Notre Dame Journal of Formal Logic
Volume VIII, Number 4, October 1967

# ON NEW THEOREMS FOR ELEMENTARY NUMBER THEORY

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*Introduction*: This paper* generalizes the classical number-theoretic notions of multiplicative function and additive function. In addition to extending classes of functions it can be shown that certain specific functions such as, e.g., Euler's totient, Möbius' function, and Liouville's function have precise analogues within the theory. Results of G. H. Hardy and S. Ramanujan are used in the study.

1. Apply the Unique Factorization Theorem (UFT) of [1], *viz.*, $n = p_1^{\alpha_1} \cdot \ldots \cdot p_m^{\alpha_m}$ ($p_i$ simply ordered, and distinct) to its own natural number exponents $>1$, and apply the UFT to the exponents $>1$ so obtained, etc. (use induction), until the process terminates, for a given $n$, with a *unique* "constellation" of prime numbers alone called a *mosaic* [2;3]. E.g., *the* mosaic of 10,000 is $2^{2^2} \cdot 5^{2^2}$. This process provides a simple revision of the standard Gaussian model of the UFT [3].

*Definition 1.* A number-theoretic function $f$ is said to be *generalized multiplicative* provided $f(a \cdot b) = f(a) \cdot f(b)$, if the mosaics of $a$ and $b$ have no prime in common.

*Lemma 1. Every standard multiplicative number-theoretic function* [1] *is generalized multiplicative, and there exists a generalized multiplicative function* $\psi^2$ *(defined as* $\psi(\psi(\cdot))$*), where* $\psi$ *is defined as follows:* $\psi(n)$ *is the product of the primes alone in the mosaic of n which is not multiplicative; i.e., the class of generalized multiplicative functions property contains the class of multiplicative functions.*

*Proof.* Clearly every multiplicative function is generalized multiplicative, since if the function "factors" when $a$ and $b$ have no prime base in common (but possibly their mosaics *may* have some prime in common) then, *a fortiori,* the function "factors" when the mosaics of $a$ and $b$ have no prime in common. Now, since $\psi$ is multiplicative $\psi^2(a \cdot b) = \psi(\psi(a) \cdot \psi(b))$. But, if the mosaics of $a$ and $b$ have no prime in common then $(\psi(a), \psi(b)) = 1$.

*This research is sponsored by the U. S. Atomic Energy Commission.

*Received August 30, 1965*

Hence $\psi^2(a \cdot b) = \psi^2(a) \cdot \psi^2(b)$. I.e., $\psi^2$ is generalized multiplicative. Finally, $\psi^2(36) = 18$, but $\psi^2(4) \cdot \psi^2(9) = 4 \cdot 6 = 24$; hence $\psi^2(4 \cdot 9) \neq \psi^2(4) \cdot \psi^2(9)$.

*Definition 2.* A number-theoretic function $g$ is said to be *generalized additive* provided $g(a \cdot b) = g(a) + g(b)$, if the mosaics of $a$ and $b$ have no prime in common.

*Lemma 2. Every standard additive function* [4] *is generalized additive, and there exists a generalized additive function* $(\psi^*(\psi(\cdot)))$, *where* $\psi$ *is defined as before and* $\psi^*$ *is defined as follows:* $\psi^*(n)$ *is the sum of the primes alone in the mosaic of n which is not additive; i.e., the class of generalized additive functions properly contains the class of additive functions.*

*Proof.* By reasoning similar to that used in Lemma 1 every additive function is generalized additive. Now, since $\psi$ is multiplicative, $\psi^*(\psi(a \cdot b)) = \psi^*(\psi(a) \cdot \psi(b))$. Hence, if the mosaics of $a$ and $b$ have no prime in common, $\psi(a)$ and $\psi(b)$ are relatively prime. Thus by the additivity of $\psi^*$, $\psi^*(\psi(a \cdot b)) = \psi^*(\psi(a)) + \psi^*(\psi(b))$. Finally, $\psi^*(\psi(36)) = 8$, but $\psi^*(\psi(4)) + \psi^*(\psi(9)) = 4 + 5 = 9$. Hence $\psi^*(\psi(4 \cdot 9)) \neq \psi^*(\psi(4)) + \psi^*(\psi(9))$.

**2.** At this point we discuss the effect of modifying a number of specific number-theoretic functions with classical significance. We shall see that, e.g., the property of generalized multiplicativity of the Möbius' function or Liouville's function is preserved under the change to their analogues.

*Definition 3.* Define $\mu^*$ (*Modified Möbius function*) as follows: $\mu^*(1) = 1$; if $n > 1$, $\mu^*(n) = 0$, if any prime is repeated in the mosaic of $n$; if $n > 1$, $\mu^*(n) = (-1)^m$, if no prime is repeated in the mosaic of $n$, where $m$ is the number of (distinct) primes in the mosaic of $n$. Define $\lambda^*$ (*Modified Liouville's function*) as follows: $\lambda^*(1) = 1$; if $n > 1$, $\lambda^*(n) = (-1)^m$, where $m$ is the number of primes in the mosaic of $n$, counting repetitions according to their multiplicities, if any. Define $\Omega_1^*$ as follows: $\Omega_1^*(n)$ is the total number of primes in the mosaic of $n$ counting repetitions according to their multiplicities, if any. Define $\Omega_2^*$ as follows: $\Omega_2^*(n)$ is the number of distinct primes in the mosaic of $n$.

