Notre Dame Journal of Formal Logic
Volume 51, Number 2, 2010

Numerical Abstraction via the Frege Quantifier

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Abstract This paper presents a formalization of first-order arithmetic characterizing the natural numbers as *abstracta* of the equinumerosity relation. The formalization turns on the interaction of a nonstandard (but still first-order) cardinality quantifier with an abstraction operator assigning objects to predicates. The project draws its philosophical motivation from a nonreductionist conception of logicism, a deflationary view of abstraction, and an approach to formal arithmetic that emphasizes the *cardinal* properties of the natural numbers over the structural ones.

1 Introduction

This paper presents a formal theory of arithmetic which is characterized by the interaction of two special devices. The first device is a binary quantifier—referred to as the "Frege" quantifier—binding one (or more) variables and taking two formulas φ and ψ as arguments, asserting that there are no more φs than ψs. Using the Frege quantifier, it is clearly possible to define the *equinumerosity* of φ and ψ by saying that there are no more φs than ψs and vice versa. The second device is an abstraction operator assigning objects to predicates (or, as Frege would say, "concepts") with the intended interpretation that the object assigned to φ is—or perhaps represents—the number of objects satisfying the formula φ. The interaction of these two devices is most notably captured in the so-called Hume's Principle (HP), the statement that the number of φs equals the number of ψs if and only if the φs are equinumerous to the ψs.

The interaction of these two fundamental devices allows a formalization of arithmetic that, while still at the first-order from a semantical point of view, follows the traditional Frege-Russell strategy of characterizing the natural numbers as *abstracta* of the equinumerosity relation. In contrast to the various alternative set-theoretic

Received April 12, 2009; accepted May 19, 2009; printed May 20, 2010
2010 Mathematics Subject Classification: Primary, 03C99, 03C80
Keywords: cardinality quantifiers, abstraction principles, arithmetic
 10.1215/00294527-2010-010

reductions, as well as directly number-theoretical approaches such as Peano Arithmetic, we regard this strategy as extremely well motivated from a philosophical point of view and basically correct in its emphasis on the *cardinal* properties of the natural numbers (as opposed to the *structural or ordinal ones*), where the cardinal properties are understood as those features that ground the role of natural numbers in answering questions of the form "How many?" and are therefore strictly connected with applications of arithmetic.[1] A similar emphasis on cardinal properties also characterizes approaches in the so-called neo-Fregean tradition of, for example, Hale and Wright [9], which also use in a crucial way a numerical abstraction operator. But such theories make heavy use of second-order resources, which are avoided in the present approach and replaced by a suitably generalized first-order quantifier.

While, as we will see, the two devices of the Frege quantifier and the abstraction operator allow us to capture the cardinal properties of the natural numbers, it is interesting to note that there does not seem to be any obvious way, using the same two devices, to give an equally direct treatment of *ordinal* notions (except perhaps trivially in finite domains, where ordinal and cardinal numbers coincide). In this respect ordinal notions, while ordinarily regarded on a par with their cardinal counterparts, would appear instead to be intrinsically more complex than the latter, and indeed quite possibly beyond the reach of a first-order treatment.

Part of the motivation for the present approach is provided by the notion, central to Frege's work, that cardinality notions enjoy a logically privileged status. The main idea of the present approach is to take this claim at face value and introduce cardinality notions as basic building blocks of a logical language—where cardinality notions are purposely differentiated from *numerical* ones, which rather pertain to the domain of abstraction. The "number-of" is of a *different type* than cardinality notions (which in the present approach are represented as embedded in the logical language): while the former assign objects to concepts, the latter represent relation between concepts. Independently of the privileged status of the abstraction operator (which has been variously criticized),[2] it is now the cardinality quantifier which helps carry the standard of Frege's conception of cardinality.[3]

This paper is organized as follows. In Section 2 we present the general view of quantifiers as second-order operators; then in Section 3 we introduce the language $\mathscr{L}_{\mathsf{F}}$ of the Frege quantifier along with its standard semantics; in Section 4 we develop a nonstandard semantics for $\mathscr{L}_{\mathsf{F}}$. Section 5 extends the language by introducing the numerical abstraction operator, and finally Section 6 delivers the promised axiomatization of arithmetic, with the proof of the interpretability of Peano Arithmetic developed in Section 7.

2 The Modern View of Quantifiers

The study of *generalized quantifiers* initiates with the work of [18] and continues with that of [17], spanning both linguistics and mathematical logic. Linguists have traditionally focused on quantifiers as tools for natural language semantics, and logicians on expressive power and properties such as axiomatizability, decidability, and so on. Building upon an idea that can be traced back to Frege's *Grundgesetze* [6], the modern study of generalized quantifiers takes a characteristically general stance, identifying quantifiers with higher-level entities.

Definition 2.1 Given a *domain of discourse* (i.e., a nonempty set) D, a quantifier Q over D is a collection of subsets of D: $\mathsf{Q} \subseteq \mathcal{P}(D)$.

This account is consistent with the traditional view of quantifiers as operators on formulas. Let us understand a formula φ in one free variable x as denoting a subset of D, namely, the collection $[\![\varphi]\!]$ of those $d \in D$ that satisfy the formula ($[\![\varphi]\!]$ can be thought of as the *extension* of φ in D, where of course we presuppose an interpretation for the nonlogical constants of the language as well as an assignment to variables other than x). Then a quantifier can indeed be identified with a collection of subsets of D. Let us look at some examples.

Example 2.2 The ordinary universal quantifier $\forall$ can be identified with the collection of subsets of D that contains D itself as its only member: $\forall = \{D\}$; a sentence of the form $\forall x\varphi(x)$ is then true over D precisely when every $d \in D$ satisfies φ, that is, when the extension $[\![\varphi]\!]$ of $\varphi(x)$ over D is D itself. Hence, $\forall$ can be identified, semantically, with $\{D\}$.

Example 2.3 Similarly (and dually) the ordinary existential quantifier $\exists$ can be identified with the collection of all nonempty subsets of D; that is, $\exists = \{X \subseteq D : X \neq \varnothing\}$. A sentence of the form $\exists x\varphi(x)$ is true precisely when some $d \in D$ satisfies $\varphi(x)$; that is, the extension of φ over D is nonempty.

Example 2.4 The quantifier "there exist exactly k," usually written $\exists!^k$ can be identified with the collection of all k-membered subsets of D; that is, $\exists!^k = \{X \subseteq D : |X| = k\}$. Then $\exists!^k x\varphi(x)$ is true precisely when there are exactly k objects in D that satisfy φ.

There are also extreme examples, which reduce to triviality. We could, for instance, consider the *empty* first-order quantifier $\mathsf{Q}_\varnothing$, that is, the empty collection of subsets of D. Then we have that $\mathsf{Q}_\varnothing x\varphi(x)$ is true precisely when $\{x \in D : \varphi(x)\} \in \mathsf{Q}_\varnothing$, that is, *never*. $\mathsf{Q}_\varnothing x\varphi(x)$ is an identically false sentence for any φ. Similarly, we could consider the quantifier $\mathsf{U} = \mathcal{P}(D)$ such that $\mathsf{U}x\varphi(x)$ is identically true for any φ.

