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Expansions of o-Minimal Structures by Iteration Sequences

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Abstract Let P be the ω-orbit of a point under a unary function definable in an o-minimal expansion $\mathfrak{R}$ of a densely ordered group. If P is monotonically cofinal in the group, and the compositional iterates of the function are cofinal at $+\infty$ in the unary functions definable in $\mathfrak{R}$, then the expansion $(\mathfrak{R}, P)$ has a number of good properties, in particular, every unary set definable in any elementarily equivalent structure is a disjoint union of open intervals and finitely many discrete sets.

The reader is assumed to be familiar with the basics of o-minimality, including the associated model theory; see, for example, Dries [2]. Throughout, "$\varnothing$-definable" means "definable without parameters", while "definable" means "definable with parameters". The set of nonnegative integers is denoted by $\mathbb{N}$; n ranges over $\mathbb{N}$.

Given a set X and a function $\phi\colon X \to X$, let ϕ_0 denote the identity on X and put $\phi_{n+1} = \phi \circ \phi_n$. For $x \in X$, put $\phi_{\mathbb{N}}(x) = \{\, \phi_n(x) : n \in \mathbb{N} \,\}$.

Until further notice, $\mathfrak{R}$ denotes an o-minimal expansion of a densely ordered group $(R, <, +, 0)$ and ϕ denotes a unary function definable in $\mathfrak{R}$. We are interested in expansions of $\mathfrak{R}$ by sets $\phi_{\mathbb{N}}(c)$ ($c \in R$), particularly when $\mathfrak{R}$ is an expansion of the real field. In this note, we deal with a special, but natural, case.

Given $c \in R$ such that the sequence $(\phi_n(c))_{n\in\mathbb{N}}$ is increasing and unbounded above (in R), define $\lambda\colon R \to R$ by

$$t \mapsto \begin{cases} \max\big((-\infty, t] \cap \phi_{\mathbb{N}}(c)\big), & t \geq c \\ c, & t < c. \end{cases}$$

We say that $\mathfrak{R}$ is *ϕ-bounded* if for each definable $f\colon R \to R$ there exists $N \in \mathbb{N}$ (depending on f) such that $f(t) \leq \phi_N(t)$ as $t \to +\infty$, in other words, if the germs

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at $+\infty$ of the compositional iterates of ϕ are cofinal in the germs at $+\infty$ of the unary functions definable in $\mathfrak{R}$.

Until further notice, assume also that $\mathfrak{R}$ is ϕ-bounded and $c \in R$ is such that $(\phi_n(c))_{n\in\mathbb{N}}$ is increasing and unbounded above.

Theorem 1 *Every n-ary set definable in $\big(\mathfrak{R}, \phi_{\mathbb{N}}(c)\big)$ is a finite union of sets of the form*

$$\{\, x \in R^n : f_1(x) = \cdots = f_M(x) = 0, g_1(x) < 0, \dots, g_N(x) < 0 \,\},$$

where the f_i and g_j are given piecewise by finite compositions of λ and functions definable in $\mathfrak{R}$. Every function definable in $\big(\mathfrak{R}, \phi_{\mathbb{N}}(c)\big)$ is given piecewise by finite compositions of λ and functions definable in $\mathfrak{R}$.

If $\mathfrak{R}$ defines a bijection between a bounded interval and an unbounded interval, then the above holds with $M = 1$.

If both ϕ and c are $\varnothing$-definable, then all of the above holds with "$\varnothing$-definable" in place of "definable".

There are some interesting consequences, especially if $\mathfrak{R}$ is an expansion of the field of real numbers, but we defer discussion.

Proof Let $\mathcal{L}_0 \supseteq \{<, +, -, 0, \phi, c\}$ be a language such that $\mathfrak{R}$ is an $\mathcal{L}_0$-structure. We shall not distinguish notationally between ϕ and c and their representing terms.

