3, $F(x)$ has no multiple factors. As before, we put

$$c = 2\max(|F_0|, |F_1|, \dots, |F_f|), \qquad \text{so that} \qquad c > 1.$$

Next we denote by

$$\Sigma = \{\kappa^{(1)}, \kappa^{(2)}, \kappa^{(3)}, \dots\}$$

a fixed infinite sequence of distinct rational numbers

$$\kappa^{(k)} = \frac{P^{(k)}}{Q^{(k)}} \neq 0$$

where $P^{(k)} \neq 0$ and $Q^{(k)} \neq 0$ are integers such that

$$(P^{(k)}, Q^{(k)}) = 1.$$

We call

$$H^{(k)} = \max(|P^{(k)}|, |Q^{(k)}|)$$

the height of $\kappa^{(k)}$. It is obvious that

(1): $$\lim_{k\to\infty} H^{(k)} = \infty.$$

For such sequences Σ we now define three properties A_d, B, and C where d is either 1 or 2 or 3.

First, Σ is said to have the property A_d if

for d=1: There exist two positive constants ρ and c_1 such that

(A_1): $$|\kappa^{(k)} - \xi| \leq c_1 H^{(k)-\rho} \qquad \text{for all } k;$$

for d=2: There exist two positive constants σ and c_2 such that

(A_2): $$|\kappa^{(k)} - \Xi|_g \leq c_2 H^{(k)-\sigma} \qquad \text{for all } k; \text{ and}$$

for d=3: There exist four positive constants ρ, σ, c_1, and c_2 such that

(A_3): $$|\kappa^{(k)} - \xi| \leq c_1 H^{(k)-\rho} \quad \text{and} \quad |\kappa^{(k)} - \Xi|_g \leq c_2 H^{(k)-\sigma} \qquad \text{for all } k.$$

The property A_3 includes therefore both properties A_1 and A_2.

If Σ has the property A_d, then its elements have for d=1 and d=3 the real limit ξ, and for d=2 and d=3 the g-adic limit σ, because $c_1H^{(k)-\rho}$ and $c_2H^{(k)-\sigma}$ tend to zero as k tends to infinity.

Secondly, Σ is said to have the property B if there exist,

(i) two integers g' and g'' satisfying

$$g' \geq 2, \quad g'' \geq 2, \quad (g', g'') = 1;$$

(ii) two real numbers λ and μ satisfying

$$0 \leq \lambda \leq 1, \quad 0 \leq \mu \leq 1; \qquad \text{and}$$

(iii) two positive constants c_3 and c_4, such that

(B): $$|P^{(k)}|_{g'} \leq c_3 H^{(k)\lambda-1} \quad \text{and} \quad |Q^{(k)}|_{g''} \leq c_4 H^{(k)\mu-1} \qquad \text{for all } k.$$

The first inequality (B) holds trivially if λ=1 as we may simply take $c_3 = 1$; and similarly for the second inequality when μ=1.

For later it is important to note that if d=2 or d=3, and if Σ has both properties A_d and B, then

$$(g, g') = 1 \quad \text{if} \quad 0 \leq \lambda < 1.$$

For $\lim_{k\to\infty} |P^{(k)}|_{g'} = 0$, while $(P^{(k)}, Q^{(k)}) = 1$, hence $|Q^{(k)}|_{g'} = 1$, and so also

$$\lim_{k\to\infty} |\kappa^{(k)}|_{g'} = 0.$$

If now g and g' had a common prime factor, p_1 say, then

$$\lim_{k\to\infty} |\kappa^{(k)}|_{p_1} = 0 \quad \text{and} \quad \lim_{k\to\infty} |\kappa^{(k)} - \xi_1|_{p_1} = 0, \quad \text{hence} \quad \xi_1 = 0,$$

contrary to the hypothesis.

Third, Σ is said to have the property C if there exists a positive constant c_5 such that

(C): $$|\kappa^{(k)}| \leqslant c_5 \qquad \text{for all } k.$$

In the two cases d=1 and d=3 the property C follows from the property A_d because

$$\lim_{k\to\infty} |\kappa^{(k)}| = |\xi|.$$

In the remaining case d=2 it is, however, independent of A_d.

Our first aim in this chapter is to prove the following result.

Main Lemma: *If the sequence*

$$\Sigma = \{\kappa^{(1)}, \kappa^{(2)}, \kappa^{(3)}, \ldots\}$$

has all three properties A_d, B, *and* C, *then*

$$\tau \leqslant \lambda + \mu$$

where

(2): $$\tau = \begin{cases} \rho & \text{if } d=1, \\ \sigma & \text{if } d=2, \\ \rho+\sigma & \text{if } d=3. \end{cases}$$

The proof of this lemma will be long and involved, and it will be indirect. It will be assumed that

(3): $$\tau = \lambda+\mu+4\epsilon \qquad \text{where} \qquad \epsilon > 0,$$

and from this hypothesis we shall deduce a contradiction.

2. The selection of the parameters.

Since the property A_d weakens when the exponents ρ and σ are decreased, we may without loss of generality assume that

(4): $$0 < \epsilon \leqslant \frac{1}{4}.$$

For the same reason we are allowed to assume that

(5): $$c_1 \geqslant 1, \; c_2 \geqslant 1, \; c_3 \geqslant 1, \; c_4 \geqslant 1, \; c_5 \geqslant 1.$$

Similar to c, $c_1, \ldots, c_5$ the letters $c_6, c_7, \ldots, C_1, C_2, C_3, T_1, T_2$, and T_3 will be used to denote certain positive constants that depend only on the sequence Σ and the algebraic numbers ξ, Ξ, or ξ and Ξ, respectively; they will, however, be independent of the numbers m, s, t, $\kappa_1, \ldots, \kappa_m$, $r_1, \ldots, r_m$

to be defined immediately. The last three constants T_1, T_2, and T_3 will not be fixed until the end of the proof.

The parameters are now selected as follows.

First, choose a positive integer m such that

$$m = \left[2\left(\frac{24f}{\epsilon}\right)^2\right] + 1,$$

and in terms of m define the positive number s by

(6): $$s = \frac{\epsilon m}{6}.$$

Secondly, choose a number t such that

(7): $$0 < t \leq 1, \quad 2^{m+1}\, t^{2^{-(m-1)}} \leq \frac{\epsilon m}{6}.$$

Third, select m distinct elements $\kappa^{(i_1)}, \kappa^{(i_2)}, \ldots, \kappa^{(i_m)}$ of Σ that satisfy certain inequality conditions to be stated at once. To simplify the notation, these elements of Σ are written as

$$\kappa^{(i_h)} = \kappa_h = \frac{P_h}{Q_h} \qquad (h = 1,2,\ldots,m),$$

where the P_h and Q_h are integers for which

$$P_h \neq 0, \quad Q_h \neq 0, \quad (P_h, Q_h) = 1.$$

Thus κ_h has the height

$$H_h = \max(|P_h|, |Q_h|).$$

The hypothesis of the main lemma imposes, for all suffixes $h=1, 2,\ldots, m$, the following inequalities:

(A_d): $$\begin{cases} |\kappa_h - \xi| \leq c_1 H_h^{-\rho} & \text{if } d=1, \\ |\kappa_h - \Xi|_g \leq c_2 H_h^{-\sigma} & \text{if } d=2, \\ |\kappa_h - \xi| \leq c_1 H_h^{-\rho} \text{ and } |\kappa_h - \Xi|_g \leq c_2 H_h^{-\sigma} & \text{if } d=3; \end{cases}$$

(B): $$|P_h|_{g'} \leq c_3 H_h^{\lambda-1} \quad \text{and} \quad |Q_h|_{g''} \leq c_4 H_h^{\mu-1};$$

(C): $$|\kappa_h| \leq c_5 \quad .$$

It is necessary for the proof to add the following conditions:

(8): $$|P_h|_{g'} \leq \frac{1}{g'} \text{ if } 0 \leq \lambda < 1 \qquad (h = 1,2,\ldots,m),$$

(9): $$\log H_{h+1} \geq \frac{2}{t} \log H_h \qquad (h = 1,2,\ldots,m-1),$$

and, depending on the suffix d,

(10): $$H_1 \geqslant \max\left((20c)^{\frac{1}{t}m^2},\ 2^{\frac{1}{t}(m-1)m(2m+1)},\ T_d\right).$$

Since the elements of Σ satisfy the limit formula (1), it is possible to choose $\kappa_1,\dots,\kappa_m$ such that all these inequalities are satisfied.