All the examples of quantifiers we have seen so far apply to a single open formula $\varphi(x)$ at a time: they are, as we will say, *unary*.[4] But, in fact, some quantifiers are not only best viewed as applying to more than one such formula, they are such that no other interpretation is possible. Consider the following examples:

1. All A are B: $\mathsf{All} = \{(A, B) : A \subseteq B\}$;
2. Some A are B: $\mathsf{Some} = \{(A, B) : A \cap B \neq \varnothing\}$;
3. Most A are B: $\mathsf{Most} = \{(A, B) : |A \cap B| > |A - B|\}$;
4. Twice as many A as B are C: $\mathsf{Twice} = \{(A, B, C) : |A \cap C| = 2 \cdot |B \cap C|\}$.

Here, as is well known, the first two quantifiers All and Some can be represented by means of *unary* quantifiers applied to Boolean combinations of their arguments. However, not all binary quantifiers can be represented in this form, that is, as a unary quantifier applied to a Boolean combination of their arguments. One example is Most (see [19, p. 468]). There is no Boolean term $F(X, Y)$ such that Most is a subset of $\{F(A, B) : A, B \subseteq D\}$ (a binary Boolean term in X and Y is a combination of X and Y by means of a finite number of applications of union, intersection, and complementation; a binary Boolean term clearly maps $\mathcal{P}(D)^2$ into $\mathcal{P}(D)$).

We insist that, from a semantical point of view, all the above quantifiers are *first-order* in that they express a relation between subsets of D; in other words, Q is a first-order binary quantifier if and only if $\mathsf{Q} \subseteq \mathcal{P}(D) \times \mathcal{P}(D)$.

Definition 2.5 An n-ary first-order quantifier Q is a subset of $\mathcal{P}(D)^n$.

According to this definition, some quantifiers are called first-order even if they are not first-order definable and therefore exceed the bounds of first-order logic as ordinarily conceived. For instance, Most is first-order but not first-order definable—and so is, as we will see, the Frege quantifier. By comparison, instead, genuine *second-order* quantifiers are collections of (or, more generally, relations among) first-order quantifiers.

To see this, consider that if Q is a first-order quantifier, then the sentence $\mathsf{Q}x\, \varphi(x)$ is true if and only if $\{x \in D : \varphi(x)\} \in \mathsf{Q}$. Analogously, if Q is second-order, then the sentence $\mathsf{Q}\, P\, \varphi(P)$ is true if and only if $\{P \in \mathcal{P}(D) : \varphi(P)\} \in \mathsf{Q}$. It follows that whereas first-order quantifiers are collections of subsets of D, second-order quantifiers are collections of collections of subsets of D, that is, collections of first-order quantifiers. The distinction between first- and second-order quantifiers is thus *semantical*, not merely notational.

A property that plays a crucial role in the modern conception of quantifiers is the following, where $\mathsf{Q}\,(A, B)$ is binary first-order:

> **Permutation invariance** if π is a permutation of D, then $\mathsf{Q}\,(A, B)$ holds if and only if $\mathsf{Q}\,(\pi[A], \pi[B])$ holds, where $\pi[X]$ (for $X \subseteq D$) is the pointwise image of X under π: $\pi[X] = \{\pi(y) : y \in X\}$.

The reason this property plays such a preeminent role is that there is a long tradition, traceable back at least to the work of [22], according to which being invariant under permutations is the hallmark of *logicality*. Logical notions deal with questions that apply to objects in the domain irrespective of their specific nature. Quantifiers are logical notions because they answer the question "How many?" with no concern for the specific nature of the objects in question. Hence, the answer should be unaffected by permutations of those objects.[5]

3 The Frege Quantifier

In this paper we introduce, as one of our two major devices, a specific binary quantifier, referred to as the "Frege quantifier."

Definition 3.1 The Frege quantifier F_D over D holds between subsets A and B of D precisely when there is an injection of A into B:

$$\mathsf{F}_D = \{(A, B) : |A| \leq |B|\}.$$

In practice, we will drop the subscripts when D is understood. The Frege quantifier F is similar, in fact, to two closely related cardinality quantifiers:

the **Härtig quantifier**: $\mathsf{I}(A, B) \Longleftrightarrow |A| = |B|$;
the **Rescher quantifier**: $\mathsf{R}(A, B) \Longleftrightarrow |A| > |B|$.

The quantifiers were first introduced by [20] and [10] and, just like the Frege quantifier, they are semantically of the first order. The defining feature of quantifiers such as these, including Frege's, is that they deal with cardinality notions *directly*, without appealing to any separately given mathematical machinery.[6]

Notice that Härtig's quantifier is definable from Rescher's, although only using the axiom of choice in an essential way: $\mathsf{I}(A, B)$ holds if and only if both $\neg\,\mathsf{R}(B, A)$ and $\neg\,\mathsf{R}(A, B)$ hold.[7] The converse is not true: Rescher's quantifier cannot be defined from Härtig's (see [19, p. 470]). Moreover, both quantifiers are *semantically* first-order in that both express binary relations between subsets of the domain. The relative advantage of the Frege quantifier, in this respect, is that Härtig's quantifier can be defined directly from it, without the axiom of choice, using only the Schröder-Bernstein theorem.

The Frege quantifier F, just like the other two related cardinality quantifiers, is permutation-invariant: If π is a permutation of D, then clearly $|A| \leq |B|$ if and only if $|\pi[A]| \leq |\pi[B]|$ (and similarly for Härtig's and Rescher's quantifiers). This fact lends additional support to the view that cardinality quantifiers express genuine logical notions.

There is, of course, also a polyadic version of the quantifier for which we use the same notation for each dimension n.

Definition 3.2 For each $n > 0$, the n-adic Frege quantifier F holds between subsets A and B of D^n if and only if there are no more n-tuples in A than there are in B.

Having defined the Frege quantifier semantically we are now going to use it as a primitive piece of logical machinery. Specifically, we introduce a formal language $\mathscr{L}_{\mathsf{F}}$ having the Frege quantifier as one of its logical primitives.

Definition 3.3 Let $\mathscr{L}_{\mathsf{F}}$ be the language built up from (individual or predicate) constants (including identity) by means of Boolean connectives ($\wedge$, $\vee$, $\neg$, and $\rightarrow$) and the quantifier F satisfying the clause "if $\varphi(\overline{x}, \overline{z})$, $\psi(\overline{x}, \overline{z})$ are formulas and $\overline{x}$, $\overline{z}$ are vectors of variables, then $\mathsf{F}\,\overline{x}(\varphi(\overline{x}, \overline{z}), \psi(\overline{x}, \overline{z}))$ is a formula."

Notice again that we are using the same notion for the primitive expression $\mathsf{F}\,\overline{x}$ and the quantifier it denotes semantically. $\mathsf{F}\,\overline{x}$ is a polyadic binary quantifier. As for the Rescher quantifier, we abbreviate $\mathsf{F}\,x(\varphi, \psi) \wedge \mathsf{F}\,x(\psi, \varphi)$ by $\mathsf{I}\,x(\varphi, \psi)$. In practice we will be mostly interested in the monadic version of F, comparing the cardinality of sets rather than that of relations, but the more general version is needed in the representation of arithmetical operations, as we will see below.

A point is worth making here. The study of generalized quantifiers is always carried out taking first-order logic for granted. Whenever logicians and linguists are interested in the properties of some quantifier Q, they explore the expressiveness of the language $\mathscr{L}(\mathsf{Q})$ obtained by adding Q to full-fledged first-order logic (see Peters and Westerståhl [19], for instance). In what follows, instead we take cardinality quantifiers as the only quantifiers in the language and explore the expressive properties of the resulting logical framework.