It suffices to consider the case that $c > 0$ and ϕ is an isomorphism of $(R, <)$ such that $\phi(t) > t$ for all $t \in R$, as we now show. Since $(\phi_n(c))_{n\in\mathbb{N}}$ is increasing and unbounded above, the set $\{\, t \in R : \phi(t) > t \,\}$ is unbounded above. By o-minimality, there exists $d > 0$ such that $\phi(t) > t$ for all $t \geq d$. By the Monotonicity Theorem, we may further assume that the restriction of ϕ to $[d, \infty)$ is strictly increasing and continuous. Since $\phi_{\mathbb{N}}(c)$ is unbounded above, there exists $N \in \mathbb{N}$ such that $\phi_N(c) \geq d$. Since there are only finitely many $x \in \phi_{\mathbb{N}}(c)$ with $x < \phi_N(c)$, we may assume that $N = 0$, that is, $c \geq d$. By replacing d with c, we may take $d = c$. Finally, replace ϕ with

$$t \mapsto \begin{cases} t + \phi(c) - c, & t < c \\ \phi(t), & t \geq c. \end{cases}$$

Since c is nonzero and $\varnothing$-definable, the complete theory $\mathrm{Th}(\mathfrak{R})$ of $\mathfrak{R}$ has definable Skolem functions, so we may reduce to the case that $\mathrm{Th}(\mathfrak{R})$ admits QE (quantifier elimination) and is universally axiomatizable, and that $\mathcal{L}_0$ has no relation symbols other than $<$. Let $\mathcal{L}$ be the result of extending $\mathcal{L}_0$ by a new unary function symbol which, for convenience, we denote also by λ. Now, $(R, <, \lambda)$ is interdefinable with $(R, <, \phi_{\mathbb{N}}(c))$, so in order to establish the first paragraph of the theorem, it suffices (by a routine compactness argument) to show that, as an $\mathcal{L}$-theory, $\mathrm{Th}(\mathfrak{R}, \lambda)$ admits QE and is universally axiomatizable. Let T be the $\mathcal{L}$-theory $\mathrm{Th}(\mathfrak{R})$ together with sentences expressing

$$\begin{aligned}
&s \le t \to \lambda(s) \le \lambda(t),\\
&t < \phi(c) \to \lambda(t) = c,\\
&\lambda(\phi(c)) = \phi(c),\\
&t \ge \phi(c) \to \lambda(t) \le t < \phi(\lambda(t)),\\
&\lambda(t) \le s < \phi(\lambda(t)) \to \lambda(s) = \lambda(t),\\
&\lambda(t) = t \leftrightarrow \lambda(\phi(t)) = \phi(t).
\end{aligned}$$

(These are somewhat redundant, but convenient in their present form.) Since $(\mathfrak{P}, \lambda\restriction P)$ embeds into every model of T, where $\mathfrak{P}$ is the prime submodel of $\mathfrak{R}$ and P its underlying set, it suffices now to show that T admits QE, for then T is also complete. Let $(\mathfrak{A}, \mu), (\mathfrak{B}, \lambda) \models T$, with $(\mathfrak{A}, \mu)$ a proper submodel of $(\mathfrak{B}, \lambda)$, and let $(\mathfrak{B}^*, \lambda^*)$ be a $\operatorname{card}(B)^+$-saturated elementary extension of $(\mathfrak{A}, \mu)$. Let A, B, B^* denote the corresponding underlying sets. Since T is universal, it suffices to show that, for some $b \in B \setminus A$, the substructure of $(\mathfrak{B}, \lambda)$ generated by b over $(\mathfrak{A}, \mu)$ embeds into $(\mathfrak{B}^*, \lambda^*)$ fixing A pointwise. We have some preliminary work to do.