Finally, select m positive integers $r_1,\dots,r_m$ such that

(11): $$r_1 \geqslant \frac{2\log H_m}{\epsilon \log H_1},$$

(12): $$r_h \geqslant r_1 \frac{\log H_1}{\log H_h} > r_h - 1 \qquad (h = 2,3,\dots,m).$$

Since, by (9) and (10), evidently

$$2 < H_1 < H_2 < \dots < H_m,$$

these formulae imply that

$$r_h \geqslant \frac{2\log H_m}{\epsilon \log H_h} \geqslant \frac{2}{\epsilon}, \quad r_h - 1 = r_h\left(1-\frac{1}{r_h}\right) \geqslant \left(1-\frac{\epsilon}{2}\right)r_h > \frac{r_h}{1+\epsilon},$$

because, by (4),

$$(1+\epsilon)\left(1-\frac{\epsilon}{2}\right) = 1+\frac{\epsilon}{2}(1-\epsilon) > 1.$$

Hence we find that

(13): $$r_1 \log H_1 \leqslant r_h \log H_h \leqslant (1+\epsilon) r_1 \log H_1 \qquad (h = 1,2,\dots,m).$$

Therefore, for arbitrary non-negative exponents $k_1,\dots,k_m$, it follows that

(14): $$H_1^{r_1 \sum_{h=1}^{m} \frac{k_h}{r_h}} \leqslant H_1^{k_1} \dots H_m^{k_m} \leqslant H_1^{(1+\epsilon) r_1 \sum_{h=1}^{m} \frac{k_h}{r_h}}.$$

We also note that, by (9), (11), and (12),

$$r_h \geqslant \frac{2}{\epsilon} > 2,\ r_h - 1 > \frac{1}{2} r_h, \quad \frac{1}{2} r_{h+1} \log H_{h+1} < r_1 \log H_1 \leqslant r_h \log H_h,$$

hence

$$\frac{r_{h+1}}{r_h} < \frac{2\log H_h}{\log H_{h+1}} \leqslant t,$$

and therefore

(15): $$r_{h+1} < r_h t \qquad (h = 1,2,\dots,m-1).$$

Thus, trivially,

(16): $$r_1 > r_2 > \dots > r_m \quad \text{and} \quad r_1 + r_2 + \dots + r_m < m r_1.$$

3. Application of Theorems 1 and 2.

The polynomial $F(x)$ has no multiple factors. Hence, by Theorem 2, applied to this polynomial and the numbers m, s, $r_1,\dots,r_m$, there is a polynomial

$$A(x_1,\dots,x_m) = \sum_{i_1=0}^{r_1} \dots \sum_{i_m=0}^{r_m} a_{i_1\dots i_m} x_1^{i_1}\dots x_m^{i_m} \not\equiv 0$$

with the following properties.

(17): The coefficients $a_{i_1\dots i_m}$ are integers such that

$$|a_{i_1\dots i_m}| \leq 5(4c)^{r_1+\dots+r_m},$$

and they vanish unless

$$\frac{1}{2}(m-s) < \sum_{h=1}^{m} \frac{i_h}{r_h} < \frac{1}{2}(m+s)\,.$$

(18): The derivative $A_{j_1\dots j_m}(x,\dots,x)$ is divisible by $F(x)$ whenever

$$0 \leq j_1 \leq r_1,\dots,0 \leq j_m \leq r_m,\ \sum_{h=1}^{m} \frac{j_h}{r_h} \leq \frac{1}{2}(m-s).$$

(19): We have the following majorants,

$$A_{j_1\dots j_m}(x_1,\dots,x_m) \ll 5(8c)^{r_1+\dots+r_m}(1+x_1)^{r_1}\dots(1+x_m)^{r_m},$$

$$A_{j_1\dots j_m}(x,\dots,x) \ll 5(8c)^{r_1+\dots+r_m}(1+x)^{r_1+\dots+r_m}.$$

By (10), (16) and (17), the height of $A(x_1,\dots,x_m)$ does not exceed

$$5(4c)^{r_1+\dots+r_m} \leq 5(4c)^{mr_1} \leq (20c)^{mr_1} \leq H_1^{\frac{1}{m}r_1 t}\,.$$

It follows then from the inequalities (10), (12), and (15) that the hypothesis of Theorem 1 is satisfied for the polynomial $A(x_1,\dots,x_m)$ and the numbers m, s, t, $\kappa_1,\dots,\kappa_m$, $r_1,\dots,r_m$. But then, by this theorem, there exist suffixes $l_1,\dots,l_m$ satisfying the inequalities

(20): $$0 \leq l_1 \leq r_1,\dots,0 \leq l_m \leq r_m,\ \sum_{h=1}^{m} \frac{l_h}{r_h} \leq 2^{m+1} t^{2^{-(m-1)}}$$

such that the function value

$$A_{l_1\dots l_m}(\kappa_1,\dots,\kappa_m) = A_{l_1\dots l_m}\left(\frac{P_1}{Q_1},\dots,\frac{P_m}{Q_m}\right), = A_{(l)} \quad \text{say,}$$

does not vanish,

(21): $$A_{(l)} \neq 0.$$

This number $A_{(l)}$ is rational and so may be written as a quotient

$$A_{(l)} = \frac{N(l)}{D(l)}$$

of two integers $N(l)$ and $D(l)$ satisfying

$$N(l) \neq 0, \quad D(l) \neq 0, \quad (N(l), D(l)) = 1.$$

In the next sections we shall establish upper and lower bounds for $|A_{(l)}|$. To express these in a simple form, it is convenient to introduce the following abbreviations,

(22): $$\Lambda = \sum_{h=1}^{m} \frac{l_h}{r_h}, \quad S_1 = \frac{1}{2}(m-s)-\Lambda, \quad S_2 = \frac{1}{2}(m+s)-\Lambda, \quad S_3 = m-\Lambda .$$

It is obvious from the formulae (4), (6), (7), and (20), that

(23): $$0 \leq \Lambda \leq \frac{\epsilon m}{6},$$

(24): $$S_1 \geq \frac{1}{4}(2-\epsilon)\,m \geq \frac{7}{16}m, \quad S_2 \geq \frac{1}{12}(6-\epsilon)m \geq \frac{23}{48}m, \quad S_3 \geq \frac{1}{6}(6-\epsilon)m \geq \frac{23}{24}m.$$

4. Upper bounds for $|A_{(l)}|$.

For real $x_1,\dots,x_m$ and arbitrary suffixes $j_1,\dots,j_m$ it follows from (16) and (19) that

$$|A_{j_1\dots j_m}(x_1,\dots,x_m)| \leq 5(8c)^{r_1+\dots+r_m}(1+|x_1|)^{r_1}\dots(1+|x_m|)^{r_m} \leq$$

$$\leq (40c)^{mr_1}\{(1+|x_1|)\dots(1+|x_m|)\}^{r_1}.$$

We apply now the property C of Σ. This property implies, first, that

(25): $$|A_{(l)}| \leq c_6^{mr_1}$$

where, for shortness,

$$c_6 = 40c(1+c_5) .$$

For

$$(1+|\kappa_1|)\dots(1+|\kappa_m|) \leq (1+c_5)^m .$$

Secondly, let d=1 or d=3. Then, again by property C,

$$|\xi| = \lim_{k\to\infty} |\kappa^{(k)}| \leq c_5$$

and hence

(26): $$|A_{j_1\dots j_m}(\xi,\dots,\xi)| \leqslant c_6^{mr_1} \quad \text{for all suffixes } j_1,\dots,j_m.$$

The inequality (25) is, of course, valid for all three values of d, but will be used only for $d=2$. A much stronger upper bound for $|A_{(l)}|$ can be proved in the other two cases $d=1$ and $d=3$, using (26).