We now turn to the task of providing a formal semantics for the language $\mathscr{L}_{\mathsf{F}}$. The Frege quantifier can be given a standard interpretation by singling out a class of models and laying down truth (in fact, satisfaction) clauses for the language. On the present semantics, no separate stipulation is needed as to what constitutes a model for $\mathscr{L}_{\mathsf{F}}$: a model $\mathfrak{M}$ with nonempty domain D provides an interpretation for the nonlogical constants of $\mathscr{L}$ in the usual way; for example, n-place predicates are mapped onto relations $\subseteq D^n$, and so on, just as in the case of standard first-order logic. What we want to say is that for any assignment s to the variables, $\mathfrak{M} \models \mathsf{F}\,x(\varphi, \psi)[s]$ if and only if there exists a 1–1 function f mapping the set $\{s(x) : \mathfrak{M} \models \varphi[s]\}$ into the set $\{s(x) : \mathfrak{M} \models \psi[s]\}$. In order to simplify the notation, first we have the following definition.

Definition 3.4 For any assignment s of objects in D to the variables of the language, let $s\frac{\overline{a}}{\overline{x}}$ be the assignment just like s except "shifted" to assign $\overline{a} = a_1, \ldots, a_k$ to $\overline{x} = x_1, \ldots, x_k$, respectively. Then define the extension of φ in $\mathfrak{M}$ relative to s as follows:

$$[\![\varphi]\!]^{\overline{x}}_s = \{\overline{a} : \mathfrak{M} \models \varphi[s\tfrac{\overline{a}}{\overline{x}}]\}.$$

Where x is the only variable free in φ, we often write simply $[\![\varphi]\!]$.

We can finally give the definition of satisfaction.[8]

Definition 3.5 Given a formula $\varphi(\overline{x})$ and a function s assigning objects from D to the variables of $\mathscr{L}$, satisfaction $\mathfrak{M} \models \varphi[s]$ is also defined in the usual way for atomic formulas and their Boolean combinations but with the additional clause,

$$\mathfrak{M} \models \mathsf{F}\,\overline{x}(\varphi, \psi)[s] \iff \exists f : [\![\varphi]\!]^{\overline{x}}_s \xrightarrow{1-1} [\![\psi]\!]^{\overline{x}}_s.$$

We have thus defined a completely rigorous semantics for the language $\mathscr{L}_\mathsf{F}$ comprising just the Frege quantifier along with connectives and nonlogical constants. As a first step in using our newly found language, we notice that the standard first-order quantifiers are expressible in $\mathscr{L}_\mathsf{F}$.

Proposition 3.6 *Ordinary first-order logic is interpretable in* $\mathscr{L}_\mathsf{F}$.

Proof It suffices to lay down the following two abbreviations:

(1) $\forall x \varphi(x) = \mathsf{F}\,x(\neg\varphi(x), x \neq x)$;
(2) $\exists x \varphi(x) = \neg\mathsf{F}\,x(\varphi(x), x \neq x)$.

The first formula expresses the fact that everything is φ if and only if there in an injection of the complement of φ into the empty set, that is, if and only if the complement of φ is itself empty, and thus if and only if everything in D falls within the extension of φ. Dually, the second formula expresses that something is φ if and only if there is no injection of φ into the empty set. □

Accordingly, from now on we will help ourselves to the abbreviations $\forall x$ and $\exists x$. But the language turns out to be much more expressive than ordinary first-order logic. For instance, while it is well known that infinity cannot be characterized using only $\forall$ and $\exists$, the situation is very much different in $\mathscr{L}_\mathsf{F}$.

Proposition 3.7 *There is an* axiom of infinity *in the pure identity fragment of* $\mathscr{L}_\mathsf{F}$.

Proof Again, it suffices to consider the sentence

AxInf: $$\exists y \mathsf{F}\,x(x = x, x \neq y),$$

which asserts that there is an injection of D into a proper subset of itself so that D is Dedekind-infinite. AxInf is then true in *all and only* the infinite models and, therefore, its negation is true in all and only the finite models. □

Corollary 3.8 *The compactness theorem fails for* $\mathscr{L}_\mathsf{F}$.

Let us abbreviate by $\mathsf{Fin}\,x\,\varphi(x)$ the statement that the set $\{x : \varphi(x)\}$ is Dedekind finite:

$$\forall y[\varphi(y) \to \neg\mathsf{F}\,x(\varphi(x), \varphi(x) \wedge x \neq y)].$$

On the present semantics, this statement completely captures the fact that the extension of φ in $\mathfrak{M}$, that is, $[\![\varphi(x)]\!] = \{a \in D : \mathfrak{M}, a \models \varphi(x)\}$, is a finite set. Using this device, it's easy to see that there is a sentence φ of the language $\mathscr{L}_{\mathsf{F}}(<)$ comprising one binary predicate symbol $<$ as a nonlogical constant that is true if and only if the interpretation of $<$ in $\mathfrak{M}$ is a relation having order type $\leq \omega$. Using such a sentence it is then possible to characterize "true" arithmetic, that is, the set of all sentences that are true in the standard model. Let the sentence $\mathsf{ModStan}$ be obtained as the conjunction of the following three clauses:

1. $<$ is a strict transitive linear order;
2. $\exists x \forall y (y \neq x \to x < y)$;
3. $\forall x\, \mathsf{Fin}\, y (y < x)$.

The last two clauses express that the ordering denoted by $<$ has a first element and that each element of the domain has only finitely many predecessors in such an ordering. Then taking the conjunction of such a sentence $\mathsf{ModStan}$ with the axiom of infinity AxInf we obtain a sentence θ that is true in a model precisely if (the interpretation of) $<$ is a countably infinite linear order. Finally, it suffices to conjoin this last sentence θ with a set of arithmetical axioms for addition and multiplication (such as, for instance, PA minus induction) in order to establish the following theorem.

Theorem 3.9 *There is a sentence of $\mathscr{L}_{\mathsf{F}}$ that characterizes the standard model $(\mathbb{N}, +, \times)$ of arithmetic up to isomorphism. Hence, the set of $\mathscr{L}_{\mathsf{F}}(+, \times)$-validities is not recursively axiomatizable.*

Since $\mathbb{N}$ is categorically definable in $\mathscr{L}_{\mathsf{F}}$, we can then define a Gödel numbering of finite sequences and finite sets, and hence *implicitly* define satisfaction. Since by Tarski's theorem satisfaction is not *explicitly* definable we have "the *Beth definability* property fails in $\mathscr{L}_{\mathsf{F}}$." The language comprising the Frege quantifier F is then quite expressive indeed.[9]

4 The General Interpretation of the Frege Quantifier

In Section 3 we introduced the standard semantics for the Frege quantifier F, according to which $\mathsf{F}\, x(\varphi, \psi)$ holds (in a model) if and only if the cardinality of $[\![\varphi]\!]$ is less than or equal to that of $[\![\psi]\!]$. But this is not the only available interpretation for the language: there is indeed an equally attractive alternative interpretation that is characterized by a tradeoff between expressibility and tractability.