Given $X \subseteq B$, let $H(X)$ be the convex hull of X in B, and $\operatorname{dcl}(X)$ be the definable closure of X in B with respect to $\operatorname{Th}(\mathfrak{R})$. Given $b \in B$, we write $0 \ll b$ if b is greater than every element of $\operatorname{dcl}(\varnothing)$. For $0 \ll b \in B$, let $[b]$ denote the convex hull in B of the set of all values $f(b)$, with f ranging over all strictly increasing and unbounded-above functions $B \to B$ that are $\varnothing$-definable in $\mathfrak{R}$ (i.e., $[b]$ is the $\operatorname{Th}(\mathfrak{R})$-level of b, as defined in Tyne [10]). Suppose that $A \cap [b] = \varnothing$. By [10], 3.11,[1] we have

$$\{\, x \in \operatorname{dcl}(A \cup \{b\}) : 0 \ll x \,\} \subseteq \bigcup_{0 \ll a \in A} [a] \cup [b]. \qquad (*)$$

(This uses only that $\operatorname{Th}(\mathfrak{R})$ is complete, o-minimal, and has definable Skolem functions.)

Given $0 \ll b \in B$, put $\phi_{\mathbb{Z}}(b) = \phi_{\mathbb{N}}(b) \cup \{\, \phi_n^{-1}(b) : n \in \mathbb{N} \,\}$. Now, ϕ is a $\varnothing$-definable isomorphism of $(R, <)$, so the same is true of each ϕ_n, as well as each compositional inverse ϕ_n^{-1}. Hence, $\phi_{\mathbb{Z}}(b) \subseteq [b]$. By ϕ-boundedness, $\phi_{\mathbb{Z}}(b)$ is not only cofinal in $[b]$, but also downward cofinal in $[b]$. It is now easy to check that $H(\phi_{\mathbb{Z}}(b)) = [b] = [\lambda(b)] = H(\phi_{\mathbb{Z}}(\lambda(b)))$. Hence, by $(*)$, we have the following lemma.

Lemma 2 *If $0 \ll b \in B$ and $A \cap H(\phi_{\mathbb{Z}}(b)) = \varnothing$, then*

$$\{\, x \in \operatorname{dcl}(A \cup \{b\}) : 0 \ll x \,\} \subseteq \bigcup_{0 \ll a \in A} H(\phi_{\mathbb{Z}}(\lambda(a))) \cup H(\phi_{\mathbb{Z}}(b)).$$

We return to the proof proper.

Suppose that $\lambda(B) \ne \mu(A)$. Fix $b \in \lambda(B) \setminus \mu(A)$. Then $0 \ll b$ and $A \cap H(\phi_{\mathbb{Z}}(b)) = \varnothing$, in particular, $b \notin A$. Let $\mathfrak{C}$, with underlying set C, be the substructure of $\mathfrak{B}$ generated by b over $\mathfrak{A}$; then $C = \operatorname{dcl}(A \cup \{b\})$. By saturation, there exists $b^* \in B^* \setminus A$ such that $\lambda^*(b^*) = b^*$ and b^* realizes the same cut in A as b. Since $\operatorname{Th}(\mathfrak{R})$ is o-minimal, there is an $\mathcal{L}_0$-embedding $e\colon C \to B^*$ fixing A pointwise such that $e(b) = b^*$. It follows easily from the lemma (and the "λ axioms") that $\lambda(C) \subseteq C$ and $e(\lambda(x)) = \lambda^*(e(x))$ for every $x \in C$. Hence, $(\mathfrak{C}, \lambda\restriction C)$ is a substructure of $(\mathfrak{B}, \lambda)$, and e is an $\mathcal{L}$-embedding as well.

The case that $\lambda(B) = \mu(A)$ is similar, but easier: Any $b \in B \setminus A$ will do, and the lemma is not needed. We omit the details. (We have now established the first paragraph of Theorem 1.)

