Notre Dame Journal of Formal Logic
Volume 52, Number 3, 2011

Automorphisms of Saturated and Boundedly Saturated Models of Arithmetic

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Abstract We discuss automorphisms of saturated models of PA and boundedly saturated models of PA. We show that Smoryński's Lemma and Kaye's Theorem are not only true for countable recursively saturated models of PA but also true for all boundedly saturated models of PA with slight modifications.

1 Introduction

In this paper our main interests are the study of groups of automorphisms of saturated and boundedly saturated models of Peano Arithmetic. Throughout this paper we will assume the existence of such structures. We will show that Smoryński's Lemma and Kaye's Theorem are true for all boundedly saturated (and in particular all saturated) models of PA. We begin by showing that these results are true for saturated models of PA (these are generalizations of results from [9]). We then show that any automorphism of a boundedly saturated model which is not saturated can be extended to an automorphism of the saturated end extension of the model. That allows us to apply the results proved for saturated models to boundedly saturated models as well.

In Section 3 for the saturated case, and in Section 6 for the boundedly saturated case, we prove the following analogue of Smoryński's Lemma [13].

Theorem 1.1 *Let M be a (boundedly) saturated model of Peano Arithmetic of cardinality λ. A cut $I \subset M$ is of the form $I_{\text{fix}}(f)$ for some $f \in \mathrm{Aut}(M)$ if and only if $I \subset M$ is an exponentially closed cut and* $\mathrm{dcf}(I) = \lambda$.

In Section 4 for saturated models, in Section 6 for boundedly saturated models which are not short, and in Section 7 for short saturated models, we prove analogues of Kaye's Theorem [4] which together give the following.

Received December 6, 2010; accepted February 22, 2011; printed July 25, 2011
2010 Mathematics Subject Classification: Primary, 03C62
Keywords: saturation, bounded saturation, automorphism groups, models of PA
 10.1215/00294527-1435483

Theorem 1.2 *Let M be a (boundedly) saturated model of Peano Arithmetic. Then H is a closed normal subgroup of* $\mathrm{Aut}(M)$ *if and only if there exists an invariant cut $I \subseteq M$ such that $H = \mathrm{Aut}(M)_{(I)}$.*

From the last theorem we obtain two important corollaries concerning automorphism groups of boundedly saturated models of PA.

Corollary 1.3 *Let M_1, M_2 be saturated models of Peano Arithmetic such that $M_1 \models \mathsf{TA}$ and $M_2 \not\models \mathsf{TA}$. Then their automorphism groups are nonisomorphic as topological groups.*

The above result is also true when in the statement above "saturated models" is replaced with "boundedly saturated models which are not short" (see Corollary 6.7).

Corollary 1.4 *Let M be a saturated model of Peano Arithmetic. Then there are short saturated elementary initial segments of M whose automorphism groups are nonisomorphic as topological groups.*

Remark 1.5 The topology used for the automorphism groups is the topology with stabilizers of finite subsets of the model as basic open sets. The first author has recently shown (yet to appear) that Corollary 1.3 is also true for saturated models with the finer topology with stabilizers of subsets of cardinality less than that of the model as basic open sets. The significance in showing this is that the finer topology has the small index property, which implies that the automorphism groups of the models in the corollary are nonisomorphic as abstract groups.

2 Basic Theorems and Definitions

We will assume that the reader is familiar with the basic facts about models of Peano Arithmetic. The book of Kaye [3] can be used to review them.

Throughout the paper, M is a model of Peano Arithmetic (PA). Let $A \subseteq B \subseteq M$. We define the following notion:

$$A \subseteq_e B \iff \forall x \in A \forall y \in B (y < x \to y \in A).$$

If $A \subseteq_e B$ we call A an *initial segment of* B, and B an *end extension of* A. A set $I \subseteq_e M$ is called a *cut* if $I \neq \varnothing$ and I is closed under the successor function.

For $A \subseteq M$ we define

$$\sup A = \{x \in M : \exists y \in A (x \leq y)\}$$

and

$$\inf A = \{x \in M : \forall y \in A (x < y)\}.$$

The standard cut will be denoted by ω. If $a \in M \models \mathsf{PA}$, then a codes the sequence $(a)_0, (a)_1, \ldots$. True Arithmetic (TA) is $\mathrm{Th}(\mathbb{N})$.

If M is a model of PA, we write $\mathrm{Aut}(M)$ for the automorphism group of M. A model M is called *recursively saturated* if it realizes all recursive types $p(x, a)$, $a \in M$. Countable recursively saturated models of PA are *ω-homogeneous* as expressed in the following lemma.

Lemma 2.1 *If M is a countable recursively saturated model of Peano Arithmetic, then two elements of M belong to the same orbit of* $\mathrm{Aut}(M)$ *if and only if they realize the same complete type.*

We say that a type over a model M is *bounded* if it contains the formula $v < a$ for some $a \in M$.

Definition 2.2 A model M is *saturated* if and only if for every $A \subseteq M$ with $|A| < |M|$, M realizes every type over A. A model M is *boundedly saturated* if and only if for every $A \subseteq M$ with $|A| < |M|$, M realizes every bounded type over A.

Clearly, every saturated model of PA is boundedly saturated. Let $M \models \mathsf{PA}$, $a \in M$. We define

$$\text{gap}(a) = \{b \in M : \text{ there is a Skolem term } t(x) \text{ such that } a \leq t(b) \text{ and } b \leq t(a)\}.$$

The Moving Gaps Lemma [5] is true for all recursively saturated models and hence for all saturated models.

Lemma 2.3 (Moving Gaps Lemma) *Let M be a (recursively) saturated model of* PA. *Suppose that $g \in \text{Aut}(M)$ and $a, b, d \in M$ are such that $g(a) \neq a < \text{gap}(b) < \text{gap}(d)$. Then there exists $c \in M$ such that $\text{gap}(b) < c < \text{gap}(d)$ and $g(c) \notin \text{gap}(c)$.*

Let $M(a) = \sup(\text{gap}(a))$. Notice that $M(a)$ is the smallest elementary initial segment of M containing a.

Definition 2.4 If $M = M(a)$ for some $a \in M$ then M is said to be *short*. If M is boundedly saturated and short then we say that M is *short saturated*.

Notice that when $M \models \mathsf{TA}$ then $\mathbb{N}$ is short saturated. We will frequently use the fact that most results concerning models of Peano Arithmetic remain true if we expand the language by a finite number of constants.