As is well known, second-order quantifiers can be given, besides a *standard* interpretation, also a so-called general interpretation (first introduced by [13]). On such a general interpretation second-order quantifiers such as $\forall P$ or $\exists P$ are taken to range not over the "true" powerset of D (or of D^n, in case P is an n-place predicate symbol), but over some previously given universe of subsets of D. So while standard models for second-order logic are indistinguishable from first-order models, general models carry, besides a domain D, also a universe of n-place relations over D (for each n). In practice, such a universe of relations will satisfy some closure conditions—it will be, for example, closed under *definability*, thereby satisfying the second-order comprehension axiom.

Somewhat surprisingly, *first-order* quantifiers can also be so interpreted, a fact that—apparently—has gone hitherto unnoticed. Perhaps the simplest example is the general interpretation of the ordinary first-order existential quantifier $\exists$. As we have seen, the ordinary existential quantifier ranges over the collection of *all* nonempty subsets of D. It is then natural to consider the "general" existential quantifier $\exists^*$ that ranges over *some* collection of nonempty subsets of D. Dually, one can also consider the general universal quantifier $\forall^*$, ranging over a collection of subsets of D containing D itself as a member. The question of whether the logic of such a quantifier can be axiomatized has a somewhat unexpected answer: it turns out that the logic of $\exists^*$ is the *positive free logic* of [15] (see [2] for details). As with the first-order existential quantifier, a *general interpretation* is available also for the Frege quantifier. In order to specify such an interpretation of F we need to single out a class of models, which will, in turn, determine a class of valid sentences—a *logic*.

Definition 4.1 A *general model* for $\mathscr{L}_{\mathsf{F}}$ is a structure $\mathfrak{M}$ providing a nonempty domain D and interpretations for the nonlogical constants, as well as a collection $\mathcal{F}$ of 1–1 functions $f : A \to B$ with $\mathrm{dom}(f) = A$, and $\mathrm{rng}(f) \subseteq B$, for $A, B \subseteq D^n$, satisfying the following *closure conditions*:[10]

Cc1: For each A, the identity map on A belongs to $\mathcal{F}$ (including the empty map on $\varnothing$).
Cc2: If $f_1, f_2 \in \mathcal{F}$, where
  (a) $f_1 : A_1 \to B_1$,
  (b) $f_2 : A_2 \to B_2$,
  (c) $A_1 \cap A_2 = \varnothing$ and $B_1 \cap B_2 = \varnothing$,
then $f_1 \cup f_2 \in \mathcal{F}$ as well.
Cc3: If $f : A \to B$ is in $\mathcal{F}$, then also $f^{-1} : f[A] \to A$ is in $\mathcal{F}$, where $f^{-1} = \{\langle y, x\rangle : \langle x, y\rangle \in f\}$.
Cc4: If $f \in \mathcal{F}$ and $f : A \to B$ and $\overline{x} \notin A$ and $\overline{y} \notin B$, then there is a $g \in \mathcal{F}$ such that $g : A \cup \{\overline{x}\} \to B \cup \{\overline{y}\}$.
Cc5: If $f : A \to C \in \mathcal{F}$ and $B \subseteq A$, then also $f \restriction B \in \mathcal{F}$.
Cc6: If $f : A \to B$ and $g : B \to C$ are in $\mathcal{F}$, then so is $f \circ g$.

Definition 4.2 Given an assignment s of objects in D to the variables of $\mathscr{L}_{\mathsf{F}}$, we define satisfaction for atomic formulas, and Boolean combinations of formulas as usual, while the satisfaction clause for the quantifier F takes the form,

$$\mathfrak{M} \models \mathsf{F}\overline{x}(\varphi, \psi)[s] \iff (\exists f \in \mathcal{F})\, f : [\![\varphi]\!]_s^{\overline{x}} \xrightarrow{1-1} [\![\psi]\!]_s^{\overline{x}}.$$

Notice that given the closure condition on the class $\mathcal{F}$ of functions associated with each general model (more specifically the fact that each model always contains the empty injection), the abbreviations for $\forall$ and $\exists$ do have their intended meaning. In other words, $\forall x\varphi(x)$ will be true precisely when nothing falls under $\neg\varphi$.

Proposition 4.3 *The following are valid in every general model satisfying the closure conditions stated above:*

1. $\mathsf{F}x(\varphi(x), \psi(x)) \wedge \forall x(\psi(x) \to \theta(x)) \to \mathsf{F}x(\varphi(x), \theta(x))$;
2. $\mathsf{Fin}\,x\ \varphi(x) \wedge \forall x(\psi(x) \to \varphi(x)) \to \mathsf{Fin}\,x\ \psi(x)$;
3. $\forall y[\mathsf{Fin}\,x\,\varphi(x) \to \mathsf{Fin}\,x(\varphi(x) \vee x = y)]$;
4. *if* $\mathsf{Fin}\,x\ \varphi(x)$ *and* $\mathsf{I}x(\varphi, \psi)$ *then* $\mathsf{Fin}\,x\ \psi(x)$).

Proof Definition 4.1 is interpreted in such a way that (1) holds: if $f : A \to B$ and $B \subseteq C$, then f itself is already an injection of A into C.

If $B \subseteq A$ and $f : B \to B$ injects B into a proper subset of itself, then $f \cup g$ injects A into a proper subset of itself, where g is the identity on $A \setminus B$. So (2) holds because of condition **Cc2** in Definition 4.1.

Next, if a set A is finite, then so is $B = A \cup \{x\}$, for if f injects B into a proper subset of itself, then $f \upharpoonright A$ injects A into a proper subset (distinguish the two cases according as $f(x) \in A$ or not). So (3) holds by condition **Cc5** in Definition 4.1.

Finally, if $f : A \to B$ and $h : B \to A$ are injections, and g injects B into a proper subset of itself, then $f \circ g \circ h$ properly injects A into itself. So (4) holds by condition **Cc6**. □

5 Numerical Abstraction

In a tradition that goes back to [5], but which has recently been revived by [9], abstraction principles have been taken to have the following general form (see [21]):

$$f(a) = f(b) \Longleftrightarrow R_f(a, b).$$

The above formula asserts that the f of a is the same as the f of b if and only if a and b are appropriately related to each other by R_f, where f is a function of the appropriate type and R_f is an equivalence relation between objects of the same type as a and b. Such principles will be referred to as *Functional Equivalences* (FEs). While a general investigation into FEs will be taken up elsewhere, it is worth mentioning that perhaps the best known of all FEs is Frege's "Hume's Principle" (HP) which, as already mentioned, governs the assignment of numbers to predicates. HP can be formulated as follows, for any formulas $\varphi(x)$ and $\psi(x)$ (possibly containing parameters):[11]

$$\mathsf{Num}\, x \,.\, \varphi(x) = \mathsf{Num}\, x \,.\, \psi(x) \Longleftrightarrow \varphi \approx \psi,$$

where $\varphi \approx \psi$ abbreviates the second-order statement that there is a bijection between φ and ψ. Notice that, strictly speaking, Num is a variable-binding operator (and sometimes we will use $\mathsf{Num}_x \varphi(x)$ as a notational variant), but when the variable x is understood, we will sometimes just write $\mathsf{Num}(\varphi)$. From a purely formal point of view, all that HP does is to assign an object—a "number"—to the equivalence classes generated by the equinumerosity relation $\approx$ in such a way that distinct equivalence classes are assigned distinct objects. We could refer to these objects as "representatives" of the equivalence classes, except that in the ordinary mathematical use of the word, representatives are picked from *within* the equivalence classes while no such assumption is made here. In fact, given the *type* of HP, assigning objects to classes, numbers cannot be viewed as representatives in the ordinary mathematical sense.