From Taylor's formula we obtain the identity

$$A(x_1,\dots,x_m) = \sum_{j_1=0}^{r_1}\dots\sum_{j_m=0}^{r_m} A_{j_1\dots j_m}(x,\dots,x)(x_1-x)^{j_1}\dots(x_m-x)^{j_m},$$

and on repeated differentiation,

(27): $$A_{l_1\dots l_m}(x_1,\dots,x_m) =$$

$$= \sum_{j_1=0}^{r_1}\dots\sum_{j_m=0}^{r_m} A_{j_1\dots j_m}(x,\dots,x)\binom{j_1}{l_1}\dots\binom{j_m}{l_m}(x_1-x)^{j_1-l_1}\dots(x_m-x)^{j_m-l_m}.$$

By putting

$$x_1 = \kappa_1,\dots,x_m = \kappa_m,\ x = \xi,$$

we find that

(28): $$A_{(l)} = \sum_{j_1=0}^{r_1}\dots\sum_{j_m=0}^{r_m} A_{j_1\dots j_m}(\xi,\dots,\xi)\binom{j_1}{l_1}\dots\binom{j_m}{l_m}(\kappa_1-\xi)^{j_1-l_1}\dots(\kappa_m-\xi)^{j_m-l_m}.$$

In this equation,

$$\binom{j_h}{l_h} = 0 \quad \text{if } j_h < l_h,$$

while, by (18),

$$A_{j_1\dots j_m}(\xi,\dots,\xi) = 0 \quad \text{if } \sum_{h=1}^{m}\frac{j_h}{r_h} \leqslant \frac{1}{2}(m-s).$$

It follows that *it suffices to extend the summation in* (28) *only over those systems of suffixes* $(j) = (j_1,\dots,j_m)$ *that belong to the set*

$$J:\ 0 \leqslant j_1-l_1 \leqslant r_1-l_1,\dots,\ 0 \leqslant j_m-l_m \leqslant r_m-l_m,\ \sum_{h=1}^{m}\frac{j_h-l_h}{r_h} > s_1.$$

It is then evident that

(29): $$|A_{(l)}| \leqslant A^*A^{**}$$

where, for shortness,

$$A^* = \sum_{j_1=0}^{r_1}\dots\sum_{j_m=0}^{r_m} |A_{j_1\dots j_m}(\xi,\dots,\xi)|\binom{j_1}{l_1}\dots\binom{j_m}{l_m}$$

and

$$A^{**} = \max_{(j)\in J} |\kappa_1 - \xi|^{j_1-l_1} \dots |\kappa_m - \xi|^{j_m-l_m} .$$

In the first expression,

$$0 < \binom{j_h}{l_h} \leqslant \sum_{l=0}^{j_h} \binom{j_h}{l} = 2^{j_h},$$

so that by (16) and (26),

$$A^* \leqslant \sum_{j_1=0}^{r_1} \dots \sum_{j_m=0}^{r_m} c_6^{mr_1} \cdot 2^{j_1+\dots+j_m} = c_6^{mr_1} (2^{r_1+1}-1) \dots (2^{r_m+1}-1) <$$

$$< c_6^{mr_1} \cdot 2^{2r_1} \dots 2^{2r_m} \leqslant (4c_6)^{mr_1} .$$

For the second expression we apply the property

(A₁): $$|\kappa_h - \xi| \leqslant c_1 H_h^{-\rho} \qquad (h = 1,2,\dots, m),$$

which holds also for $d=3$; here, by (5), $c_1 \geqslant 1$. Hence

$$A^{**} \leqslant \max_{(j)\in J} c_1^{(j_1-l_1)+\dots+(j_m-l_m)} \cdot \max_{(j)\in J} (H_1^{j_1-l_1} \dots H_m^{j_m-l_m})^{-\rho},$$

where

$$\max_{(j)\in J} c_1^{(j_1-l_1)+\dots+(j_m-l_m)} \leqslant c_1^{r_1+\dots+r_m} \leqslant c_1^{mr_1} .$$

Further, by the left-hand side of (14) and the definition of J,

$$\max_{(j)\in J} (H_1^{j_1-l_1} \dots H_m^{j_m-l_m})^{-\rho} \leqslant \max_{(j)\in J} H_1^{-\rho r_1 \sum_{h=1}^{m} \frac{j_h-l_h}{r_h}} \leqslant H_1^{-\rho S_1 r_1} ,$$

so that

$$A^{**} \leqslant c_1^{mr_1} H_1^{-\rho S_1 r_1} .$$

We finally put

$$c_7 = 4c_1 c_6$$

and substitute the upper bounds for A^* and A^{**} in (29). We so find that

(30): $$|A_{(l)}| \leqslant c_7^{mr_1} H_1^{-\rho S_1 r_1} \qquad \text{for } d=1 \text{ and } d=3.$$

Here, by (24), the exponent of H_1 is negative; hence this upper bound is smaller than that given by (25). However, no explicit use of this fact will be made.

5. An upper bound for $|A_{(l)}|_g$.

In the two cases d=2 and d=3 there exists an upper bound for $|A_{(l)}|_g$ very similar to that for $|A_{(l)}|$ which has just been proved.

In both cases the sequence Σ has the g-adic limit Ξ, and so

$$\lim_{k\to\infty} |\kappa^{(k)}|_g = |\Xi|_g.$$

There exists then a constant $c_8 \geqslant 1$ depending only on Σ such that

$$|\kappa^{(k)}|_g \leqslant c_8 \qquad \text{for all } k,$$

and therefore also

$$|\Xi|_g \leqslant c_8.$$

This time we substitute the values

$$x_1 = \kappa_1, \dots, x_m = \kappa_m, \ x = \Xi$$

in the identity (27), so obtaining the equation

$$(31): \quad A_{(l)} = \sum_{j_1=0}^{r_1} \dots \sum_{j_m=0}^{r_m} A_{j_1 \dots j_m}(\Xi,\dots,\Xi)\binom{j_1}{l_1}\dots\binom{j_m}{l_m}(\kappa_1-\Xi)^{j_1-l_1}\dots(\kappa_m-\Xi)^{j_m-l_m}.$$

Here, just as in (28), *it suffices to extend the summation only over all systems of suffices* $(j) = (j_1,\dots,j_m)$ *in* J.

The binomial coefficients in (31) are integers, hence their g-adic values are not greater than 1. It follows then from the non-Archimedean property of the g-adic pseudo-valuation that

$$(32): \qquad |A_{(l)}|_g \leqslant B^* B^{**}$$

where, for shortness,

$$B^* = \max_{(j)\in J} |A_{j_1\dots j_m}(\Xi,\dots,\Xi)|_g$$

and

$$B^{**} = \max_{(j)\in J} |\kappa_1 - \Xi|_g^{j_1-l_1} \dots |\kappa_m - \Xi|_g^{j_m-l_m}.$$

The polynomials $A_{j_1\dots j_m}(x,\dots,x)$ have integral coefficients and are at most of degree $r_1+\dots+r_m \leqslant mr_1$; therefore

$$|A_{j_1\dots j_m}(\Xi,\dots,\Xi)|_g \leqslant c_8^{mr_1}$$

and hence also

$$B^* \leqslant c_8^{mr_1}.$$

Next, for the second factor, we apply the property

$$(A_2): \qquad |\kappa_h - \Xi|_g \leqslant c_2 H_h^{-\sigma} \qquad (h = 1, 2, \dots, m)$$

which holds also for $d=3$; here again, by (5), $c_2 \geqslant 1$. It follows that

$$B^{**} \leqslant \max_{(j)\in J} c_2^{(j_1-l_1)+\dots+(j_m-l_m)} \cdot \max_{(j)\in J} (H_1^{j_1-l_1} \dots H_m^{j_m-l_m})^{-\sigma}.$$

Here

$$\max_{(j)\in J} c_2^{(j_1-l_1)+\dots+(j_m-l_m)} \leqslant c_2^{r_1+\dots+r_m} \leqslant c_2^{mr_1},$$

while, by the left-hand side of (14) and the definition of J,

$$\max_{(j)\in J} (H_1^{j_1-l_1} \dots H_m^{j_m-l_m})^{-\sigma} \leqslant \max_{(j)\in J} H_1^{-\sigma r_1 \sum_{h=1}^{m} \frac{j_h-l_h}{r_h}} \leqslant H_1^{-\sigma S_1 r_1}.$$

Hence

$$B^{**} \leqslant c_2^{mr_1} H_1^{-\sigma S_1 r_1}.$$

Put

$$c_9 = c_2 c_8$$

and substitute the upper bounds for B^* and B^{**} in (32). We so find that

$$(33): \qquad |A_{(l)}|_g \leqslant c_9^{mr_1} H_1^{-\sigma S_1 r_1} \qquad \text{for } d=2 \text{ and } d=3.$$

6. An upper bound for $|D_{(l)}|$.

In this and the next sections we shall establish an upper bound for $|D_{(l)}|$ and a lower bound for $|N_{(l)}|$; by combining these, a lower bound for $|A_{(l)}|$ will be obtained.