3 Initial Segments

In this section, we prove an analogue of Smoryński's Lemma. Let $M \models \mathsf{PA}$. If $g \in \text{Aut}(M)$, then

$$I_{\text{fix}}(g) = \{x \in M : \forall y < x(g(y) = y)\}.$$

Thus $I_{\text{fix}}(g)$ is the largest cut pointwise fixed by g. We say that $I \subseteq_e M$ is an *exponentially closed cut* if whenever $a \in I$ then $2^a \in I$.

Lemma 3.1 ([5]) *Let M be a model of Peano Arithmetic. If $I = I_{\text{fix}}(g)$ for some $g \in \text{Aut}(M)$, then I is an exponentially closed cut.*

Smoryński [13] proved the converse of Lemma 3.1 for countable recursively saturated models.

Lemma 3.2 (Smoryński's Lemma [13]) *Let M be a countable recursively saturated model of Peano Arithmetic and let $I \subseteq_e M$ be a cut. If I is an exponentially closed cut then there is $g \in \text{Aut}(M)$ such that $I_{\text{fix}}(g) = I$.*

The proof of Smoryński's Lemma is based on the following two propositions.

Proposition 3.3 ([5], [8], [13]) *Let M be recursively saturated and let $a, b, c \in M$ be such that for all $x < 2^{2^c}$: $(M, x, a) \equiv (M, x, b)$. Then for each a' there is b' such that, for all $x < c$, $(M, x, a, a') \equiv (M, x, b, b')$.*

Proposition 3.4 ([13]) *Let M be recursively saturated. Then for every $a \in M$ and every nondefinable d there are $b, c < 2^{d^2}$ such that $b \neq c$ and for all $x < d$, $(M, x, a, b) \equiv (M, x, a, c)$.*

Recall from [10] that when $M \models \mathsf{PA}$ is saturated and $|M| = \lambda$ then λ is regular and $2^\kappa \leq \lambda$ whenever $\kappa < \lambda$. For the rest of this section we fix M to be a saturated model of Peano Arithmetic of cardinality λ and $G = \mathrm{Aut}(M)$.

Let $I \subset M$ be a cut. By $\mathrm{dcf}(I)$ we denote the least cardinality κ of a downward cofinal set $A \subset M \setminus I$. By $\mathrm{cf}(I)$ we denote the least cardinality μ of a cofinal set $B \subset I$.

Theorem 3.5 *A cut $I \subset M$ is of the form $I_{\mathrm{fix}}(f)$ for some $f \in G$ if and only if I is exponentially closed and* $\mathrm{dcf}(I) = \lambda$.

Proof $\Longrightarrow$ First, by Lemma 3.1, I is closed under exponentiation. Now we prove that $\mathrm{dcf}(I) = \lambda$. We notice that if $I = \omega$ then $\mathrm{dcf}(I) = \lambda$. Assume that $\mathrm{dcf}(I) \neq \lambda$ (in particular, $I \neq \omega$). Then there is a set $A \subset M \setminus I$ which is downward cofinal in $M \setminus I$ and $\mathrm{card}(A) < \lambda$. We can choose A in such a way that, for every $a' \in A$, $f(a') \neq a'$. Because $\mathrm{card}(A) < \lambda$, by the saturation of M there exist $c \in I$ and $a \in M$ such that for every $a' \in A$ there is $n < c$ with $(a)_n = a'$. Using saturation again we can assume that $(a)_i \geq (a)_j$ when $i < j < c$. Define

$$J = \sup\{n : n < c \text{ and } (a)_n \in A\}.$$

Let $b = f(a)$. By the assumptions on a, b, c, I, and J we have $(b)_i = f((a)_i) = (a)_i$ when $c > i > J$. Also for every $i \in J$ there is $j < c$ such that $i \leq j$ and $(a)_j \in A$; hence $(b)_j = f((a)_j) \neq (a)_j$. Thus J is definable from a, b, and c:

$$n \in J \Longleftrightarrow n < c \text{ and there exists } j < c \text{ such that } n \leq j \text{ and } (a)_j \neq (b)_j.$$

Define m to be the least element such that $m \notin J$. Then $m - 1$ is the largest element in J, and $(a)_{m-1}$ is the least element in A. Hence $(a)_{m-1} \notin I$ and $(a)_{m-1} - 1 \in I$, which is impossible.

$\Longleftarrow$ We will prove this direction by a back-and-forth argument. Order M in order type λ (that is, let $M = \{m_j : j < \lambda\}$). We construct a sequence $B_i = \{b_k : k < i\}$ and $C_i = \{c_k : k < i\}$ such that at each step we have $a_i > I$ with the property,

$$\text{for all formulas } \theta,\ \text{all } \bar{b} \in B_i, \bar{c} \in C_i\ M \models \forall x < a_i(\theta(x, \bar{b}) \leftrightarrow \theta(x, \bar{c})).$$

For the forth direction at any nonlimit step, let $b_i = m_j$, where j is the least one with m_j in $M \backslash B_i$. By Proposition 3.3, for every $\bar{b} \in B_i$ with corresponding $\bar{c} \in C_i$ there exists $c' \in M$ such that for all formulas $\theta(x, \bar{y}, y)$,

$$M \models \forall x < a_{i+1}(\theta(x, \bar{b}, b_i) \leftrightarrow \theta(x, \bar{c}, c_i)),$$

where $a_{i+1} = \log_2(\log_2(a_i)) > I$. Since M is saturated there exists c_i such that for all formulas θ and for all $\bar{b} \in B_i, \bar{c} \in C_i$,

$$M \models \forall x < a_{i+1}(\theta(x, \bar{b}, b_i) \leftrightarrow \theta(x, \bar{c}, c_i)).$$

Similarly the back direction.

At limit step define $B_i = \bigcup_{j<i} B_j$, $C_i = \bigcup_{j<i} C_j$ and because $\mathrm{dcf}(I) = \lambda$ we have $a > I$ with the property, for all formulas θ, all $\bar{b} \in B_i, \bar{c} \in C_i$

$$M \models \forall x < a(\theta(x, \bar{b}) \leftrightarrow \theta(x, \bar{c})).$$

To guarantee that $I_{\text{fix}}(f) = I$ add one more step. Given B_i, C_i consider any $d > I$. We show how to find $b_i \leq d$: $b_i \notin I_{\text{fix}}(f)$. By Proposition 3.4 and using saturation of M, we can find $a' > I, b'', b' < d, b'' \neq b'$ such that for all $x < a'$, $(M, x, B_i, b') \equiv (M, x, B_i, b'')$. By Proposition 3.3 and using saturation of M we can find $a'' > I$ and c_i such that, for all $x < a''$, $(M, x, B_i, b') \equiv (M, x, C_i, c_i)$. Now let a_i be the smaller of a', a'', and pick $b_i = b'$ if $b' \neq c_i$; otherwise choose $b_i = b''$. □

One can notice that we actually proved a little stronger result.