It is important to notice that HP represents exactly the kind of abstraction principle that is needed in order to formalize arithmetic. To this end, we further extend the language $\mathscr{L}_{\mathsf{F}}$ to a language $\mathscr{L}_{\mathsf{F}}{}^{\mathsf{Num}}$ by adding Num explicitly as new basic variable-binding operator and stipulating that Num takes formulas into terms: if $\varphi(x)$ is a formula then $\mathsf{Num}\, x \,.\, \varphi(x)$ is a term.

Semantically, we need to specify what a model for the language $\mathscr{L}_{\mathsf{F}}{}^{\mathsf{Num}}$ would look like. Here again, we have a choice as to whether to adopt the *general* interpretation of F or the *standard* one. But in either case, the semantic device needed to account for Num is the same.

Definition 5.1 A *general model* for $\mathcal{L}_{\mathsf{F}}^{\mathsf{Num}}$ is a structure $\mathfrak{M}$ providing a nonempty domain D and interpretations for the nonlogical constants as well as

1. a collection $\mathcal{F}$ of 1–1 functions $f : A \to B$ such that $(\mathfrak{M}, \mathcal{F})$ is a general model for $\mathcal{L}_{\mathsf{F}}$,
2. a function $\eta : \mathcal{P}(D) \to D$ providing an interpretation for the abstraction operator.

A *standard model* for $\mathcal{L}_{\mathsf{F}}^{\mathsf{Num}}$ does not specify any class $\mathcal{F}$ of functions (or equivalently, $\mathcal{F}$ is the class of *all* injections between subsets of D).

Notice that in either case nothing special is said about the function η except that it has type $\eta : \mathcal{P}(D) \to D$. Any further constraints will be specified by assuming special axioms in Section 6. For now, we just lay down the semantic definitions. Given the abstraction operator, satisfaction and reference will have to be defined by simultaneous recursion. In particular, we need to specify a referent $[\![t]\!]_s$ in $\mathfrak{M}$ for each term t of the language.

Definition 5.2 Given a general model for $\mathcal{L}_{\mathsf{F}}^{\mathsf{Num}}$ and a function s assigning objects from D to the individual variables of the language, we define satisfaction and reference simultaneously, where the two crucial clauses are as follows:

1. $\mathfrak{M} \models \mathsf{F}\,\overline{x}(\varphi, \psi)[s] \iff (\exists f \in \mathcal{F})\; f : [\![\varphi]\!]_s^{\overline{x}} \xrightarrow{1-1} [\![\psi]\!]_s^{\overline{x}}$;
2. $[\![\mathsf{Num}\, x.\varphi(x)]\!]_s = \eta([\![\varphi]\!]_s^x)$.

In the case of a standard model the bound on the existential quantifier can be dropped from the first clause.

Notice that we only use the monadic version of Num, binding one variable at a time, although, of course, a polyadic version is also possible.

6 Formalizing Arithmetic: The Axioms

We now turn to the task of providing axioms for arithmetic in $\mathcal{L}_{\mathsf{F}}{}^{\mathsf{Num}}$ using the two devices the language gives us: the Frege quantifier F and the abstraction operator. One option, of course, since $\mathcal{L}_{\mathsf{F}}$ already interprets first-order logic, is just to reproduce the Peano-Dedekind axioms; this would not require using the abstraction operator at all. But nothing could be farther from the spirit of the current enterprise. Rather, we want to establish arithmetic firmly on the ground of the cardinal properties of numbers, as expressed through a judicious use of the abstraction operator, exploiting as much as we can the expressive power of the Frege quantifier.

The axioms we consider are formulated in the language containing the standard connectives, the Frege quantifier F, and the identity predicate $=$ as logical constants, while the *extra-logical* constants comprise a primitive one-place predicate symbol $\mathbb{N}$ and a primitive relational symbol $\leq$. The strict ordering relation $<$ and the Härtig quantifier I are taken to be definitional abbreviations in the usual way: $x < y$ stands for $x \leq y \wedge \neg y \leq x$ and $\mathsf{I}x(\varphi(x), \psi(x))$ stands for $\mathsf{F}x(\varphi(x), \psi(x)) \wedge \mathsf{F}x(\psi(x), \varphi(x))$.

The axioms can be conveniently divided into three groups. Group **A** comprises axioms that do not have significant existential import. These axioms either lay down the basic definitions or impose identity conditions on the entities involved. Group **B** comprises axioms that more directly characterize the arithmetical universe by specifying which entities are required, either directly or as a result of closure conditions, on the universe itself. Finally, group **C** comprises the definitions of addition and multiplication (which require the dyadic form of the Frege quantifier). We lay down the axioms first and then describe their function in detail.

A.1: $\mathsf{Num}(\varphi) = \mathsf{Num}(\psi) \leftrightarrow \mathsf{I}z(\varphi(z), \psi(z))$.
A.2: $\mathsf{Succ}(\varphi, \psi) \leftrightarrow \exists x(\psi(x) \wedge \mathsf{I}y(\varphi(y), \psi(y) \wedge y \neq x)$.
A.3: $\mathsf{Num}(\varphi) \leq \mathsf{Num}(\psi) \leftrightarrow \mathsf{F}\,z(\varphi(z), \psi(z))$.
A.4: $\forall x(\mathbb{N}(x) \leftrightarrow \mathsf{Fin}\,y(\mathbb{N}(y) \wedge y < x) \wedge x = \mathsf{Num}(\mathbb{N}(y) \wedge y < x))$.
B.1: $\forall x(\varphi(x) \rightarrow \exists! y(\psi(y) \wedge \theta(x, y))) \rightarrow \mathsf{F}\,x(\varphi(x), \psi(x))$.
B.2: $[\exists x(\mathbb{N}(x) \wedge \varphi(x)) \wedge \mathsf{Fin}\,x(\mathbb{N}(x) \wedge \varphi(x))] \rightarrow$
$\exists y[(\mathbb{N}(y) \wedge \varphi(y)) \wedge \forall x(\mathbb{N}(x) \wedge \varphi(x) \rightarrow x \leq y)]$.
C.1: $\mathsf{Prod}(\varphi, \psi, \theta) \iff \mathsf{I}xy(\varphi(x) \wedge \psi(y), \theta(x) \wedge x = y)$.
C.2: $\mathsf{Sum}(\varphi, \psi, \theta) \iff$
$\mathsf{I}xy((x = \mathsf{Num}(\eta_1) \wedge \varphi(y)) \vee (x = \mathsf{Num}(\eta_2) \wedge \psi(y)), \theta(x) \wedge x = y)$;

We take up these axioms in turn. Axiom **A.1** is just Hume's Principle, and its formulation in the present context is just as natural as in the standard, second-order version. Axiom **A.2** defines the notion "the number of ψs succeeds the number of φs"; this is the typical Frege-Russell definition providing a convenient abbreviation for a complex notion: $\mathsf{Succ}(\varphi, \psi)$ holds precisely when there is a ψ such that there are as many φs as there are ψs other than it. Axiom **A.3** provides a definition of the "less-than" relation; notice that it *does not* just provide an abbreviation: $\leq$ is a primitive symbol of the language and **A.3** specifies its meaning when it occurs between terms of the form $\mathsf{Num}(\varphi)$—that is, between two *abstracta*—and it says nothing about $x \leq y$ when one or both of x and y are not numerical *abstracta*. Having $\leq$ as a primitive symbol allows us to use it with quantified variables (which might in turn be instantiated with *abstracta*). From **A.3** and HP (i.e., **A.1**) it immediately follows that when applied to *abstracta* $\leq$ is antisymmetric (and conversely the antisymmetry of $\leq$ when so applied, together with **A.3**, implies HP). Also notice that as a consequence of our definitions we immediately have the following.