*Lemma 3.* $\mu^*$ *is generalized multiplicative, but not multiplicative;* $\lambda^*$ *is multiplicative, but not completely so;* $\Omega_1^*$ *is additive, but not completely so;* $\Omega_2^*$ *is generalized additive, but not additive.*

*Proof.* To establish that $\mu^*$ is generalized multiplicative suppose that the mosaics of $a$ and $b$ have no prime in common. Then, if a prime is repeated in either mosaic it is repeated in the mosaic of $a \cdot b$. If a prime is repeated in a mosaic of $a \cdot b$, it is repeated in the mosaic of $a$ or in the mosaic of $b$. For this case $\mu^*(a \cdot b) = \mu^*(a) \cdot \mu^*(b) = 0$. Hence we need only consider the case where the mosaics of $a$ and $b$ have no prime repeated within themselves individually, and no prime in common between them. However, then $\mu^*(a \cdot b) = \mu^*(a) \cdot \mu^*(b)$. Now, $\mu^*(225) = 0$; but $\mu^*(9) = 1$ and $\mu^*(25) = 1$. Hence $\mu^*$ is *not* multiplicative.

To establish that $\lambda^*$ is multiplicative, suppose that $(a, b) = 1$. Then, the summation of the number of primes in the mosaics of $a$ and $b$ act independently in total summations. Hence $\lambda^*(a \cdot b) = \lambda^*(a) \cdot \lambda^*(b)$. But $\lambda^*(16) = -1$, whereas $\lambda^*(4) \cdot \lambda^*(4) = 1$; thus $\lambda^*$ is not completely multiplicative.

To show that $\Omega_1^*$ is additive, recall that when $(a, b) = 1$ the total of primes in the mosaics act independently. Hence $\Omega_1^*(a \cdot b) = \Omega_1^*(a) + \Omega_1^*(b)$. Now $\Omega_1^*(16) = 3$, but $\Omega_1^*(4) + \Omega_1^*(4) = 4$; hence $\Omega_1^*$ is *not* completely additive.

To prove that $\Omega_2^*$ is generalized additive recall that when mosaics have no prime in common the numbers of their distinct primes act independently. Finally $\Omega_2^*(72) = 2$, but $\Omega_2^*(8) + \Omega_2^*(9) = 2 + 2 = 4$; hence $\Omega_2^*$ is not additive.

We close by noting that there are infinitely many generalized multiplicative arithmetic functions which are not multiplicative. This result follows since if $f_1$ and $f_2$ are *different* multiplicative functions which are not completely multiplicative, then $f_1(\ (\cdot))$ and $f_2(\ (\cdot))$ are *different* generalized multiplicative functions which are not multiplicative. But there are infinitely many multiplicative functions which are not completely multiplicative. Thus, $F_1(n) = \Sigma_{d/n}\sigma(d)$, $F_2(n) = \Sigma_{d/n}F_1(d), \ldots$, where $\sigma(d)$ is the sum of the divisors of $d$.

**3.** Now we relate some of these ideas to work by G. H. Hardy and S. Ramanujan [5;6].

*Definition 4.* Let $\Omega_1(n)$ be the total number of prime factors of $n$, and let $\Omega_2(n)$ be the number of *distinct* prime factors of $n$. Then we have, clearly,

*Lemma 4. For every natural number $n$, $\Omega_2(n) \le \Omega_2^*(n) \le \Omega_1^*(n) \le \Omega_1(n)$. Further, $\Omega_2(q) = \Omega_2^*(q) = \Omega_1^*(q) = \Omega_1(q)$, if $q$ is square-free.*

*Proof.* Every distinct prime factor of $n$ is a distinct prime in the mosaic of $n$; hence $\Omega_2(n) \le \Omega_2^*(n)$. Clearly $\Omega_2^*(n) \le \Omega_1^*(n)$ since the number of distinct primes in a mosaic cannot exceed the total number of primes in the mosaic. Finally $\Omega_1^*(n) \le \Omega_1(n)$, since for every prime in a mosaic there is at least one prime factor (and generally many prime factors). The last result follows because under this condition a mosaic is a product of distinct primes, and thus the canonic form of $q$ and the mosaic of $q$ coincide.

*Lemma 5. The normal order* [5;6] *of both $\Omega_1^*(n)$ and $\Omega_2^*(n)$ is* $\log(\log n)$.

*Proof.* Apply lemma 4, and recall Hardy's and Ramanujan's result that the normal order of both $\Omega_1(n)$ and $\Omega_2(n)$ is $\log(\log n)$.

## BIBLIOGRAPHY

[1] Fraenkel, A. A., *Integers and Theory of Numbers*, Yeshiva University, New York, 1955.

[2] Mullin, A. A., "Some related number-theoretic functions," *Bulletin of the American Mathematical Society*, Vol. 69 (1963), pp. 446-447.

[3] Mullin, A. A., "Models of the Fundamental Theorem of Arithmetic," *Proceedings of the National Academy of Sciences, USA*, Vol. 50 (1963), pp. 604-606.

[4] Erdös, P., "On the density of some sequences of numbers," *Journal of the London Mathematical Society*, Vol. 10 (1935), pp. 120-125.

[5] Hardy, G. H., and S. Ramanujan, "Proof that almost all numbers $n$ are composed of about log (log ($n$)) prime factors," *Proceedings of the London Mathematical Society*, Ser. 2, Vol. 16 (1917), pp. x-xii.

[6] Hardy, G. H. and S. Ramanujan, "The normal number of prime factors of a number $n$," *Quarterly Journal of Mathematics*, Vol. 48, (1917), pp. 76-92.

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