Lemma 3.6 *Let $I \subset M$ be an exponentially closed cut such that dcf$(I) = \lambda$, let $A \subseteq M$ have cardinality less than λ, and let $h : A \to M$ be such that $(M, x, a)_{a \in A} \equiv (M, x, h(a))_{a \in A}$ for all $x \in I$. Then there is $f \in G$ such that $f \supseteq h$ and $I_{\text{fix}}(f) = I$.*

Saturation of M easily implies the next lemma.

Lemma 3.7 *Let I be a cut such that* $\text{cf}(I) \neq \lambda$. *Then* $\text{dcf}(I) = \lambda$.

Corollary 3.8 *If $I \subset M$ is an exponentially closed cut and* $\text{cf}(I) < \lambda$ *then there exists $f \in G$ such that $I_{\text{fix}}(f) = I$.*

Corollary 3.9 *If $a \in M$ then there exists $f \in G$ such that $I_{\text{fix}}(f) = M(a)$.*

4 Closed Normal Subgroups

If M is a model of Peano Arithmetic, we can consider its automorphism group $\text{Aut}(M)$ as a topological group by letting the stabilizers of finite subsets of M be the basic open subgroups.

Let $A \subset M$ and let H be a subgroup of $\text{Aut}(M)$. Then we define

$$H_{(A)} = \{g \in H : g(a) = a \text{ for every } a \in A\};$$

in particular, $\text{Aut}(M)_a$ is the *stabilizer of a point $a \in M$*.

Let I be a cut in a model M. We say I is *invariant* if for every $f \in \text{Aut}(M)$, $f(I) = I$. We will leave the proof of the following lemma to the reader.

Lemma 4.1 *Let $M \models$ PA be saturated and let $I \subset M$ be a cut. Then I is invariant if and only if there is a sequence of definable elements in M which is cofinal in I or a sequence of definable elements in M which is downward cofinal in $M \setminus I$.*

We note that the above result is not true for short saturated models. In particular, if $M(a)$ is short saturated and $M(a) \neq M(0)$, there are other invariant initial segments to the model, as will be discussed in Section 7.

It is not difficult to see that if I is an invariant cut, then $\text{Aut}(M)_{(I)}$ is a closed normal subgroup in $\text{Aut}(M)$. Kaye [4] showed that for countable recursively saturated models the converse is true (another proof can be found in [11]).

Theorem 4.2 (Kaye's Theorem) *Let M be a countable recursively saturated model of Peano Arithmetic. A subgroup H of* $\text{Aut}(M)$ *is a closed normal subgroup if and only if there is an invariant cut $I \subseteq_e M$ such that $H =$* $\text{Aut}(M)_{(I)}$.

For the rest of this section we fix M to be a saturated model of Peano Arithmetic of cardinality λ and $G = \text{Aut}(M)$.

Lemma 4.3 *Let $a \neq b \in M$, let $g \in G$ satisfy $g(a) = b$, let $I = I_{\mathrm{fix}}(g)$, and let $J = \inf_{n \in \omega} j_n$, where*

$$j_n = \max\{j \in M : \forall \bar{x} < j \bigwedge_{\theta < n} (\theta(\bar{x}, a) \leftrightarrow \theta(\bar{x}, b))\}.$$

Then $I \subset J$.

Proof Clearly, $I \subseteq J$. By Theorem 3.5, $\mathrm{dcf}(I) = \lambda$. By the definition of J we have that J is ω-coded from above. Hence, $J \neq I$. □

If $I \subseteq M$ is a cut, we define

$$\mathrm{Aut}(M)_{(>I)} = \{g \in \mathrm{Aut}(M) : I \subsetneq I_{\mathrm{fix}}(g)\}.$$

Lemma 4.4

1. *Suppose I is a cut, $\mathrm{dcf}(I) = \lambda$, and I is closed under exponentiation. Then $G_{(>I)} \neq G_{(I)}$ and the closure of $G_{(>I)}$ is $G_{(I)}$.*
2. *Suppose I is an exponentially closed cut and $\mathrm{dcf}(I) \neq \lambda$. Then $G_{(>I)} = G_{(I)}$.*

Proof

1. By Theorem 3.5, $G_{(>I)} \neq G_{(I)}$. Clearly, $\overline{G_{(>I)}} \leq G_{(I)}$. Let $g \in G_{(I)}$ and $g(a) = b$. We must show that there is $h \in G_{(>I)}$ with $h(a) = b$. Let J be the cut defined in the previous lemma, so $I \subset J$, and let $c \in J \setminus I$. Define $2^c_0 = c$, $2^c_1 = 2^c$, and $2^c_{n+1} = 2^{2^c_n}$. Let $2^c_\omega = \sup\{2^c_n : n \in \omega\}$. Notice that 2^c_ω is closed under exponentiation. Also, $\mathrm{cf}(2^c_\omega) = \omega$. Hence, by Lemma 3.7 $\mathrm{dcf}(2^c_\omega) = \lambda$. Then by Lemma 3.6 we can find h such that $h \in G_{(2^c_\omega)}$ and $h(a) = b$. Since $I < c$, we have $I \subset 2^c_\omega$, and we are done.
2. Clearly, $G_{(>I)} \leq G_{(I)}$. Now if $f \in G_{(I)}$, then by Theorem 3.5 $I_{\mathrm{fix}}(f) \supset I$, because $\mathrm{dcf}(I) \neq \lambda$. Therefore, $f \in G_{(>I)}$. □

We will need the following lemma from [11].

Lemma 4.5 *Let N be a countable recursively saturated model of* PA. *Let $g \in \mathrm{Aut}(N)$, $I = I_{\mathrm{fix}}(g)$ and suppose there exist arbitrarily small $x > I$ such that $g(x) < x$. Suppose $a < b \in N$ and $h \in G_{(I)}$ are such that $b = h(a)$. Then there exist $u, v, w \in N$ such that*

$$g(v) = u < v, \mathrm{tp}(u, v) = \mathrm{tp}(u, w) \textit{ and } \mathrm{tp}(v, w) = \mathrm{tp}(a, b).$$

In order to prove the analogue of this lemma in the saturated case, we use the fact that whenever M is recursively saturated (and in particular when M is saturated), for any automorphisms g, h of M and any $\bar{a} \in M$, there is a countable recursively saturated model N, with $g', h' \in \mathrm{Aut}(N)$, such that $(N, g', h', \bar{a}) \prec (M, g, h, \bar{a})$. This fact can be proven by a downward Skolem-Löwenheim type argument.