Suppose now that $\mathfrak{R}$ defines a bijection between a bounded interval and an unbounded interval. We show that we may take $M = 1$ in the statement of Theorem 1. By definability of Skolem functions, there is a $\varnothing$-definable bijection between a bounded interval and an unbounded interval. By Peterzil and Starchenko [9], there exist binary operations $\oplus, \odot$ on R that are $\varnothing$-definable in $\mathfrak{R}$ such that $(R, \oplus, \odot, 0, c)$ is a real closed field with additive identity 0 and multiplicative identity c. Hence, for all $r_1, \dots, r_M \in R$, we have $r_1 = \cdots = r_M = 0$ if and only if the sum of the squares of the r_k, taken with respect to $\oplus$ and $\odot$, is equal to 0. The final paragraph of the theorem is immediate by examination of the proof so far. □

The proof of the theorem is quite similar to that of Miller [8], Proposition 8.6 (which was inspired by Dries [1], Theorem II), but [10], 3.11, replaces the use of the Wilkie Inequality from Dries [3], Theorem C.

In general, $\big(\mathfrak{R}, \phi_{\mathbb{N}}(c)\big)$ does not admit QE (in an extension of $\mathcal{L}_0$ by a new unary predicate): It is easy to check that every unary quantifier-free definable set in $\big(\mathfrak{R}, \phi_{\mathbb{N}}(c)\big)$ has either nonempty interior or only finitely many limit points. If $\mathfrak{R}$ expands a field, then every $x \in \phi_{\mathbb{N}}(c)$ is a limit point of the definable set $\{\, x + (1/y) : x, y \in \phi_{\mathbb{N}}(c), y \neq 0 \,\}$.

We now collect some consequences of the theorem.

Corollary 3 (of the proof) $\mathrm{Th}\big(\mathfrak{R}, \phi_{\mathbb{N}}(c)\big)$ *is axiomatized relative to* $\mathrm{Th}(\mathfrak{R}, \phi, c)$ *by axioms expressing that*

$$\big(\phi_{\mathbb{N}}(c), <, c, \phi{\restriction}\phi_{\mathbb{N}}(c)\big) \equiv \big(\mathbb{N}, <, 0, n \mapsto n+1\big)$$
$$\forall x \geq c\; \exists y \in \phi_{\mathbb{N}}(c),\;\; y \leq x < \phi(y).$$

A first-order theory extending the theory of dense linear orders without endpoints is *d-minimal* (short for “discrete-minimal”) if, in every model, every unary definable set either has interior or is a finite union of discrete sets, and the underlying set of the model is definably connected (in the model). An expansion of a dense linear order without endpoints is d-minimal if its complete theory is d-minimal. (We regard these definitions as provisional; it is not yet clear that they capture the notion of “the next best thing to o-minimality” for expansions of densely ordered structures by infinite discrete sets.) For expansions of the real line, especially of the real field, a number of interesting properties follow from d-minimality; see [8], §3.4. The situation is less understood otherwise; indeed, it is not clear how to define the right analogues for some of the properties that hold when working over the real line (but see also Miller [7]).

Corollary 4 $\big(\mathfrak{R}, \phi_{\mathbb{N}}(c)\big)$ *is d-minimal.*

Proof With $\mathcal{L}$ as in the proof of the theorem, let $\mathcal{L}_R$ be the expansion of $\mathcal{L}$ by constants for elements of R. By induction on complexity, for every finite set Σ of unary $\mathcal{L}_R$-terms there exist $m \in \mathbb{N}$, $f : R^{m+1} \to R$ definable in $\mathfrak{R}$, and $S \subseteq R$ such that

1. S is a finite union of discrete sets definable in $\big(\mathfrak{R}, \phi_{\mathbb{N}}(c)\big)$;
2. $R \setminus S$ is a disjoint union of open intervals with endpoints in $S \cup \{\pm\infty\}$;

3. if $-\infty \le a < b \le +\infty$ and $(a,b) \cap S = \varnothing$, then for each $\sigma \in \Sigma$ there exists $x \in R^m$ such that $\sigma\restriction(a,b) = f(x,\cdot)\restriction(a,b)$.