From the definition of $A(x_1, \dots, x_m)$,

$$A_{l_1\dots l_m}(x_1, \dots, x_m) = \sum_{i_1=0}^{r_1} \dots \sum_{i_m=0}^{r_m} a_{i_1\dots i_m} \binom{i_1}{l_1} \dots \binom{i_m}{l_m} x_1^{i_1-l_1} \dots x_m^{i_m-l_m},$$

so that, in particular,

$$(34): \quad A_{(l)} = \frac{N_{(l)}}{D_{(l)}} = \sum_{i_1=0}^{r_1} \dots \sum_{i_m=0}^{r_m} a_{i_1\dots i_m} \binom{i_1}{l_1} \dots \binom{i_m}{l_m} \left(\frac{P_1}{Q_1}\right)^{i_1-l_1} \dots \left(\frac{P_m}{Q_m}\right)^{i_m-l_m}.$$

In this equation,

$$\binom{i_h}{l_h} = 0 \text{ if } i_h < l_h,$$

while, by (17),

$$a_{i_1 \dots i_m} = 0 \qquad \text{unless} \qquad \frac{1}{2}(m-s) < \sum_{h=1}^{m} \frac{i_h}{r_h} < \frac{1}{2}(m+s).$$

It follows then that the summation in (34) need be extended only over those systems of suffixes $(i) = (i_1, \dots, i_m)$ that belong to the set

$$I:\ 0 \leq i_1 - l_1 \leq r_1 - l_1, \dots, 0 \leq i_m - l_m \leq r_m - l_m,\ S_1 < \sum_{h=1}^{m} \frac{i_h - l_h}{r_h} < S_2.$$

Each single term in (34) has a denominator

$$Q_1^{i_1 - l_1} \dots Q_m^{i_m - l_m}.$$

Therefore the least common denominator $D_{(l)}$ satisfies the inequality

$$|D(l)| \leq \operatorname*{lcm}_{(i) \in I} Q_1^{i_1 - l_1} \dots Q_m^{i_m - l_m}, \ = D \qquad \text{say,}$$

where the symbol "lcm" stands for the least common multiple.

We apply now the second half

$$|Q_h|_{g''} \leq c_4 H_h^{\mu - 1} \qquad (h = 1, 2, \dots, m)$$

of the property B. By this property, Q_h is divisible by an integral power of g'' that is easily proved to be not smaller than

$$(c_4 g'' H_h^{\mu - 1})^{-1},$$

but may be larger. For each suffix $h = 1, 2, \dots, m$ it is then certainly possible to find a factorisation

$$Q_h = Q_h^* Q_h^{**}$$

of Q_h where Q_h^* is that integral power of g'' which is defined by the inequalities

$$(35): \qquad \frac{1}{c_4 g''} H_h^{1-\mu} \leq Q_h^* < \frac{1}{c_4} H_h^{1-\mu} \leq c_{10} H_h^{1-\mu}$$

and where we have put

$$c_{10} = \max\left(1, \frac{1}{c_4}\right).$$

The complementary factor Q_h^{**} then satisfies the inequality

$$(36): \qquad |Q_h^{**}| = |Q_h| Q_h^{*-1} \leq H_h \left(\frac{1}{c_4 g''} H_h^{1-\mu}\right)^{-1} = c_4 g'' H_h^{\mu}.$$

From the factorisations of the Q_h it is obvious that

$$(37): \qquad D \leq D^* D^{**}$$

where

$$D^* = \operatorname*{lcm}_{(i)\in I} Q_1^{*i_1-l_1} \dots Q_m^{*i_m-l_m}$$

and

$$D^{**} = \operatorname*{lcm}_{(i)\in I} Q_1^{**i_1-l_1} \dots Q_m^{**i_m-l_m} .$$

The first factor D^* is the least common multiple of certain integral powers of g'' and hence is equal to their maximum,

$$D^* = \max_{(i)\in I} Q_1^{*i_1-l_1} \dots Q_m^{*i_m-l_m}.$$

Therefore, from (35),

$$D^* \leqslant \max_{(i)\in I} c_{10}^{(i_1-l_1)+\dots+(i_m-l_m)} \cdot \max_{(i)\in I} (H_1^{i_1-l_1} \dots H_m^{i_m-l_m})^{1-\mu}.$$

Here $c_{10} \geqslant 1$ so that

$$\max_{(i)\in I} c_{10}^{(i_1-l_1)+\dots+(i_m-l_m)} \leqslant c_{10}^{r_1+\dots+r_m} \leqslant c_{10}^{mr_1} .$$

Further, by the definition of I and by the right-hand side of (14),

$$\max_{(i)\in I} (H_1^{i_1-l_1} \dots H_m^{i_m-l_m})^{1-\mu} \leqslant \max_{(i)\in I} H_1^{(1-\mu)(1+\epsilon)r_1 \sum_{h=1}^{m} \frac{i_h-l_h}{r_h}} \leqslant H_1^{(1-\mu)(1+\epsilon)S_2 r_1} .$$

Hence

$$D^* \leqslant c_{10}^{mr_1} H_1^{(1-\mu)(1+\epsilon)S_2 r_1} .$$

For the second factor D^{**} we use the trivial estimate

$$D^{**} \leqslant \left| Q_1^{**r_1-l_1} \dots Q_m^{**r_m-l_m} \right| \leqslant (c_4 g'' H_1^{\mu})^{r_1-l_1} \dots (c_4 g'' H_m^{\mu})^{r_m-l_m}.$$

Here $c_4 g'' \geqslant 1$ and hence

$$(c_4 g'')^{(r_1-l_1)+\dots+(r_m-l_m)} \leqslant (c_4 g'')^{r_1+\dots+r_m} \leqslant (c_4 g'')^{mr_1}.$$

Further, again by the definition of I and by the right-hand side of (14),

$$(H_1^{r_1-l_1} \dots H_m^{r_m-l_m})^{\mu} \leqslant H_1^{\mu(1+\epsilon)r_1 \sum_{h=1}^{m} \frac{r_h-l_h}{r_h}} = H_1^{\mu(1+\epsilon)S_3 r_1} .$$

Thus it follows that

$$D^{**} \leqslant (c_4 g'')^{mr_1} H_1^{\mu(1+\epsilon)S_3 r_1} .$$

Finally put

$$c_{11} = c_4 g'' \cdot c_{10},$$

and substitute in (37) the upper bounds for D^* and D^{**} just obtained. Since $|D_{(l)}| \leq D$, we so find the inequality

(38): $$|D_{(l)}| \leq c_{11}^{mr_1} H_1^{(1-\mu)(1+\epsilon)S_2r_1+\mu(1+\epsilon)S_3r_1}.$$

7. Lower bounds for $|N_{(l)}|$.

We again apply the equation (34) of the last section; by means of it, we shall determine integral powers N^* of g' and N^{**} of g that are divisors of $N_{(l)}$.

These two powers are relatively prime, so that their product likewise divides $N_{(l)}$. However, in certain cases it becomes necessary to take N^* or N^{**} equal to 1, and it may even be convenient to allow lower estimates for these numbers that are smaller than 1.

Assume for the moment that

$$0 \leq \lambda < 1.$$

Then, by the hypothesis (8), all numerators P_h are divisible by g', and hence all denominators Q_h are prime to g'. The first half

$$|P_h|_{g'} \leq c_3 H_h^{\lambda-1} \qquad (h = 1,2,\ldots, m)$$

of the property B implies that P_h is divisible by an integral power of g', P_h^* say, which is easily seen to satisfy the inequality

(39): $$P_h^* \geq (c_3 g' H_h^{\lambda-1})^{-1};$$

here $c_3 g' \geq 1$. On the other hand, the denominators

$$Q_1^{i_1-l_1}\ldots Q_m^{i_m-l_m}$$

of the terms of $A_{(l)}$ and hence also their least common denominator $D_{(l)}$ is relatively prime to g'. It follows that the numerator $N_{(l)}$ of $A_{(l)}$ is divisible by that power N^* of g' which is defined by

$$N^* = \gcd_{(i)\in I} P_1^{*i_1-l_1}\ldots P_m^{*i_m-l_m};$$

here the symbol "gcd" denotes the greatest common divisor. All products

$$P_1^{*i_1-l_1}\ldots P_m^{*i_m-l_m}$$

are, however, integral powers of g'. Their greatest common divisor is then equal to their minimum,

$$N^* = \min_{(i)\in I} P_1^{*i_1-l_1}\ldots P_m^{*i_m-l_m}.$$

It follows therefore from (39) that

$$N^* \geq \min_{(i)\in I}\left(\frac{1}{c_3 g'}\right)^{(i_1-l_1)+\ldots+(i_m-l_m)} \cdot \min_{(i)\in I}(H_1^{i_1-l_1}\ldots H_m^{i_m-l_m})^{1-\lambda}.$$