Lemma 4.6 *Let $g \in G$, $I = I_{\mathrm{fix}}(g)$ and suppose there exist arbitrarily small $x > I$ such that $g(x) < x$. Suppose $a < b \in M$ and $h \in G_{(I)}$ are such that $b = h(a)$. Then there exist $u, v, w \in M$ such that*

$$g(v) = u < v, \mathrm{tp}(u, v) = \mathrm{tp}(u, w) \textit{ and } \mathrm{tp}(v, w) = \mathrm{tp}(a, b).$$

Proof There exists $(N, g', h') \prec (M, g, h)$ such that $a, b \in N$ and N is recursively saturated and countable. By Lemma 4.5 there are $u, v, w \in N$ such that $g'(v) = u < v, \mathrm{tp}(u, v) = \mathrm{tp}(u, w)$, and $\mathrm{tp}(v, w) = \mathrm{tp}(a, b)$. Since $(N, g', h') \prec (M, g, h)$ we are done. $\square$

Now we prove an analogue of Kaye's Theorem.

Theorem 4.7 *Let $H \leq \mathrm{Aut}(M)$. Then H is a closed normal subgroup if and only if there exists an invariant cut $I \subset M$ such that $H = \mathrm{Aut}(M)_{(I)}$.*

Proof $\Longleftarrow$ It is not difficult to see that if I is an invariant cut, then $G_{(I)}$ is a closed normal subgroup in G.

$\Longrightarrow$ We prove this direction exactly the same way as in [11]. Suppose $g \in G$ and let $I = I_{\text{fix}}(g)$. Without loss of generality, we can assume that there exist arbitrarily small $x > I$ such that $g(x) < x$. We know I is exponentially closed and $\mathrm{dcf}(I) = \lambda$. Let $J \subseteq I$ be the largest invariant exponentially closed cut (such J is well defined). $G_{(J)}$ is a closed normal subgroup, $g \in G_{(J)}$. So it is sufficient to show that $G_{(J)}$ is the closure of $\{g^{-f_1} g^{f_2} : f_1, f_2 \in G\}$.

Let $h_1 \in G_{(J)}$ and let $c, d \in M$ be such that $h_1(c) = d$. If $c = d$, there is no problem. Assume $c < d$. Then it follows from Lemma 4.4 that there are $h_2, f, h \in \mathrm{Aut}(M)$ such that $h_2(c) = d$ and $h = h_2^f \in G_{(I)}$. Let $a = f^{-1}(c)$, $b = f^{-1}(d)$. Then $a < h(a) = b$, so there exist u, v, w as in Lemma 4.6. Let $f_3, f_4 \in G$ be such that

$$f_3(u) = u,\ f_3(v) = w,\ f_4(v) = a,\ f_4(w) = b.$$

Let $f_1 = f_3^{-1} f_4^{-1} f^{-1}$ and $f_2 = f_4^{-1} f^{-1}$. Then $g^{-f_1} g^{f_2}(c) = d$. The case $c > d$ is similar. $\square$

Ω_ω is the set of all elements greater than the standard cut and smaller than any nonstandard definable element. In general Ω_ω in models of PA might be empty. Using saturation one can show that if $\mathrm{Th}(M) \neq \mathsf{TA}$ (by our assumption M is saturated) then $\Omega_\omega \neq \varnothing$. The next lemma easily follows from Theorem 4.7.

Lemma 4.8 *If $\mathrm{Th}(M) \neq \mathsf{TA}$ then $G_{(\Omega_\omega)}$ is the largest proper closed normal subgroup in G.*

From Theorem 4.7 we obtain a corollary.

Corollary 4.9 *Let M_1, M_2 be two saturated models of Peano Arithmetic of cardinality λ such that $M_1 \models \mathsf{TA}$ and $M_2 \not\models \mathsf{TA}$. Then their automorphism groups are nonisomorphic as topological groups.*

Proof Because $M_1 \models \mathsf{TA}$, M_1 does not have any nonstandard definable elements. Since $M_2 \not\models \mathsf{TA}$, there are nonstandard definable elements in M_2. Thus, by Theorem 4.7, $\mathrm{Aut}(M_1)$ has no nontrivial closed normal subgroups and $\mathrm{Aut}(M_2)$ has nontrivial closed normal subgroups (consider, for example, a subgroup $\mathrm{Aut}(M_2)_{(\Omega_\omega)}$ from Lemma 4.8). Therefore, $\mathrm{Aut}(M_1)$ cannot be topologically isomorphic to $\mathrm{Aut}(M_2)$. $\square$

5 Bounded Saturation

In this section we show that every automorphism of a boundedly saturated model of PA which is not saturated can be extended to an automorphism of the saturated end extension of the model.

It is not hard to show that when M is a saturated model and $N \prec_{\text{end}} M$ then N is boundedly saturated. Moreover, in a recent unpublished paper, Schmerl showed, using a back-and-forth argument, that if N is boundedly saturated and M is a saturated model with $N \equiv M$ and $|N| = |M|$, then N is isomorphic to an elementary initial segment of M. Hence, we get the following.

Proposition 5.1 *N is boundedly saturated if and only if $N \prec_{\text{end}} M$ for some saturated model M.*

For the rest of the paper we fix a saturated model M, $\lambda = |M|$, $G = \text{Aut}(M)$.

Proposition 5.2 *Let $N \prec_{\text{end}} M$. Then N is saturated if and only if* $\text{cf}(N) = \lambda$.