Let $A \subseteq R$ be definable in $\big(\mathfrak{R}, \phi_{\mathbb{N}}(c)\big)$. Since A is quantifier-free definable in $(\mathfrak{R}, \lambda)$, there exist (by the above) $S \subseteq R$, $m \in \mathbb{N}$, and $B \subseteq R^{m+1}$ such that

1. S is a finite union of discrete sets definable in $(\mathfrak{R}, \lambda)$;
2. $R \setminus S$ is a disjoint union of open intervals with endpoints in $S \cup \{\pm\infty\}$;
3. B is definable in $\mathfrak{R}$;
4. if $-\infty \le a < b \le +\infty$ and $(a,b) \cap S = \varnothing$, then there exists $x \in R^m$ such that $(a,b) \cap A = (a,b) \cap B_x$.

Hence, A is a union of disjoint open intervals and finitely many discrete sets definable in $\big(\mathfrak{R}, \phi_{\mathbb{N}}(c)\big)$. The argument is the same in arbitrary models of $\mathrm{Th}\big(\mathfrak{R}, \phi_{\mathbb{N}}(c)\big)$, so $\big(\mathfrak{R}, \phi_{\mathbb{N}}(c)\big)$ is d-minimal. □

Remark We analyzed only the unary $\mathcal{L}_R$-terms. As the reader might imagine, something stronger (at least, on the face of it) than d-minimality can be established by analyzing arbitrary terms, but we shall not pursue this matter here.

Corollary 5 *If $R = \mathbb{R}$, then the expansion of $\mathfrak{R}$ by any collection of subsets of any Cartesian powers of $\phi_{\mathbb{N}}(c)$ is d-minimal.*

Proof See [8], §3.4. □

Now assume that $\mathfrak{R}$ expands the field of real numbers, and drop the assumption that $\mathfrak{R}$ is ϕ-bounded. Suppose that $\mathfrak{R}$ is polynomially bounded (equivalently, x^2-bounded) and $\phi(t)/t$ is unbounded above as $t \to +\infty$. By Miller [6], there exist $a > 0$ and $r > 1$ such that the power function x^r is definable in $\mathfrak{R}$ and $\lim_{t\to+\infty} \phi(t)/t^r = a$. For each n, we then have

$$\lim_{t\to+\infty} \phi_n(t)/t^{r^n} = a^{(r^n-1)/(r-1)},$$

so $\mathfrak{R}$ is ϕ-bounded. By the Monotonicity Theorem, there exists $C \in \mathbb{R}$ such that, for each $c \ge C$, the sequence $(\phi(c))_{n\in\mathbb{N}}$ is strictly increasing and unbounded above. Hence, for every sufficiently large c, Theorem 1 applies to $\big(\mathfrak{R}, \phi_{\mathbb{N}}(c)\big)$. Let us examine the situation further. Write $\phi_n(c) = b_n a^{(r^n-1)/(r-1)} c^{r^n}$. Note that

$$\frac{\phi_{n+1}^{1/r}(c)}{a^{1/r}\phi_n(c)} = \frac{b_{n+1}^{1/r}}{b_n},$$

so $\big(\mathfrak{R}, \phi_{\mathbb{N}}(c)\big)$ defines the set $B := \{\, b_{n+1}^{1/r}/b_n : n \in \mathbb{N}\,\}$. By Corollary 4, $\big(\mathfrak{R}, \phi_{\mathbb{N}}(c)\big)$ is d-minimal, so there are some obvious limitations on the nature of B; for example, unless it is finite, it is not of the form $f(\mathbb{N})$ for any unary f definable in $\mathfrak{R}$ (since otherwise, by monotonicity, $\big(\mathfrak{R}, \phi_{\mathbb{N}}(c)\big)$ would define $\mathbb{N}$, hence all real projective sets). For a deeper analysis, see Miller [5].

Remark The assumption that $\phi(t)/t$ be unbounded above at $+\infty$ is not necessary, but the situation is more delicate otherwise. For example, with $a > 1$, $c > 0$, and $\phi = ax$, the conclusion of Theorem 1 holds for $\big(\mathfrak{R}, \phi_{\mathbb{N}}(c)\big)$ if and only if $\mathfrak{R}$ defines no irrational power functions; see [8], §3.4 for details.