Proposition 6.1 *For any formulas φ and ψ,* $\mathsf{Num}(\varphi) < \mathsf{Num}(\psi)$ *holds if and only if* $[\mathsf{F}\,z(\varphi(z), \psi(z)) \wedge \neg\mathsf{F}\,z(\psi(z), \varphi(z))]$.

Similarly, HP implies that we could have just defined $\mathsf{Num}(\varphi) < \mathsf{Num}(\psi)$ as $\mathsf{Num}(\varphi) \leq \mathsf{Num}(\psi) \wedge \mathsf{Num}(\varphi) \neq \mathsf{Num}(\psi)$.

The last axiom in this group, **A.4**, characterizes the set $\mathbb{N}$ of the natural numbers. Notice that, just like **A.3**, this is not a definition in the standard sense—it does not provide an abbreviation for $\mathbb{N}(x)$. Rather, $\mathbb{N}$ is a *primitive* symbol of the language, and Axiom **A.4** should rather be viewed as providing something like an *implicit* definition of $\mathbb{N}$. The axiom says that x is a natural number if and only if it numbers its predecessors (in $\mathbb{N}$), and, moreover, the set of such predecessors is finite. In particular, it will follow that if $\mathbb{N}(x)$, where x is a variable, then x can be instantiated by a term of the form $\mathsf{Num}(\varphi)$, allowing, for instance, use of HP.

We now come to the second group of axioms. Axiom **B.1**, the "infinitary axiom," expresses the closure of the set $\mathcal{F}$ of injections under definability, and therefore subsumes the existence of the empty and identity maps. Axiom **B.2** expresses a particular form of the principle of induction, to the effect that "Every finite, nonempty set of numbers has a maximum" (this form will be shown to imply the standard one).

The two axioms in group **C** provide abbreviations for addition and multiplication: the formula $\mathsf{Prod}(\varphi, \psi, \theta)$ expresses that "the number of θ equals the number of φ

multiplied by the number of ψ" by saying that there are as many objects in θ (more exactly, along the diagonal of $\theta \times \theta$) as there are pairs whose first component is in φ and the second one in ψ. Similarly, the formula $\mathsf{Sum}(\varphi, \psi, \theta)$ represents the fact that "the number of θ equals the number of φ plus the number of ψ" by saying that there are as many objects in θ (again, along the diagonal) as there are pairs whose first member is $\mathsf{Num}(\eta_1)$ and second member is φ, or whose first member is $\mathsf{Num}(\eta_2)$ and second member is ψ. Here $\mathsf{Num}(\eta_1)$ and $\mathsf{Num}(\eta_2)$ only function as markers; η_1 and η_2 are fixed formulas applying to 0 and 1 objects, respectively.[12]

7 Representing Peano Arithmetic

These axioms allow the representation of a great many facts about arithmetic. In fact, on the standard interpretation of F, they are categorical and completely characterize the standard model of arithmetic. In this section we proceed to establish a number of arithmetical facts based on the general interpretation—the results will, of course, apply also to the standard interpretation. We will focus on the theory of successor—it will be clear how the present treatment can be extended to the operations of addition and multiplication using the axioms of the **C** group.

In this section by "valid" we mean "valid in every general model of $\mathscr{L}_{\mathsf{F}}{}^{\mathsf{Num}}$ satisfying the axioms **A.1** through **C.2**." Similarly, "if φ then ψ" means $\varphi \models_{\mathscr{L}_{\mathsf{F}}{}^{\mathsf{Num}}} \psi$; that is, for any general model $\mathfrak{M}$, if φ is true in it, then so is ψ.

Definition 7.1 Let 0 abbreviate $\mathsf{Num}_y(y \neq y)$.

Theorem 7.2 $\mathbb{N}(0)$ *is valid; that is,* 0 *is a number.*

Proof By Axiom **A.4**, we need to establish both the following:

$$0 = \mathsf{Num}_y(\mathbb{N}(y) \wedge y < 0), \tag{1}$$
$$\mathsf{Fin}_y(\mathbb{N}(y) \wedge y < 0). \tag{2}$$

To establish the first of these, by Hume's Principle, it suffices to prove $\mathsf{I}\, y(\mathbb{N}(y) \wedge = y < 0, y \neq y)$, which in turn abbreviates the conjunction of

$$\mathsf{F}\, y\, (y \neq y, \mathbb{N}(y) \wedge y < 0), \tag{3}$$
$$\mathsf{F}\, y\, (\mathbb{N}(y) \wedge y < 0, y \neq y). \tag{4}$$

The sentence (3) is clearly valid. To establish (4), it suffices to prove that $\forall y(\mathbb{N}(y) \rightarrow \neg y < 0)$. So assume $\mathbb{N}(y)$; in particular, y is an *abstractum*: $y = \mathsf{Num}_z(\mathbb{N}(z) \wedge z < y))$. Hence, if $y < 0$, then by Proposition 6.1 we have $\neg Fz(z \neq z, \mathbb{N}(z) \wedge z < y)$. But the latter is impossible—for its negation is valid, just like (3). So we have shown (4), which in turn suffices for (1) as well.

All that is left is to prove (2). But we just established that $\forall y \neg(\mathbb{N}(y) \wedge y < 0)$, and since the empty set is certainly finite, $\mathsf{Fin}_y(\mathbb{N}(y) \wedge y < 0)$ immediately follows. □

Definition 7.3 Whenever $\mathbb{N}(p)$ and $\mathbb{N}(q)$, let $\mathsf{Succ}(p, q)$ abbreviate $\mathsf{Succ}(\mathbb{N}(x) \wedge x < p, \mathbb{N}(x) \wedge x < q)$.

Notice that whenever p is a natural number then it is the abstractum of $\mathbb{N}(x) \wedge x < p$; so to say that some number q falls under the concept that p abstracts is just to say $q < p$.

Proposition 7.4 *Suppose p, q, and r are numbers, then*

(a) *if $p \leq q$, then either $p < q$ or $p = q$,*

(b) *if* $p < r$ *and* $\mathsf{Succ}(p, q)$*, then* $q \leq r$*.*

Proof Part (a) is immediate from the definitions. Part (b) requires a bit more work. Working in a general model $\mathfrak{M}$, assume $p < r$ and let $f \in \mathcal{F}$ witness the inequality; that is, f witnesses $\mathsf{F}x(x < p, x < r)$. Then there must be $y < r$ such that $y \notin \mathrm{rng}(f)$, for if not then f^{-1} would witness $Fx(x < r, x < p)$, whence $p = r$ against the assumption.