Proof It is not hard to show that if N is saturated, $\text{cf}(N) = \lambda$. Conversely, suppose that $\text{cf}(N) = \lambda$. Let $p(v)$ be a type with $\kappa < \lambda$ many parameters finitely realized in N. Enumerate all finite conjunctions of formulas from $p(v)$, $\{\Phi_i\}_{i\in\kappa}$. Let b_i be an element realizing the finite conjunction Φ_i in N. Since $\text{cf}(N) = \lambda$, there is $b \in N$ such that $b > b_i$ for all $i \in \kappa$. Then clearly the bounded type

$$q(v) = p(v) \cup \{v < b\}$$

is finitely realized. But since $N \prec_{\text{end}} M$, by the previous proposition, N is boundedly saturated, so this type is realized in N by some c. Thus, $c \in N$ realizes $p(v)$. □

Notice that whenever $N \prec_{\text{end}} M$, if $f \in \text{Aut}(M)$ and $f(N) = N$, then $f|_N \in \text{Aut}(N)$. We now proceed to show that every automorphism of a boundedly saturated model $N \prec_{\text{end}} M$ which is not saturated can be extended to an automorphism of M. This result is in contrast to a result from [12] which states that there are continuum many automorphisms of any countable short recursively saturated model which cannot be extended to automorphisms of the recursively saturated elementary end extension of the model.

Let $N \subseteq M$. By $\text{Cod}(M/N)$ we denote the set of all subsets of N which are coded in M. For any $c \in M$, by $x \in c$ we denote that x is an element of the set coded by c in M. By $\text{lh}(c)$ we denote the length of the sequence coded by c.

Proposition 5.3 *Let $N \prec_{\text{end}} M$ be such that* $\text{cf}(N) < \lambda$. *Let* $f \in \text{Aut}(N)$. *Then* $f \in \text{Aut}((N, \text{Cod}(M/N)))$.

Proof Let $X \subset N$ be coded in M by d. Let $\kappa = \text{cf}(N)$ and let $\{e_i\}_{i\in\kappa}$ be a cofinal sequence in N. For all $i \in \kappa$ let c_i be the smallest element in M coding $X \cap [0, e_i]$ (such element must be in N). Notice that if $i < j \in \kappa$, for all $r < \text{lh}(c_i)$, $(c_i)_r = (c_j)_r$. Now $f(x) \in f(X) \iff x \in X \iff x \in c_i \iff f(x) \in f(c_i)$ for some $i \in \kappa$. Since f is a bijection on N, $f(x) \in f(X) \iff f(x) \in f(c_i)$ for some $i \in \kappa$ implies that $x \in f(X) \iff x \in f(c_i)$ for some $i \in \kappa$.

Now let $p(v)$ be the following type.

$$p(v) = \{\forall r < \text{lh}(f(c_i)), (v)_r = (f(c_i))_r : i \in \kappa\}.$$

Since for any $i < j \in \kappa$, for all $r < \text{lh}(f(c_i))$, $(f(c_i))_r = (f(c_j))_r$, any finite collection of formulas from $p(v)$ is realized by some $f(c_j)$, where $j \in \kappa$ is the largest j used in the finite collection. Since there are $\kappa < \lambda$ many parameters, $p(v)$ is realized in M by some c, and hence $f(X)$ is coded in M. Therefore, f fixes $\mathrm{Cod}(M/N)$ setwise. □

The next proposition can be found in [7] for the countable recursively saturated case. The proof in the saturated case is similar.

Proposition 5.4 *Suppose that $N \prec_{\mathrm{end}} M$ and that $\mathrm{dcf}(N) = \lambda$. Then every $f \in \mathrm{Aut}((N, \mathrm{Cod}(M/N)))$ extends to an automorphism of M.*

Proof Let $f \in \mathrm{Aut}((N, \mathrm{Cod}(M/N)))$. Let $M = \{m_i : i < \lambda\}$. We will construct an automorphism g of M extending f by back-and-forth. We will only do the forth direction since the back is similar. Assume that for $i < \lambda$ we have $A_i = \{a_k : k < i\} \subset M$ and $B_i = \{b_k : k < i\} \subset M$ such that for all $\bar{a} \subset A_i$ and all $x \in N$,

$$(M, \bar{a}, x) \equiv (M, \bar{b}, f(x)),$$

where $\bar{b}$ is a sequence of elements b_k in B_i with the same indices as those of the a_ks in A_i (we shall call this the $\bar{b}$ corresponding to $\bar{a}$). Let j be the least $j \in \lambda$ such that $m_j \in M \backslash A_i$. Let $a = m_j$. We will find $b \in M$ such that for all $\bar{a} \subset A_i$ and their corresponding $\bar{b} \subset B_i$ and all $x \in N$,

$$(M, \bar{a}, a, x) \equiv (M, \bar{b}, b, f(x)).$$

Let $c' \in N$ be nonstandard. For any finite $\bar{a} \subset A_i$, we can assign (from the outside) a unique $s_{\bar{a}} < c'$. This can be done since there are less than λ many finite $\bar{a} \subset A_i$, and since there are λ many elements in $[0, c')$.

Let $e > N$ and let α be an element such that for every finite sequence $\bar{a} \subset A_i$ and all $\varphi \in \mathcal{L}_{\mathsf{PA}}$

$$M \models \forall x < e(\langle \ulcorner \varphi \urcorner, s_{\bar{a}}, x\rangle \in \alpha \iff \varphi(\bar{a}, a, x)).$$

Such α exists because M is saturated and because $|A_i| < \lambda$. The reason that we are using $\langle \ulcorner \varphi \urcorner, s_{\bar{a}}, x\rangle$, as opposed to $\langle \ulcorner \varphi \urcorner, \bar{a}, x\rangle$, is that $\langle \ulcorner \varphi \urcorner, s_{\bar{a}}, x\rangle$ enables us to use the inductive assumption. Let $S = \alpha \cap N$. Since S is coded in M, by our assumption, $S' = f(S)$ is also coded in M by some β.

Now for every nonstandard $c \in N$ with $c > c'$ and for every $\bar{a} \subset A_i$ and any $\varphi \in \mathcal{L}_{\mathsf{PA}}$,

$$M \models \exists u \forall x < c(\langle \ulcorner \varphi \urcorner, s_{\bar{a}}, x\rangle \in S \cap [0, 2^{c^4}] \iff \varphi(\bar{a}, u, x)).$$

Note that a is a witness for the existence of such u. Also note that the set $S \cap [0, 2^{c^4}]$ is coded in N. Let $s_{\bar{b}} = f(s_{\bar{a}})$. Thus, by our inductive assumption,

$$M \models \exists u \forall x < f(c)(\langle \ulcorner \varphi \urcorner, s_{\bar{b}}, x\rangle \in S' \cap [0, 2^{f(c)^4}] \iff \varphi(\bar{b}, u, x)).$$