Here

$$\min_{(i)\in I}\left(\frac{1}{c_3 g'}\right)^{(i_1-l_1)+\ldots+(i_m-l_m)} \geqslant \left(\frac{1}{c_3 g'}\right)^{r_1+\ldots+r_m} \geqslant \left(\frac{1}{c_3 g'}\right)^{mr_1}.$$

Further, by the left-hand side of (14) and by the definition of I,

$$\min_{(i)\in I}\left(H_1^{i_1-l_1}\ldots H_m^{i_m-l_m}\right)^{1-\lambda} \geqslant \min_{(i)\in I} H_1^{(1-\lambda)r_1\sum_{h=1}^{m}\frac{i_h-l_h}{r_h}} \geqslant H_1^{(1-\lambda)S_1 r_1}.$$

Therefore, finally,

(40): $$N^* \geqslant \left(\frac{1}{c_3 g'}\right)^{mr_1} H_1^{(1-\lambda)S_1 r_1}.$$

In the case $\lambda=1$ so far excluded the right-hand side of this inequality does not exceed 1; *hence* (40) *remains valid without the restriction on* λ.

We put

(41): $$N^{**} = 1 \qquad \text{if } d=1.$$

Next let $d=2$ or $d=3$. Then $|A_{(l)}|_g$ possesses the upper bound (33). This upper bound implies that $N_{(l)}$ is divisible by an integral power of g, N^{**} say, which satisfies the inequality

(42): $$N^{**} \geqslant \left(g\cdot c_9^{mr_1} H_1^{-\sigma S_1 r_1}\right)^{-1} \geqslant \left(\frac{1}{c_9 g}\right)^{mr_1} H_1^{\sigma S_1 r_1} \quad \text{if } d=2 \text{ or } d=3.$$

First assume again that

$$0 \leqslant \lambda < 1.$$

As was shown in §1, g and g' are in this case relatively prime, and so the same is true for their powers N^{**} and N^*. Hence N^*N^{**} is a divisor of $N_{(l)}$, whence

$$|N_{(l)}| \geqslant N^* N^{**}.$$

This inequality still remains valid for $\lambda=1$ provided N^* is then replaced by its lower bound from (40).

Therefore, depending on the value of d, a lower bound for $|N_{(l)}|$ is given by the products of either the right-hand sides of (40) and (41), or the right-hand sides of (40) and (42). Hence, on introducing the new constant

$$c_{12} = c_3 g_9 g g',$$

we arrive at the following lower estimates,

(43): $$|N_{(l)}| \geqslant \begin{cases} (c_3 g')^{-mr_1} H_1^{(1-\lambda)S_1 r_1} & \text{for } d=1, \\ c_{12}^{-mr_1} H_1^{(1-\lambda+\sigma)S_1 r_1} & \text{for } d=2 \text{ or } d=3. \end{cases}$$

8. Conclusion of the proof of the Main Lemma.

Put

$$c_{13} = c_3 c_{11} g', \quad c_{14} = c_{11} c_{13}$$

and

$$C_1 = c_7 c_{13}, \qquad C_2 = c_6 c_{14}, \qquad C_3 = c_7 c_{14}.$$

The two inequalities (38) for $|D_{(l)}|$ and (43) for $|N_{(l)}|$ immediately lead to a lower bound for

$$|A_{(l)}| = \frac{|N_{(l)}|}{|D_{(l)}|}$$

which, naturally, depends on d. The result is as follows:

$$|A_{(l)}| \geqslant \begin{cases} c_{13}^{-mr_1} H_1^{(1-\lambda)S_1 r_1 - (1-\mu)(1+\epsilon)S_2 r_1 - \mu(1+\epsilon)S_3 r_1} & \text{for } d=1, \\ c_{14}^{-mr_1} H_1^{(1-\lambda+\sigma)S_1 r_1 - (1-\mu)(1+\epsilon)S_2 r_1 - \mu(1+\epsilon)S_3 r_1} & \text{for } d=2, \text{ or } d=3. \end{cases}$$

On the other hand, the formulae (25) and (30) asserted that

$$|A_{(l)}| \leqslant \begin{cases} c_7^{mr_1} H_1^{-\rho S_1 r_1} & \text{for } d=1 \text{ or } d=3, \\ c_6^{mr_1} & \text{for } d=2. \end{cases}$$

On combining these inequalities, it follows that

$$c_{13}^{-m} H_1^{(1-\lambda)S_1 - (1-\mu)(1+\epsilon)S_2 - \mu(1+\epsilon)S_3} \leqslant c_7^m H_1^{-\rho S_1} \qquad \text{for } d=1,$$

$$c_{14}^{-m} H_1^{(1-\lambda+\sigma)S_1 - (1-\mu)(1+\epsilon)S_2 - \mu(1+\epsilon)S_3} \leqslant c_6^m \qquad \text{for } d=2,$$

$$c_{14}^{-m} H_1^{(1-\lambda+\sigma)S_1 - (1-\mu)(1+\epsilon)S_2 - \mu(1+\epsilon)S_3} \leqslant c_7^m H_1^{-\rho S_1} \qquad \text{for } d=3.$$

These three formulae may be put into exactly the same form,

(44): $$H_1^{E_d} \leqslant C_d^m \qquad (d=1,2,3),$$

where, for shortness,

(45): $$E_d = (1-\lambda+\tau)S_1 - (1-\mu)(1+\epsilon)S_2 - \mu(1+\epsilon)S_3 \qquad (d=1,2,3),$$

and τ denotes the number which was defined in the Main Lemma. We can write the expression for E_d also as

$$E_d = \left(\frac{\tau-\lambda-\mu}{2} - \frac{1+\mu}{2}\epsilon\right) m - \left(\frac{2-\lambda-\mu+\tau}{2} + \frac{1-\mu}{2}\epsilon\right) s - (\tau-\lambda-\epsilon)\Lambda .$$

Here the coefficient of m is equal to

$$\frac{\tau-\lambda-\mu}{2} - \frac{1+\mu}{2}\epsilon \geqslant 2\epsilon - \frac{1+1}{2}\epsilon = \epsilon,$$

that of $-s$ is equal to

$$\frac{2-\lambda-\mu+\tau}{2} + \frac{1-\mu}{2}\epsilon \leq (1+2\epsilon) + \frac{1-0}{2}\epsilon < 1 + 4\epsilon \leq 2,$$

and that of $-\Lambda$ is equal to

$$\tau-\lambda-\epsilon = \mu+3\epsilon \leq 1+3\cdot\frac{1}{4} < 2;$$

here we have applied the former assumptions

$$\tau = \lambda+\mu+4\epsilon, \quad 0 < \epsilon \leq \frac{1}{4}, \quad 0 \leq \mu \leq 1.$$

It follows therefore in all three cases $d=1, 2$, or 3 that

$$E_d \geq \epsilon m - 2\cdot\frac{\epsilon m}{6} - 2\cdot\frac{\epsilon m}{6} = \frac{\epsilon m}{3}.$$

We finally choose the remaining constants T_d such that

$$T_d > C_d^{\frac{3}{\epsilon}} \qquad (d=1, 2, 3).$$

Since, by (10),

$$H_1 \geq T_d \qquad (d=1, 2, 3),$$

it follows that

$$H_1^{E_d} > \left(C_d^{\frac{3}{\epsilon}}\right)^{\frac{\epsilon m}{3}} = C_d^m \qquad (d=1, 2, 3),$$

contrary to (44).

This proves that the original hypothesis (3) leads to a contradiction and so shows that the Main Lemma is true.

9. The first form of the First Approximation Theorem.

It is now easy to deduce from the main lemma a more general result which we call the *First Approximation Theorem.* This theorem will be stated in two different forms which, however, are equivalent.

The first form of the theorem is nearly identical with the main lemma, except that the condition C is omitted.