Corollary 6 *If $\mathfrak{R}$ is polynomially bounded, then for each $a > 0$, $c > 1$, and $r > 1$ such that the power function x^r is definable in $\mathfrak{R}$, the expansion of $\mathfrak{R}$ by the set*

$$\left\{ a^{(r^n-1)/(r-1)} c^{r^n} : n \in \mathbb{N} \right\}$$

is d-minimal.

Remark We could analyze similarly the case that $\mathfrak{R}$ is an o-minimal expansion of $(\mathbb{R}, <, +)$ that does not define multiplication, but the extra generality is illusory except in the rather degenerate case that every unary function definable in $\mathfrak{R}$ is ultimately linear; see, for example, the discussion following the statement of Friedman and Miller [4], Theorem 3. We shall not pursue this matter here.

If $\mathfrak{R}$ is not polynomially bounded, then it is exponential (i.e., it defines the function e^x) [6]; we close with an application to this case.

Corollary 7 *If $\mathfrak{R}$ is exponential and exponentially bounded, then for each $c > 1$, the expansion of $\mathfrak{R}$ by the set of towers $\{c, c^c, c^{c^c}, \dots\}$ is d-minimal.*

(We use the established terminology "exponentially bounded" rather than "e^x-bounded".)

The only previously known d-minimal expansions of the real exponential field were obtained from sequences having much faster growth rates; see [4] for information.

Note

1. The second author is currently preparing a manuscript for publication, entitled "T-height in weakly o-minimal structures," that includes a generalization of [10], 3.11.

References

[1] Dries, L. van den, "The field of reals with a predicate for the powers of two," *Manuscripta Mathematica*, vol. 54 (1985), pp. 187–95. Zbl 0631.03020. MR 808687. 96

[2] Dries, L. van den, "o-minimal structures," pp. 137–85 in *Logic: From Foundations to Applications (Staffordshire, 1993)*, edited by W. Hodges, M. Hyland, and C. Steinhorn, Oxford Science Publications, Oxford University Press, New York, 1996. Zbl 0861.03028. MR 1428004. 93

[3] Dries, L. van den, "T-convexity and tame extensions. II," *The Journal of Symbolic Logic*, vol. 62 (1997), pp. 14–34. Zbl 0922.03055. MR 1450511. 96

[4] Friedman, H., and C. Miller, "Expansions of o-minimal structures by fast sequences," *The Journal of Symbolic Logic*, vol. 70 (2005), pp. 410–18. MR 2140038. 98

[5] Miller, C., "Avoiding the projective hierarchy in expansions of the real field by sequences," forthcoming in *Proceedings of the American Mathematical Society*. 97

[6] Miller, C., "Exponentiation is hard to avoid," *Proceedings of the American Mathematical Society*, vol. 122 (1994), pp. 257–59. Zbl 0808.03022. MR 1195484. 97, 98

[7] Miller, C., "Expansions of dense linear orders with the intermediate value property," *The Journal of Symbolic Logic*, vol. 66 (2001), pp. 1783–90. Zbl 1001.03039. MR 1877021. 96

[8] Miller, C., "Tameness in expansions of the real field," pp. 281–316 in *Logic Colloquium '01*, vol. 20 of *Lecture Notes in Logic*, edited by M. Baaz, S. D. Friedman, and J. Krajicek, Association for Symbolic Logic/A. K. Peters, Urbana/Natick, 2005. Zbl 02222021. MR 2143901. 96, 97

[9] Peterzil, Y., and S. Starchenko, "A trichotomy theorem for o-minimal structures," *Proceedings of the London Mathematical Society*, Third Series, vol. 77 (1998), pp. 481–523. Zbl 0904.03021. MR 1643405. 96

[10] Tyne, J., *T-levels and T-convexity*, Ph.D. thesis, University of Illinois at Urbana-Champaign, 2003. 95, 96, 98

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