Since f is an automorphism of N we get that for every nonstandard $c \in N$ such that $f(c) > c'$ and for every finite sequence $\bar{b} \subset B_i$ and any $\varphi \in \mathcal{L}_{\mathsf{PA}}$,

$$M \models \exists u \forall x < c(\langle \ulcorner \varphi \urcorner, s_{\bar{b}}, x\rangle \in \beta \cap [0, 2^{c^4}] \iff \varphi(\bar{b}, u, x)).$$

(Notice that for each $c \in N$, $S' \cap [0, 2^{c^4}] = \beta \cap [0, 2^{c^4}]$.) Let $d_{\varphi, \bar{b}}$ be the largest d such that

$$M \models \exists u \forall x < d(\langle \ulcorner \varphi \urcorner, s_{\bar{b}}, x\rangle \in \beta \cap [0, 2^{d^4}] \iff \varphi(\bar{b}, u, x)).$$

By overspill, $d_{\varphi,\bar{b}}$ must be in $M \backslash N$. Since $\mathrm{dcf}(N) = \lambda$ and since there are less than λ many finite subsets of B_i, there must be $d \in M$ such that $N < d < d_{\varphi,\bar{b}}$ for all $\varphi \in \mathcal{L}_{\mathsf{PA}}$ and $\bar{b} \subset B_i$. Thus, by saturation there is $b \in M$ such that for every $\bar{b} \subset B_i$ and any $\varphi \in \mathcal{L}_{\mathsf{PA}}$

$$M \models \forall x < d(\langle \ulcorner \varphi \urcorner, s_{\bar{b}}, x\rangle \in \beta \cap [0, 2^{d^4}] \iff \varphi(\bar{b}, b, x)).$$

Hence, for any $x \in N$, any $\bar{a} \subset A_i$ and its corresponding $\bar{b} \subset B_i$, and any $\varphi \in \mathcal{L}_{\mathsf{PA}}$,

$$M \models \varphi(\bar{a}, a, x) \iff \langle \ulcorner \varphi \urcorner, s_{\bar{a}}, x\rangle \in \alpha \iff \langle \ulcorner \varphi \urcorner, s_{\bar{b}}, x\rangle \in \beta \iff \varphi(\bar{b}, b, x).$$

Finally, for any limit ordinal $i \leq \lambda$ let $A_i = \bigcup_{j<i} A_j$ and $B_i = \bigcup_{j<i} B_j$. □

Since whenever N is a cut of M with $\mathrm{cf}(N) < \lambda$, $\mathrm{dcf}(N) = \lambda$ (see Lemma 3.7), Propositions 5.3 and 5.4 imply the following theorem.

Theorem 5.5 *Let $N \prec_{\mathrm{end}} M$ be such that* $\mathrm{cf}(N) < \lambda$ *and let $f \in \mathrm{Aut}(N)$. Then f can be extended to an automorphism of M.*

The above result shows that any boundedly saturated submodel of M which is not saturated can be extended to M. We will now show that this is not true for some saturated submodels.

We say that a type is *rare* if any element that realizes it in a model of PA is the only element in its gap that does. It was shown in [6] that every recursively saturated model has elements realizing rare types which are not selective. If $b \in M$ realizes such type, there is $c \in \mathrm{Scl}(b) \backslash \mathrm{Scl}(0)$ with $c < \mathrm{gap}(b)$.

For $b \in M$ let $M[b] = M(b) \backslash \mathrm{gap}(b)$. Notice that $M[b]$ is an elementary initial segment of M and $\mathrm{dcf}(M[b]) = \omega$. Therefore, $\mathrm{cf}(M[b]) = \lambda$ and hence $M[b]$ is saturated.

Proposition 5.6 *Let $b \in M$ realize a rare type which is not selective; then there is $f \in$* $\mathrm{Aut}(M[b])$ *which cannot be extended to an automorphism of M.*

Proof By the remark before the proposition, there is $c = t(b)$ for some Skolem term t, with $c \notin \mathrm{Scl}(0)$ and $c < \mathrm{gap}(b)$. Thus, c is a nondefinable element in $M[b]$, so there is $f \in \mathrm{Aut}(M[b])$ which moves c. But such f cannot be extended because f fixes $M[b]$ setwise but moves c, and any $g \in G$ which fixes $M[b]$ setwise must fix b (since b realizes a rare type), and so it must fix $c = t(b)$. □

6 Initial Segments of Boundedly Saturated Models

In this section we show that the Moving Gaps Lemma and Smoryński's Lemma apply to all boundedly saturated models of PA. We then show that Kaye's Theorem applies to boundedly saturated models which are not short. The proof for the short case is given in Section 7.

Fix N a boundedly saturated elementary initial segment of M with $\mathrm{cf}(N) = \kappa < \lambda$. We begin with the Moving Gaps Lemma.

Lemma 6.1 *Suppose that $g \in$* $\mathrm{Aut}(N)$*, and $a, b, d \in N$ are such that $g(a) \neq a <$* $\mathrm{gap}(b) < \mathrm{gap}(d)$*. Then there exists $c \in N$ such that* $\mathrm{gap}(b) < c < \mathrm{gap}(d)$ *and* $g(c) \notin \mathrm{gap}(c)$*.*

Proof By Theorem 5.5 there is an automorphism $g' \in \mathrm{Aut}(M)$ extending g. Since $a \in N$, $g(a) = g'(a)$, so $g'(a) \neq a < \mathrm{gap}(b) < \mathrm{gap}(d)$. Hence, by the Moving

Gaps Lemma, there exists $c \in M$ such that $\mathrm{gap}(b) < c < \mathrm{gap}(d)$ and $g'(c) \notin \mathrm{gap}(c)$. Since $c < d$, $c \in N$, so $g(c) = g'(c) \notin \mathrm{gap}(c)$. □

Notice that if N has no last gap (i.e., when N is not short), every automorphism of N moves λ many gaps. However, if N has a last gap, there are 2^λ many automorphisms of N which do not move any gaps. These are the automorphisms which fix all elements below the last gap.

We now show that Smoryński's Lemma applies to boundedly saturated models as well.

Theorem 6.2 *A cut $I \subset N$ is $I_{\mathrm{fix}}(f)$ for some $f \in \mathrm{Aut}(N)$ if and only if I is an exponentially closed cut such that $\mathrm{dcf}(I) = \lambda$.*

Proof Let $I \subset N$ be an exponentially closed cut such that $\mathrm{dcf}(I) = \lambda$. Since $\mathrm{cf}(N) = \kappa$, there is a set A of cardinality κ cofinal in N. Then by Lemma 3.6, there is $g \in G$ such that $g(A) = A$ and $I_{\mathrm{fix}}(g) = I$. But since $g(A) = A$, $g(N) = N$, and therefore $g|_N$ is an automorphism of N. Let $f = g|_N$. Then $I_{\mathrm{fix}}(f) = I$.