First Approximation Theorem (I): *Let* $\xi \neq 0$ *be a real algebraic number and* $\Xi \leftrightarrow (\xi_1,\ldots,\xi_r)$, *where* $\xi_1 \neq 0,\ldots,\xi_r \neq 0$, *a* g-*adic algebraic number. Let* $\rho, \sigma, \lambda, \mu$ *be real constants satisfying*

$$\rho > 0, \quad \sigma > 0, \quad 0 \leq \lambda \leq 1, \quad 0 \leq \mu \leq 1;$$

let c_1, c_2, c_3, c_4 *be positive constants; and let* $g' \geq 2$ *and* $g'' \geq 2$ *be fixed integers. Finally let* $\Sigma = \{\kappa^{(1)}, \kappa^{(2)}, \kappa^{(3)},\ldots,\}$ *be an infinite sequence of distinct rational numbers*

$$\kappa^{(k)} = \frac{P^{(k)}}{Q^{(k)}} \neq 0, \text{ where } P^{(k)} \neq 0,\ Q^{(k)} \neq 0,\ (P^{(k)},Q^{(k)}) = 1,\ H^{(k)} = \max(|P^{(k)}|, |Q^{(k)}|),$$

with the following two properties.

(A_d): *For all* k,

$$|\kappa^{(k)} - \xi| \leqslant c_1 H^{(k)-\rho} \qquad \text{if } d=1,$$

$$|\kappa^{(k)} - \Xi|_g \leqslant c_2 H^{(k)-\sigma} \qquad \text{if } d=2,$$

$$|\kappa^{(k)} - \xi| \leqslant c_1 H^{(k)-\rho} \text{ and } |\kappa^{(k)} - \Xi|_g \leqslant c_2 H^{(k)-\sigma} \qquad \text{if } d=3.$$

(B): *For all* k,

$$|P^{(k)}|_{g'} \leqslant c_3 H^{(k)\lambda-1} \quad \text{and} \quad |Q^{(k)}|_{g''} \leqslant c_4 H^{(k)\mu-1}.$$

Then

$$\rho \leqslant \lambda+\mu \qquad \text{for } d=1,$$

$$\sigma \leqslant \lambda+\mu \qquad \text{for } d=2,$$

$$\rho+\sigma \leqslant \lambda+\mu \qquad \text{for } d=3.$$

Proof: We mentioned already in §1 that, for d=1 and d=3, a sequence Σ with the properties A_d and B has the real limit ξ and so possesses the third property C trivially. The assertion is therefore in these two cases contained in the main lemma. There remains then only the case d=2 in which the assertion has yet to be proved.

First assume that Σ contains an infinite subsequence

$$\Sigma_1 = \{\kappa^{(i_1)}, \kappa^{(i_2)}, \kappa^{(i_3)}, \ldots\}, \quad \text{where } i_1 < i_2 < i_3 < \ldots,$$

such that, for all k and some positive constant c_5,

$$|\kappa^{(i_k)}| \leqslant c_5.$$

The main lemma may then be applied to Σ_1 and gives the assertion.

Secondly let Σ contain no such subsequence Σ_1. Then

$$\lim_{k\to\infty} |\kappa^{(k)}| = \infty,$$

and hence the sequence of the reciprocals

$$\Sigma_0 = \{\kappa_0^{(1)}, \kappa_0^{(2)}, \kappa_0^{(3)}, \ldots\}, \qquad \text{where} \qquad \kappa_0^{(k)} = \kappa^{(k)-1} = \frac{Q^{(k)}}{P^{(k)}},$$

has the property C,

$$|\kappa_0^{(k)}| \leqslant c_5' \qquad (k = 1,2,3,\ldots)$$

for some positive constant c_5'. It is obvious that $\kappa^{(k)}$ and $\kappa_0^{(k)}$ are of the same height $H^{(k)}$. Hence Σ_0 has also the property B_0 which is analogous to B, except that λ and μ, and also c_3 and c_4, are interchanged.

We finally show that Σ_0 has the property A_2. By hypothesis, Σ has this property, (46): $$|\kappa^{(k)} - \Xi|_g \leqslant c_2 H^{(k)-\sigma} \qquad (k = 1,2,3,\ldots).$$

Therefore Σ has the g-adic limit Ξ, and so, for $j=1,2,\ldots,r$, this sequence has also the p_j-adic limits ξ_j. But then the reciprocal sequence Σ_0 has for all j the p_j-adic limits ξ_j^{-1}, and hence Σ_0 has also the g-adic limit

$$\Xi^{-1} \leftrightarrow (\xi_1^{-1}, \ldots, \xi_r^{-1}).$$

Therefore, in particular,

$$\lim_{k\to\infty} |\kappa_0^{(k)}|_g = |\Xi^{-1}|_g,$$

and hence there is a positive constant c_0 such that

$$|\kappa_0^{(k)}|_g \leqslant c_0 \qquad \text{for all } k.$$

From (46) and the identity

$$\kappa_0^{(k)} - \Xi^{-1} = -\kappa_0^{(k)} \Xi^{-1} (\kappa^{(k)} - \Xi)$$

it follows finally that

$$|\kappa_0^{(k)} - \Xi^{-1}|_g \leqslant c_2' H^{(k)-\sigma}, \quad \text{where} \quad c_2' = c_0 |\Xi^{-1}|_g c_2.$$

We apply now the main lemma to the sequence Σ_0 instead of Σ and find that $\sigma \leqslant \mu + \lambda$, giving the assertion.

10. Polynomials in a field with a valuation.

The second form of the First Approximation Theorem makes a statement on the values of a polynomial assumed in a sequence of rational numbers. Before enunciating and proving this theorem, it is necessary to discuss first a property of fields with a valuation.

Let K be a field with a valuation $w(a)$, and let K_w again be the completion of K with respect to w. We say that K has the property D if the following compactness condition is satisfied:

(D): *Every infinite sequence of elements of* K *that is bounded with respect to* w *contains an infinite subsequence which is a fundamental sequence with respect to* w, *hence has a limit in* K_w.

Let K have this property D, and let

$$F(x) = F_0 x^f + F_1 x^{f-1} + \ldots + F_f, \quad \text{where} \quad f \geqslant 1,\ F_0 \neq 0,$$

be a polynomial with coefficients in K which has no multiple zero in K_w. Put

$$G(x) = F_0^{-1} F(x) = x^f + G_1 x^{f-1} + G_2 x^{f-2} + \ldots + G_f, \quad \gamma = 1 + w(G_1) + w(G_2) + \ldots + w(G_f),$$

so that

$$\gamma \geqslant 1,$$

and also $G(x)$ has no multiple zero in K_w. Assume now that x is an element of K_w such that

$$w(x) > \gamma \qquad \text{and hence} \qquad w(x) > 1.$$

Then

$$w(G_1x^{f-1}+G_2x^{f-2}+\ldots+G_f) \leqslant (\gamma-1)\max(w(x)^{f-1}, w(x)^{f-2}, \ldots, w(x), 1) \leqslant$$
$$\leqslant (\gamma-1)w(x)^{f-1},$$

and therefore

$$w(G(x) \geqslant w(x^f) - w(G_1x^{f-1}+G_2x^{f-2}+\ldots+G_f) \geqslant w(x)^f-(\gamma-1)w(x)^{f-1} =$$
$$= \{1 + (w(x)-\gamma)\}w(x)^{f-1} > 1.$$

Conversely, it follows that

$$\textit{if} \quad w(F(x)) \leqslant w(F_0), \quad \textit{then} \quad w(x) \leqslant \gamma,$$

because the first inequality implies that

$$w(G(x)) = w(F_0^{-1}F(x)) \leqslant w(F_0^{-1})w(F_0) = 1.$$

Consider now an infinite sequence $\Sigma = \{\kappa^{(1)}, \kappa^{(2)}, \kappa^{(3)}, \ldots\}$ of elements of K satisfying

$$\lim_{k\to\infty} w(F(\kappa^{(k)})) = 0.$$

This assumption implies that

$$w(F(\kappa^{(k)})) \leqslant w(F_0) \qquad \text{and hence} \qquad w(\kappa^{(k)}) \leqslant \gamma$$

for all sufficiently large k. Thus the sequence Σ is bounded with respect to w and so, by the property D of K, it contains an infinite subsequence $\Sigma' = \{\kappa^{(i_1)}, \kappa^{(i_2)}, \kappa^{(i_3)}, \ldots\}$, where $i_1 < i_2 < i_3 < \ldots$, which is a fundamental sequence with respect to w and so has a limit

$$\lim_{k\to\infty} \kappa^{(i_k)}\,(w), \qquad = \xi \qquad \text{say,}$$

in K_w. However, polynomials in $K[x]$ are continuous functions with respect to the metric on K defined by w, and therefore

$$F(\xi) = \lim_{k\to\infty} F(\kappa^{(i_k)}) = \lim_{k\to\infty} F(\kappa^{(k)}) = 0.$$

This means that ξ is a zero of $F(x)$, hence that $F(x)$ is divisible by the linear polynomial $x-\xi$,

$$F(x) = (x-\xi)F_1(x)$$

where $F_1(x)$ is a polynomial with coefficients in K_w. It is obvious that

$$F_1(\xi) \neq 0$$

because otherwise ξ would be a multiple zero of $F(x)$. From the continuity of the polynomial $F_1(x)$, it follows that

$$\lim_{k\to\infty} F_1(\kappa^{(i_k)}) = F_1(\xi) \neq 0 \quad (w),$$

and hence that

$$F_1(\kappa^{(i_k)}) \neq 0 \qquad \text{for all sufficiently large } k.$$

There is no loss of generality in assuming that this inequality holds for all suffixes k. Hence a positive constant γ_1 exists such that

$$w(F_1(\kappa^{(i_k)})) \geq \gamma_1^{-1} \qquad (k = 1,2,3,\ldots).$$

The equation

$$F(\kappa^{(i_k)}) = (\kappa^{(i_k)} - \xi)F_1(\kappa^{(i_k)})$$

leads therefore at once to the following result.