Now given $\mathsf{Succ}(p, q)$, there is z such that $z < q$ but $z \not< p$. Then $g = f \cup \{\langle z, y\rangle\}$ is in $\mathcal{F}$, witnessing $Fx(x < q, x < r)$; that is, $q \leq r$ as desired. □

Theorem 7.5 *Every number has a successor; that is, for all p there is a q such that* $\mathsf{Succ}(p, q)$.

Proof Assume p is in $\mathbb{N}$ so that $p = \mathsf{Num}_x(\mathbb{N}(x) \wedge x < p)$. Put $q = \mathsf{Num}_x\varphi(x)$, where $\varphi(x)$ abbreviates

$$(\mathbb{N}(x) \wedge x < p) \vee x = p.$$

We want to show

$$\mathsf{Succ}(\mathbb{N}(x) \wedge x < p, \varphi(x)), \tag{1}$$

$$\mathbb{N}(\mathsf{Num}_x\varphi(x)). \tag{2}$$

The first of these is equivalent to

$$\exists x[\varphi(x) \wedge \mathsf{I}_y(\mathbb{N}(y) \wedge y < p, \varphi(y) \wedge y \neq x)],$$

which clearly holds (with p itself witnessing the existential quantifier). In order to establish (2) as well, we need that $\mathsf{Num}_x\varphi(x)$ is in $\mathbb{N}$, which in turn by **A.4** requires both

$$\mathsf{Fin}_x(\mathbb{N}(x) \wedge x < \mathsf{Num}_y\varphi(y)); \tag{3}$$

$$\mathsf{Num}_x\varphi(x) = \mathsf{Num}_x(\mathbb{N}(x) \wedge x < \mathsf{Num}_y\varphi(y)). \tag{4}$$

For (3), we first notice that by Proposition 4.3, part (3) we have

$$\mathsf{Fin}_x((\mathbb{N}(x) \wedge x < p) \vee x = p), \tag{5}$$

and again by Proposition 4.3, part (2), it suffices to show, for $x \in \mathbb{N}$,

$$x < \mathsf{Num}_y\varphi(y) \to x < p \vee x = p. \tag{6}$$

But by definition of φ and the fact that $x \in \mathbb{N}$, we have

$$x < \mathsf{Num}_y\varphi(y) \to \mathsf{F}z(z < x, z < p \vee z = p). \tag{7}$$

So in the model there is an injection f of numbers less than x into the set of numbers $< p$ or $= p$. If $p \in \mathrm{rng}\, f$ then $x \leq p$, and if $p \notin \mathrm{rng}\, f$ then $x < p$, which gives (6), and immediately (3) as desired.

Now we turn our attention to (4). First observe that we can't have $p = \mathsf{Num}_x\varphi(x)$, or else we would have an injection of $(\mathbb{N}(x) \wedge x < p) \vee x = p$ into $x < p$ contradicting (5).

Next, to establish (4), clearly it suffices to show that the formulas within the Num operators are equiextensional; that is, after unpacking φ,

$$((\mathbb{N}(x) \wedge x < p) \vee x = p) \leftrightarrow (\mathbb{N}(x) \wedge x < \mathsf{Num}_y\varphi(y)).$$

For the right-to-left direction, observe that by (7) and Axiom **A.3**, we have $x < \mathsf{Num}_y\varphi(y) \to x \leq p$.

For the left-to-right direction, we proceed by cases, according as $x < p$ or $x = p$.

Case 1 If $x < p$, then in particular $\mathsf{F}z(z < x, z < p)$, which implies also $\mathsf{F}z(z < x, z < p \vee z = p)$. But by closure condition **Cc5**, also $\neg\mathsf{F}z(z < p \vee z = p, z < x)$ (or else there would be an injection of numbers $< p$ into numbers $< x$ against the assumption $x < p$). So we have $x < \mathsf{Num}_y((\mathbb{N}(y) \wedge y < p) \vee y = p$; that is, $x < \mathsf{Num}_y\varphi(y)$, as desired.

Case 2 If $x = p$, obviously it suffices to prove $p < \mathsf{Num}_y\varphi(y)$ (p is strictly less than its successor). Now clearly $\mathsf{F}z(z < p, z < p \vee z = p)$, which gives $p \leq \mathsf{Num}_x\varphi(x)$. But it can't also be $\mathsf{Num}_x\varphi(x) \leq p$ or else $p = \mathsf{Num}_x\varphi(x)$, which we have seen is impossible. So $p < \mathsf{Num}_x\varphi(x)$. □

Corollary 7.6 *If* $\mathsf{Succ}(p, q)$, *then* $p < q$.

Proposition 7.7 *The following hold:*

1. Succ *is a function: if* $\mathsf{Succ}(p, q)$ *and* $\mathsf{Succ}(p, r)$, *then* $q = r$;
2. Succ *is injective: if* $\mathsf{Succ}(p, r)$ *and* $\mathsf{Succ}(q, r)$, *then* $p = q$.

Proof For part (1), unpacking the two hypotheses using Definition 7.3 and Axiom **A.2**, we obtain

$$(\exists x < q)\,\mathsf{I}y(y < p, y < q \wedge y \neq x),$$
$$(\exists z < q)\,\mathsf{I}y(y < p, y < r \wedge y \neq z).$$

So by closure under composition, **Cc6**, it follows

$$\mathsf{I}y(y < q \wedge y \neq x, y < r \wedge y \neq z),$$

and, by **Cc4**, $\mathsf{I}y(y < q, y < r)$, which gives $\mathsf{Num}_y(y < q) = \mathsf{Num}_y(y < r)$; that is, $q = r$ as desired.

For part (2), it is similar but easier: the hypotheses give

$$(\exists x < r)\,\mathsf{I}y(y < p, y < r \wedge y \neq x),$$
$$(\exists z < r)\,\mathsf{I}y(y < q, y < r \wedge y \neq z),$$

and closure under composition immediately gives $\mathsf{I}y(y < p, y < q)$, whence $p = q$. □

Corollary 7.8 $\neg\mathsf{Fin}\,_x\mathbb{N}(x)$.

Proof Succ defines an injection of $\mathbb{N}$ into a proper subset of itself. The infinitary axiom **B.1** provides the desired witness to the claim. □

Theorem 7.9 *The following two forms of the induction schema are valid:*

$$\forall n((\forall m < n)\varphi(m) \to \varphi(n)) \to \forall n\,\varphi(n);$$
$$\varphi(0) \wedge \forall n \forall m(\mathsf{Succ}(n, m) \wedge \varphi(n) \to \varphi(m)) \to \forall n\,\varphi(n),$$

where quantifiers such as $\forall n(\ldots)$ *abbreviate* $\forall x(\mathbb{N}(x) \to \ldots)$.

Proof Since the latter standardly follows from the former, that's the one we deal with. Assume the antecedent of the induction principle; that is,

$$\forall n((\forall m < n)\varphi(m) \to \varphi(n)), \tag{1}$$

but suppose the consequence fails: $\neg\forall n\varphi(n)$. Choose n such that $\neg\varphi(n)$. In particular, $\mathsf{Fin}\,y(\mathbb{N}(y) \wedge y < n)$.

Now let $S = \{m < n : (\forall z < m)\varphi(z)\}$. Vacuously, $0 \in S$, so it follows that S is nonempty. By the assumption (1), $\forall x \in S(\mathbb{N}(x) \wedge \varphi(x))$. Moreover, since by definition $S \subseteq \{y : \mathbb{N}(y) \wedge y < n\}$ and $\mathsf{Fin}_y(\mathbb{N}(y) \wedge y < n)$, also $\mathsf{Fin}\, S$ (by Proposition 4.3).