Conversely, let $f \in \mathrm{Aut}(N)$ be a nontrivial automorphism of N. By Theorem 5.5 there is an automorphism $g \in G$ extending f. Since $I = I_{\mathrm{fix}}(g) = I_{\mathrm{fix}}(f)$, by Theorem 3.5, I is an exponentially closed cut such that $\mathrm{dcf}(I) = \lambda$. □

We now proceed to show that Kaye's Theorem is true for boundedly saturated models which are not short.

We begin by sketching a proof of the following result which is based on a proof from the same unpublished paper of Schmerl. Suppose N is not short and $a, b \in N$ are such that $\mathrm{tp}(a) = \mathrm{tp}(b)$. Let A be a cofinal sequence in N of elements $\{a_i\}_{i \in \kappa}$ realizing minimal type. Therefore, $(M, a, A) \equiv (M, b, A)$. By the homogeneity of M, there is an automorphism of M sending a to b and fixing A. Since A is cofinal in N, the restriction of the automorphism to N is an automorphism of N sending a to b. This proves the following.

Proposition 6.3 *Every boundedly saturated model of* PA *which not short is ω-homogeneous.*

In the proof of Lemma 4.5 from [11], the types used were not bounded. However, one can guarantee that for any $\alpha > \mathrm{gap}(b)$, the existence of the elements u, v, and w is below α, by adding the formulas $x < \alpha$ and $y < \alpha$ to the sequences of formulas and types constructed, as well as replacing $\exists z$ with $\exists z < \alpha$ in $\psi_n(x, y)$ in the proof. Therefore, Lemma 4.6 can be modified as follows.

Lemma 6.4 *Let $g \in G$, $I = I_{\mathrm{fix}}(g)$ and suppose there exist arbitrarily small $x > I$ such that $g(x) < x$. Suppose $a < b \in M$ and $h \in G_{(I)}$ are such that $b = h(a)$. Let $\alpha > \mathrm{gap}(b)$. Then there exist $u, v, w < \alpha$ such that*

$$g(v) = u < v, \mathrm{tp}(u, v) = \mathrm{tp}(u, w), \text{ and } \mathrm{tp}(v, w) = \mathrm{tp}(a, b).$$

Lemma 6.5 *Let $g \in \mathrm{Aut}(N)$, $I = I_{\mathrm{fix}}(g)$ and suppose there exist arbitrarily small $x > I$ such that $g(x) < x$. Suppose $a < b \in N$ and $h \in \mathrm{Aut}(N)_{(I)}$ are such that $b = h(a)$. Suppose further that there is $\alpha \in N$ with $\alpha > \mathrm{gap}(b)$. Then there exist $u, v, w < \alpha$ such that*

$$g(v) = u < v, \mathrm{tp}(u, v) = \mathrm{tp}(u, w), \text{ and } \mathrm{tp}(v, w) = \mathrm{tp}(a, b).$$

Proof Since $g \in \mathrm{Aut}(N)$, by Theorem 5.5 there is $g' \in G$ extending g. Thus, $I_{\mathrm{fix}}(g') = I$ and there exist arbitrarily small $x > I$ such that $g'(x) < x$. Also, by Theorem 5.5, there is $h' \in G$ extending h, so $h' \in \mathrm{Aut}(N)_{(I)}$ and $b = h(a)$. Thus, by Lemma 6.4, there are $u, v, w < \alpha$ such that

$$g'(v) = u < v, \mathrm{tp}(u, v) = \mathrm{tp}(u, w) \text{ and } \mathrm{tp}(v, w) = \mathrm{tp}(a, b).$$

Since $\alpha \in N$, $u, v, w \in N$ and $g(v) = u < v$, and the result follows. □

Theorem 6.6 *Suppose that N is not short. Let $H \leq \mathrm{Aut}(N)$. Then H is a closed normal subgroup if and only if there exists an invariant cut $I \subset N$ such that $H = \mathrm{Aut}(N)_{(I)}$.*

Proof Exactly the same as the proof of Theorem 4.7, with N replacing M, $\mathrm{Aut}(N)$ replacing G, and Lemma 6.5 replacing Lemma 4.6. □

Now Corollary 4.9 can be extended to all boundedly saturated models of PA which are not short.

Corollary 6.7 *Let M_1, M_2 be two boundedly saturated models of Peano Arithmetic of cardinality λ which are not short, with $M_1 \models$ TA and $M_2 \not\models$ TA. Then their automorphism groups are nonisomorphic as topological groups.*

7 Short Saturation

In this section we show that Kaye's Theorem is true for short saturated models of PA. This result, together with Theorem 4.7 and Theorem 6.6, imply that Kaye's Theorem is true for all boundedly saturated models of PA. We then use this result to show that any saturated model of PA has short elementary initial segments whose automorphism groups are nonisomorphic as topological groups.

For the rest of the paper let $a \in M$. Then $M(a)$ is a short saturated elementary initial segment of M. Let $G(a) = \mathrm{Aut}(M(a))$.

Since $\mathrm{gap}(a)$ is the last gap in $M(a)$, it must be fixed setwise by all automorphisms of the model. Moreover, if an automorphism of M fixes $\mathrm{gap}(a)$ setwise, its restriction to $M(a)$ is an automorphism of $M(a)$. This implies the following.

Proposition 7.1 *Let $f \in G$. The restriction of f to the domain of $M(a)$, $f|_{M(a)}$, is in $G(a)$ if and only if $f(\mathrm{gap}(a)) = \mathrm{gap}(a)$.*

Notice that since the last gap of $M(a)$ must be fixed setwise, if $b \in \mathrm{gap}(a)$ and $c \notin \mathrm{gap}(a)$ realize the same type, there is no automorphism sending b to c. Thus, if $\mathrm{gap}(a) \neq \mathrm{gap}(0)$, $M(a)$ is not ω-homogeneous. However, if $b, c \in \mathrm{gap}(a)$ and $\bar{d}, \bar{e} \in M(a)$ are such that $(M(a), \bar{d}, b) \equiv (M(a), \bar{e}, c)$, then by elementarity $(M, \bar{d}, b) \equiv (M, \bar{e}, c)$, and since M is saturated, there is an automorphism h of M sending $\bar{d}$ to $\bar{e}$ and b to c. Since b and c are in $\mathrm{gap}(a)$, h preserves $\mathrm{gap}(a)$ setwise, so by Proposition 7.1 its restriction is an automorphism of $M(a)$. Thus, we have the following proposition.