Lemma 1: *Assume that* K *has the property* D. *Let* $F(x)$ *be a polynomial in* $K[x]$ *which has no multiple zeros in* K_w, *and let* $\Sigma = \{\kappa^{(1)}, \kappa^{(2)}, \kappa^{(3)},\ldots\}$ *be an infinite sequence in* K *such that*

$$\lim_{k\to\infty} w(F(\kappa^{(k)})) = 0.$$

There exist an infinite subsequence $\Sigma' = \{\kappa^{(i_1)}, \kappa^{(i_2)}, \kappa^{(i_3)},\ldots\}$ *of* Σ, *a zero* ξ *of* $F(x)$ *in* K_w, *and a constant* $\gamma_1 > 0$ *such that*

$$w(\kappa^{(i_k)} - \xi) \leq \gamma_1 w(F(\kappa^{(i_k)})) \qquad (k = 1,2,3,\ldots).$$

11. Two applications of Lemma 1.

In Lemma 1 choose for K the rational field Γ and for $w(a)$ either the absolute value $|a|$ or any p-adic value $|a|_p$ where p is an arbitrary prime. The completion K_w becomes then either the real field, or the p-adic field.

Both the real field and every p-adic field have the compactness property D[1]. Hence the following two results are contained in Lemma 1.

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[1] This is a classical theorem in the real case, and it may be proved in the p-adic case as follows.

Denote by $\Sigma = \{\kappa^{(1)}, \kappa^{(2)}, \kappa^{(3)}, \ldots\}$ any bounded sequence of p-adic numbers. Let its elements, without loss of generality, be p-adic integers; thus they can be written as series

$$\kappa^{(k)} = a_0^{(k)} + a_1^{(k)} p + a_2^{(k)} p^2 + \ldots \ (p) \qquad (k = 1,2,3,\ldots)$$

Lemma 2: *Let* F(x) *be a polynomial with rational coefficients which has no multiple zeros, and let* $\Sigma = \{\kappa^{(1)}, \kappa^{(2)}, \kappa^{(3)}, \ldots\}$ *be an infinite sequence of rational numbers such that*

$$\lim_{k\to\infty} |F(\kappa^{(k)})| = 0.$$

There exist an infinite subsequence $\Sigma' = \{\kappa^{(i_1)}, \kappa^{(i_2)}, \kappa^{(i_3)}, \ldots\}$ *of* Σ, *a real zero* ξ *of* F(x), *and a constant* $\gamma_1 > 0$, *such that*

$$|\kappa^{(i_k)} - \xi| \leq \gamma_1 |F(\kappa^{(i_k)})| \qquad \text{for all } k.$$

Lemma 2': *Let* F(x) *be as in Lemma 2, and let* $\Sigma = \{\kappa^{(1)}, \kappa^{(2)}, \kappa^{(3)}, \ldots\}$ *be an infinite sequence of rational numbers such that*

$$\lim_{k\to\infty} |F(\kappa^{(k)})|_p = 0.$$

There exist an infinite subsequence $\Sigma'' = \{\kappa^{(j_1)}, \kappa^{(j_2)}, \kappa^{(j_3)}, \ldots\}$ *of* Σ, *a* p-*adic zero* ξ_p *of* F(x), *and a constant* $\gamma_2 > 0$, *such that*

$$|\kappa^{(j_k)} - \xi_p|_p \leq \gamma_2 |F(\kappa^{(j_k)})|_p \qquad \text{for all } k.$$

The second lemma may be extended to g-adic values and g-adic numbers,

Lemma 3: *Let* F(x) *be as in Lemma 2, and let* $\Sigma = \{\kappa^{(1)}, \kappa^{(2)}, \kappa^{(3)}, \ldots\}$ *be an infinite sequence of rational numbers such that*

$$\lim_{k\to\infty} |F(\kappa^{(k)})|_g = 0.$$

There exist an infinite subsequence $\Sigma''' = \{\kappa^{(h_1)}, \kappa^{(h_2)}, \kappa^{(h_3)}, \ldots\}$ *of* Σ, *a* g-*adic zero* Ξ *of* F(x), *and a constant* $\gamma_3 > 0$, *such that*

$$|\kappa^{(h_k)} - \Xi|_g \leq \gamma_3 |F(\kappa^{(h_k)})|_g \qquad \text{for all } k.$$

Proof: From the hypothesis and the definition of the g-adic value, it follows that also

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where the digits $a_n^{(k)}$ assume only the values o, 1,. . ., p−1. The set of the first n digits of each $\kappa^{(k)}$ has thus only p^n possibilities. It follows that it is possible to select successively

an infinite subsequence $\Sigma_1 = \{\kappa_1^{(1)}, \kappa_1^{(2)}, \kappa_1^{(3)}, \ldots\}$ of Σ,

an infinite subsequence $\Sigma_2 = \{\kappa_2^{(1)}, \kappa_2^{(2)}, \kappa_2^{(3)}, \ldots\}$ of Σ_1,

an infinite subsequence $\Sigma_3 = \{\kappa_3^{(1)}, \kappa_3^{(2)}, \kappa_3^{(3)}, \ldots\}$ of Σ_2, etc.,

such that, for every n, the n first digits of all elements of Σ_n are identical. The diagonal sequence $\Sigma' = \{\kappa_1^{(1)}, \kappa_2^{(2)}, \kappa_3^{(3)}, \ldots\}$ is still a subsequence of Σ, and it has the property that, for every n, the first n digits of all but finitely many of its elements are identical. Hence Σ' is a fundamental sequence, as asserted.

$$\lim_{k\to\infty} |F(\kappa^{(k)})|_{p_j} = 0 \qquad (j = 1,2,\ldots, r).$$

We now apply Lemma 2' repeatedly, once for each prime factor p_j of g. First, there exist an infinite subsequence $\Sigma_1 = \{\kappa^{(h_{11})}, \kappa^{(h_{12})}, \kappa^{(h_{13})},\ldots\}$ of Σ, a p_1-adic zero ξ_1 of $F(x)$, and a constant $\gamma^{(1)} > 0$, such that

$$|\kappa^{(h_{1k})} - \xi_1|_{p_1} \leq \gamma^{(1)} |F(\kappa^{(h_{1k})})|_{p_1} \qquad \text{for all } k.$$

Secondly, there exist an infinite subsequence $\Sigma_2 = \{\kappa^{(h_{21})}, \kappa^{(h_{22})}, \kappa^{(h_{23})},\ldots\}$ of Σ_1, a p_2-adic zero ξ_2 of $F(x)$, and a constant $\gamma^{(2)} > 0$, such that

$$|\kappa^{(h_{2k})} - \xi_2|_{p_2} \leq \gamma^{(2)} |F(\kappa^{(h_{2k})})|_{p_2} \qquad \text{for all } k,$$

while, naturally, also

$$|\kappa^{(h_{2k})} - \xi_1|_{p_1} \leq \gamma^{(1)} |F(\kappa^{(h_{2k})})|_{p_1} \qquad \text{for all } k.$$

Continuing in this manner, we obtain for every suffix $j=1, 2, \ldots r$ an infinite sequence $\Sigma_j = \{\kappa^{(h_{j1})}, \kappa^{(h_{j2})}, \kappa^{(h_{j3})},\ldots\}$, where

$$\Sigma_1 \supseteq \Sigma_2 \supseteq \ldots \supseteq \Sigma_r,$$

a p_j-adic zero ξ_j of $F(x)$, and a constant $\gamma_j > 0$, such that

$$|\kappa^{(h_{jk})} - \xi_i|_{p_i} \leq \gamma^{(i)} |F(\kappa^{(h_{jk})})|_{p_i} \qquad \text{for } i=1, 2,\ldots, j \text{ and for all } k.$$