By the induction axiom **B.2**, let $p = \max S$. Since $p \in S$, also $p < n$. By Theorem 7.9, p has a successor, q, such that $q \leq n$ (by part (b) of Proposition (7.4)) and, moreover, $q \notin S$. From $q \leq n$ we distinguish two subcases:

Case $q < n$; then $q \in \{m < n : (\forall z < m)\varphi(z)\} = S$, which is impossible.

Case $q = n$; then again $(\forall z < q)\varphi(z)$, which by the assumption (1) gives $\varphi(q)$, that is, $\varphi(n)$, also impossible. □

The following is a version of the "axiom of counting" that says that for any number p the number of $x \leq p$ is the successor of p.

Proposition 7.10 $\mathbb{N}(x) \rightarrow \mathsf{Succ}_x(x, \mathsf{Num}_z(z \leq x))$.

Proof Immediate by induction on x. □

Theorem 7.11 *Peano Arithmetic is interpretable in the theory with axioms* ***A.1*** *through* ***C.2*** *in* $\mathscr{L}_{\mathsf{F}}^{\mathsf{Num}}$.

Proof We already have all the facts we need. We just record their proof:

1. 0 is a number: Theorem 7.2.
2. Every number has a unique successor: Theorem 7.5 and Corollary 7.7.
3. Every number other than 0 is a successor: immediate proof by induction.
4. The successor function is injective: Corollary 7.7.
5. The induction schema: Theorem 7.9. □

8 Conclusion and Further Developments

The account just developed exploits the interaction of a nonstandard, but still first-order cardinality quantifier and a numerical abstraction operator to provide an interpretation of arithmetic. The cardinality quantifier can be given two equally attractive interpretations. On the "standard" interpretation we obtain an account of arithmetic that is *categorical* in that it characterizes the standard model up to isomorphism. But we have shown that Peano Arithmetic can be obtained on the general interpretation as well.

The Frege quantifier is rather natural and attractive, and its use in providing a formalization of arithmetic allows us to avoid using second- or higher-order logic. The semantical interpretation of F is, as we have seen, completely at the first order. It is true that such an interpretation involves reference to functions of a certain kind, but such functions are *not* available as genuine objects of quantification: one cannot say, for instance, that the composition of two such functions is again a function of the desired kind—part of the reason why such a closure condition had to be built into the semantics.

Another potential issue is the "logical character" of the Frege quantifier. As we have seen, the Frege quantifier is logically invariant—that is, invariant under permutations of the domain—a feature it shares with a number of various notions with quite explicit mathematical or set-theoretical content, for example, the quantifier "continuum many." While such invariance is a necessary condition for the logical character of some notion (or so could one convincingly argue), the approach of the present paper does not require a separate argument for the logical character of F. The approach is based on the assumption that *cardinality is already a logical notion* (at least in the limited way in which F allows us to compare cardinalities) and explores the extent to which arithmetic can be developed based on such an assumption.

A thread that runs throughout the paper is the distinction between *cardinality* and *number*. While the former can be taken to be, in some sense, a logical notion, the latter is not, in a strong sense, a logical notion. Numbers figure in the present approach in two distinct ways. They are introduced by abstraction, as denotations of terms of the form $\mathsf{Num}(\varphi)$, but the properties of such abstracta are not logical, and they have to be explicitly posited by means of axioms whose nature is extra-logical. The second way in which numbers enter the picture is through the "definition" of the set $\mathbb{N}$ of the natural numbers. As we have seen, this "definition," as given in Axiom **A.4**, is not really an explicit definition in the standard sense, whereas the Fregean and neo-Fregean definition of $\mathbb{N}$ really is an explicit definition. But the Fregean definition of $\mathbb{N}$ essentially requires second-order tools that are not available in the present approach. Therefore we have to content ourselves with the kind of implicit definition provided by Axiom **A.4**.

Let us conclude by briefly pointing out avenues for further research. Besides the already-mentioned development of the theory of addition and multiplication using the **C** axioms, there remains to pinpoint more exactly the complexity of $\mathscr{L}_{\mathsf{F}}^{\mathsf{Num}}$ on the general interpretation (possibly exploring alternative sets of closure conditions for $\mathscr{F}$) as well as the strength of the given axiomatization of arithmetic on the general interpretation. The general motivation and the philosophical and conceptual underpinnings of the present approach will be more extensively explored in [3], whereas a general theory of Functional Equivalences (such as HP) and their properties will be developed and explored elsewhere.

Notes

1. For more on the cardinal properties of the natural numbers, see [12].

2. See, e.g., [4; 11; 23].

3. The philosophical motivation for the approach is presented more at length in an accompanying paper, [3].

4. They are also referred to as *monadic*, but we'd rather reserve the term "monadic" for quantifiers that apply to formulas having only one free variable, "dyadic" for quantifiers that apply to formulas with two free variables, and so on.

5. See [16] for a general account of logical notions.

6. Compare this to the situation in set theory, where in order to express certain relationships between the cardinality of two given sets, one has to appeal to the existence of certain *other objects* in the domain of quantification—such objects are, in turn, sets of a certain kind, containing ordered pairs as members and satisfying certain further conditions. Alternatively, one can express such cardinality notions at the second order, by asserting the existence of *relations* satisfying certain further constraints.

7. The axiom of choice, in the form of the trychotomy principle, is needed in order to go from $|A| \not> |B|$ to $|A| \leq |B|$.

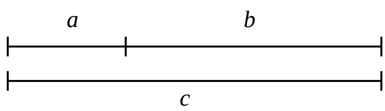
8. Strictly speaking, Definitions 3.4 and 3.5 should be given by simultaneous recursion.

9. The above results were originally proved in connection with Härtig's quantifier in the 1970s and subsequently rediscovered by the author when preparing the present paper. See [14] for a systematic treatment of Härtig's quantifier and its expressive power—including the fact that, perhaps surprisingly, the set of $\mathscr{L}_{\mathsf{I}}(<)$-validities *is* decidable.

   As further evidence of the expressive power of F, consider the following, adapted from [14]. While it is well known that addition is not definable in first-order logic over the structure $(\mathbb{N}, <)$, addition is so definable in $\mathscr{L}_{\mathsf{F}}$:

$$a + b = c \Longleftrightarrow (\mathbb{N}, <) \models \mathsf{I}x(x < b, a \leq x < c).$$

   The proof, "without words," only requires the diagram:



10. A further condition we might adopt is closure of $\mathcal{F}$ under *definability*; in order to keep the basic semantic framework as simple as possible, an axiom to that effect will be introduced in Section 6.

11. Antonelli and May [1] use a similar device, but there abstraction was represented by means of a heterogeneous *predicate* $\mathsf{VR}(x, \varphi)$ expressing the fact that x is the value range of φ. Since the abstraction principles deal directly with extensions, care was taken that not all φs were assigned a value-range, lest a paradox would arise. In keeping with the *non*-logical character of abstraction, that paper contains a particular, well-motivated choice for concepts having value ranges.

12. An alternative definition of addition using only monadic F is given in note 9 using $\leq$ and $<$ as primitive.

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Acknowledgments

I am grateful to Albert Visser, Ed Zalta, and an anonymous referee for the *NDJFL* for helpful comments and suggestions.

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