Proposition 7.2 *If $b, c \in \mathrm{gap}(a)$ and $\bar{d}, \bar{e} \in M(a)$ are such that $\mathrm{tp}(\bar{d}, b) = \mathrm{tp}(\bar{e}, c)$, then there is an automorphism $g \in G(a)$ such that $g(\bar{d}) = \bar{e}$ and $g(b) = c$.*

In order to show that Kaye's theorem [4] is true for short saturated models, we need the following result which is due to Blass [1] and Gaifman [2].

Lemma 7.3 (Blass and Gaifman Lemma) *Let K be a model of* PA. *Let $a < b \in K$. If $b \in \text{gap}(a)$ then there is a Skolem term $t(x)$ such that $K \models a < b \leq t(a) = t(b)$.*

The above lemma implies the following proposition whose proof can be found in [7].

Proposition 7.4 *Let K be a model of* PA. *Let $f \in \text{Aut}(K)$ and let $a \in K$. If $f(a) \in \text{gap}(a)$, then there is $c \in \text{gap}(a)$ such that $f(c) = c$.*

Lemma 7.5 *Let $g \in G(a)$, $I = I_{\text{fix}}(g) \subset M(a)$ and suppose there exists an arbitrarily small $x > I$ such that $g(x) < x$. Suppose $a' < b' \in M(a)$ and $h \in G(a)_{(I)}$ are such that $b' = h(a')$. Then there exist $u, v, w \in M(a)$ and $e \in \text{gap}(a)$ such that*

$$g(v) = u < v, \text{tp}(u, v, e) = \text{tp}(u, w, e), \text{ and } \text{tp}(v, w, e) = \text{tp}(a', b', e).$$

Proof Since $\text{gap}(a)$ is fixed setwise by all automorphisms, Proposition 7.4 implies that both g and h fix some elements in $\text{gap}(a)$. Therefore, by the Blass and Gaifman Lemma, there are some elements which are fixed by both g and h (since if $r \in \text{gap}(a)$ is fixed by g and $s \in \text{gap}(a)$ is fixed by h, if $t(r) = t(s)$ then $t(r)$ is fixed by both g and h). Thus, there is $e \in \text{gap}(a)$ such that $g(e) = h(e) = e$. Now consider the saturated structure, (M, e). Since g and h are automorphisms of the short saturated initial segment $(M(a), e)$, by Theorem 5.5 there are automorphisms $g', h' \in \text{Aut}((M, e))$ extending g and h, respectively. Since $I = I_{\text{fix}}(g) = I_{\text{fix}}(g')$, and since $h' \in \text{Aut}((M, e))_{(I)}$ and $b' = h(a') = h'(a')$, by Lemma 4.6 there exist u, v, and w such that

$$g'(v) = u < v, \text{tp}(u, v) = \text{tp}(u, w), \text{ and } \text{tp}(v, w) = \text{tp}(a', b').$$

But since e is in the language of this structure we conclude that $\text{tp}(u, v, e) = \text{tp}(u, w, e)$ and $\text{tp}(v, w, e) = \text{tp}(a', b', e)$. Since $e \in \text{gap}(a)$ and $a', b' \in M(a)$, then $\text{tp}(v, w, e) = \text{tp}(a', b', e)$ implies that $v, w \in M(a)$, and since $u < v$, $u \in M(a)$. Thus, $g(v) = g'(v) = u$ and the result follows. □

Notice that the above lemma is almost identical to Lemma 4.6, but here we have added the element $e \in \text{gap}(a)$. The reason we have added it is as follows. In the saturated case, $\text{tp}(u, v) = \text{tp}(u, w)$ and $\text{tp}(v, w) = \text{tp}(a', b')$ imply the existence of the automorphisms $f_3, f_4 \in G$ with $f_3(u) = u$, $f_3(v) = w$, $f_4(v) = a'$, and $f_4(w) = b'$. This is not necessarily true in the short case. However, if $\text{tp}(u, v, e) = \text{tp}(u, w, e)$ and $\text{tp}(v, w, e) = \text{tp}(a', b', e)$ for some $e \in \text{gap}(a)$, by Proposition 7.2 there are such $f_3, f_4 \in G(a)$.

Theorem 7.6 *Let $H \leq G(a)$. Then H is a closed normal subgroup if and only if there exists an invariant cut $I \subset M(a)$ such that $H = G(a)_{(I)}$.*

Proof Exactly the same as the proof of Theorem 4.7, with a' replacing a, b' replacing b, $M(a)$ replacing M, $G(a)$ replacing G, and Lemma 7.5 replacing Lemma 4.6. □

Recall that a type is rare if any element that realizes it in a model of PA is the only element in its gap that does. It follows from [2] that any saturated model has gaps with elements realizing rare types (*labeled gaps*), while it follows from [6] that any saturated model has also gaps with no elements realizing rare types (*nonlabeled gaps*). Since the last gap of a short saturated model is fixed setwise, all the elements defined by elements realizing rare types in the last gap must be fixed by all automorphisms. This implies that when the last gap of a short saturated model $M(a)$ is labeled, there are infinitely many invariant initial segments cofinally high in the model, which implies that there is no least proper closed normal subgroup. On the other hand, it was shown in [6] that the type of every element in a nonlabeled gap of a countable recursively saturated model is ubiquitous; that is, the type is realized cofinally high and cofinally low in the gap. However, the proof of this fact does not make use of the countability of the model and hence is true for all recursively saturated models and in particular all saturated models. Thus, when a short saturated model $M(b)$ has a nonlabeled last gap, it has no invariant initial segments in its last gap. Therefore, it has a largest invariant initial segment, namely, $\inf(\mathrm{gap}(b))$, which implies a least proper closed normal subgroup. This proves the following.

Corollary 7.7 *Let M be a saturated model, and let $a, b \in M$ be such that* $\mathrm{gap}(a)$ *is labeled and* $\mathrm{gap}(b)$ *is nonlabeled. Then* $\mathrm{Aut}(M(a)) \ncong \mathrm{Aut}(M(b))$ *as topological groups.*

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Acknowledgments

The authors would like to thank Roman Kossak and the anonymous referee for their useful suggestions.

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