Let Σ''' be the sequence Σ_r; further put

$$\gamma_3 = \max\left(\gamma^{(1)\frac{\log g}{e_1 \log p_1}}, \ldots, \gamma^{(r)\frac{\log g}{e_r \log p_r}}\right),$$

and denote by Ξ the g-adic number

$$\Xi \longleftrightarrow (\xi_1,\ldots, \xi_r),$$

which is algebraic and a zero of $F(x)$. We have then

$$\max_{i=1,2,\ldots,r} \left(|\kappa^{(h_{rk})} - \xi_i|_{p_i}^{\frac{\log g}{e_i \log p_i}}\right) \leq \max_{i=1,2,\ldots,r} \left\{\left(\gamma^{(i)} |F(\kappa^{(h_{rk})})|_{p_i}\right)^{\frac{\log g}{e_i \log p_i}}\right\}$$

for all k

and hence

$$|\kappa^{(h_{rk})} - \Xi|_g \leq \gamma_3 |F(\kappa^{(h_{rk})})|_g \qquad \text{for all } k,$$

whence the assertion.

12. The property A'_d.

As earlier in this chapter, let again

$$F(x) = F_0x^f + F_1x^{f-1} + \dots + F_f, \qquad \text{where} \qquad f \geq 1,\ F_0 \neq 0,\ F_f \neq 0,$$

be a polynomial with integral coefficients which does not vanish at $x=0$ and has no multiple factors, hence also no multiple zeros in any extension field of the rational field. Further denote again by ξ a real zero and by Ξ a g-adic zero of $F(x)$, and by ρ and σ two positive constants. Finally let again $\Sigma = \{\kappa^{(1)}, \kappa^{(2)}, \kappa^{(3)}, \dots\}$ be a sequence of distinct rational numbers

$$\kappa^{(k)} = \frac{P^{(k)}}{Q^{(k)}} \neq 0 \qquad \text{of heights} \qquad H^{(k)} = \max(|P^{(k)}|,\ |Q^{(k)}|)$$

such that

$$P^{(k)} \neq 0,\ Q^{(k)} \neq 0,\ (P^{(k)}, Q^{(k)}) = 1.$$

For $d=1, 2,$ or 3, we define a property A'_d of Σ as follows.

The sequence Σ is said to have the property A'_d if for $d=1$: There exist two positive constants ρ and c'_1 such that

$$(A'_1): \qquad |F(\kappa^{(k)})| \leq c'_1 H^{(k)-\rho} \qquad \text{for all } k;$$

for $d=2$: There exist two positive constants σ and c'_2 such that

$$(A'_2): \qquad |F(\kappa^{(k)})|_g \leq c'_2 H^{(k)-\sigma} \qquad \text{for all } k; \text{ and}$$

for $d=3$: There exist four positive constants ρ, σ, c'_1, and c'_2 such that

$$(A'_3): \qquad |F(\kappa^{(k)})| \leq c'_1 H^{(k)-\rho} \quad \text{and} \quad |F(\kappa^{(k)})|_g \leq c'_2 H^{(k)-\sigma} \qquad \text{for all } k.$$

The property A'_3 includes therefore both properties A'_1 and A'_2.

The two properties A_d and A'_d are closely connected, as the following lemma shows.

Lemma 4: *If the sequence Σ has the property A_d with respect to ξ, or Ξ, or ξ and Ξ, then it also has the property A'_d with respect to $F(x)$. Conversely, if Σ has the property A'_d with respect to $F(x)$, then there exist an infinite subsequence Σ' of Σ and either a real zero ξ of $F(x)$, or a g-adic zero Ξ of $F(x)$, or both, such that Σ' has the property A_d with respect to ξ, or to Ξ, or to both ξ and Ξ.*

Proof: First let Σ has the property A_d. The quotient

$$\Phi(x,y) = \frac{F(x)-F(y)}{x-y}$$

is a polynomial in x and y with integral coefficients. Evidently

$$|F(\kappa^{(k)})| = |\kappa^{(k)}-\xi|\ |\Phi(\kappa^{(k)},\xi)| \qquad \text{if } d=1 \text{ or } 3,$$

$$|F(\kappa^{(k)})|_g = |\kappa^{(k)}-\Xi|_g |\Phi(\kappa^{(k)}, \Xi)|_g \qquad \text{if } d=2 \text{ or } 3.$$

Further, by the hypothesis,

$$\kappa^{(k)} \text{ has the real limit } \xi \qquad \text{if } d=1 \text{ or } 3,$$

$$\kappa^{(k)} \text{ has the g-adic limit } \Xi \qquad \text{if } d=2 \text{ or } 3.$$

This means that Σ is a bounded sequence with respect to the absolute or g-adic values, and hence that the numbers

$$|\Phi(\kappa^{(k)},\xi)| \quad \text{for } d=1 \text{ or } 3, \text{ and } \quad |\Phi(\kappa^{(k)},\Xi)|_g \quad \text{for } d=2 \text{ or } 3$$

are bounded. Let their upper bounds by Γ_1 and Γ_2, respectively; it follows then that Σ has the property A_d' with the constants

$$c_1' = c_1\Gamma_1 \qquad \text{and} \qquad c_2' = c_2\Gamma_2,$$

respectively.

Secondly let Σ have the property A_d'. If d=1 or d=2, the assertion is contained in Lemmas 2 and 3, respectively. If, however, d=3, both lemmas must be applied one after the other. First, by Lemma 2, there is a real zero ξ of F(x) and a subsequence Σ_1 of Σ which has the property A_1 with respect to ξ. Secondly, by Lemma 3, there exists also a g-adic zero Ξ of F(x) and a subsequence Σ' of Σ_1 which has the property A_2 with respect to Ξ. Since Σ' still has the property A_1 with respect to ξ, it has then the property A_3 with respect to both ξ and Ξ, whence the assertion.

13. The second form of the First Approximation Theorem.

By combining the lemma just proved with the first form of the First Approximation Theorem we immediately obtain the following second form of the theorem.

First Approximation Theorem (II): *Let* F(x) *be a polynomial with integral coefficients which does not vanish for* x=0 *and has no multiple factors. Let* ρ, σ, λ, μ *be real constants satisfying*

$$\rho > 0, \quad \sigma > 0, \quad 0 \leq \lambda \leq 1, \quad 0 \leq \mu \leq 1;$$

let c_1', c_2', c_3, c_4 *be positive constants; and let* $g' \geq 2$ *and* $g'' \geq 2$ *be fixed integers. Finally let* $\Sigma = \{\kappa^{(1)}, \kappa^{(2)}, \kappa^{(3)},\ldots\}$ *be an infinite sequence of distinct rational numbers*

$$\kappa^{(k)} = \frac{P^{(k)}}{Q^{(k)}} \neq 0, \quad \textit{where} \quad P^{(k)} \neq 0,\ Q^{(k)} \neq 0,\ (P^{(k)},Q^{(k)}) = 1,$$

$$H^{(k)} = \max(|P^{(k)}|,\ |Q^{(k)}|),$$

with the following two properties.

(A_d'): *For all* k,

$$|F(\kappa^{(k)})| \leq c_1' H^{(k)-\rho} \qquad \textit{if } d=1,$$

$$|F(\kappa^{(k)})|_g \leq c_2' H^{(k)-\sigma} \qquad \textit{if } d=2,$$

$|F(\kappa^{(k)})| \leq c_1' H^{(k)-\rho}$ *and* $|F(\kappa^{(k)})|_g \leq c_2' H^{(k)-\sigma}$ if d=3.

(B): *For all* k,

$|P^{(k)}|_{g'} \leq c_3 H^{(k)\lambda-1}$ *and* $|Q^{(k)}|_{g''} \leq c_4 H^{(k)\mu-1}$.

Then

$$\rho \leq \lambda + \mu \quad \text{for } d=1,$$
$$\sigma \leq \lambda + \mu \quad \text{for } d=2,$$
$$\rho+\sigma \leq \lambda + \mu \quad \text{for } d=3.$$

Proof: It suffices to apply the first form of the theorem to the sequence Σ' and the zero or zeros ξ, Ξ obtained by Lemma 4.- By the same lemma, the new second form of the theorem implies also the original first form; both forms are thus equivalent.